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ALGORITHM FOR COMPUTING A COMMON SOLUTION OF EQUILIBRIUM AND FIXED POINT PROBLEMS WITH SET-VALUED DEMICONTRACTIVE OPERATORS

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Abstract. We introduce an iterative algorithm based on the well-known Krasnoselskii–Mann's method for finding a common element of the set of fixed points of multivalued demicontractive mapping and the set of solutions of an equilibrium problem in a real Hilbert space. Then, the strong convergence of the scheme to a common element of the two sets is proved without imposing any compactness condition on the mapping or the space. We further apply our results to solve some optimization problems. Our results improve many recent results using Krasnoselskii–Mann's algorithm for solving nonlinear problems.

1. Introduction

Let C be a nonempty set and let f be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the real numbers. The equilibrium problem for f is to find $x \in C$ such that

$$f(x, y) \geq 0, \quad \text{for all } y \in C.$$

The set of solutions is denoted by $EP(f)$. Equilibrium problems introduced by Fan [8] and Blum and Oettli [1], have had a great impact and influence on the development of several branches of pure and applied sciences. However, there were few iterative algorithms developed for the approximation of solutions of equilibrium problems; see [1, 5, 16, 24–26] and the references therein.

Let K be a nonempty subset of a real Hilbert space H and let $T : K \to 2^K$ be a multivalued mapping. An element $x \in K$ is called a fixed point of T if $x \in Tx$.

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The fixed point set of T is denoted by $Fix(T) := \{x \in K : x \in Tx\}$. It is easy to see that the single-valued mapping is a particular case of multivalued mappings. For several years, the study of fixed point theory for *single-valued and multivalued nonlinear mappings* has attracted, and continues to attract, the interest of several well-known mathematicians (see, for example, Brouwer [2], Kakutani [14], Nash [21, 22], Geanakoplos [10], Nadla [20], Downing and Kirk [7], Sow, Djitté, and Chidume [28], Markin [18], Lim [15], and Gorniewicz [6, 11, 12]).

Interest in the study of fixed point theory for multivalued nonlinear mappings stems, perhaps, is mainly from its usefulness in real-world applications such as game theory and market economy and in other areas of mathematics, such as in nonsmooth differential equations and differential inclusions, optimization theory.

Let $CB(K)$ and $P(K)$ denote the family of nonempty closed bounded subsets and nonempty proximinal bounded subsets of K, respectively. The *Pompeiu Hausdorff metric* on $CB(K)$ is defined by

$$D(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\}$$

for all $A, B \in CB(K)$. A multivalued mapping $T : K \to CB(K)$ is called *L-Lipschitzian* if there exists $L > 0$ such that

$$D(Tx,Ty) \le L\|x-y\| \quad \text{for all } x,y \in K,$$

and if $L = 1$ T is called a nonexpansive mapping.

A multivalued map T is called quasi-nonexpansive if

$$D(Tx,Tp) \le \|x-p\|$$

holds for all $x \in K$ and $p \in Fix(T)$. It is easy to see that the class of multivalued quasi-nonexpansive mappings properly includes that of multivalued nonexpansive maps with fixed points.

A multivalued mapping $T : K \to CB(K)$ is said to be *k-strictly pseudo-contractive*, if there exists $k \in (0,1)$ such for all $x, y \in K$, we have

$$\Big(D(Tx,Ty)\Big)^2 \le \|x-y\|^2 + k\|(x-u)-(y-v)\|^2 \quad \text{for all } u \in Tx,\ v \in Ty. \tag{1.1}$$

If $k = 1$ in (1.1), the map T is said to be pseudo-contractive.

A map $T : K \to 2^K$ is said to be demicontractive if $Fix(T) \neq \emptyset$ and for all $p \in Fix(T), x \in K$ there exists $k \in [0,1)$ such that

$$\Big(D(Tx,Tp)\Big)^2 \le \|x-p\|^2 + kd(x,Tx)^2. \tag{1.2}$$

If $k = 1$ in (1.2), the map T is said to be hemicontractive.

Remark 1.1. It is easily seen that any multivalued nonexpansive, quasi-nonexpansive, and k-strictly pseudo-contractive mappings are k-demicontractive for any $k \in [0,1)$. Moreover the inverse is not true (see, for example, Isiogugu and Osilike [13]).

Over the last years, one may see an increasing interest in the study of equilibrium and fixed point problems; see, for instance, [1, 4, 12, 24, 29].

For nonlinear mappings with fixed points, the Mann iterative method [17] is a valuable tool to study them. However, only the weak convergence is guaranteed in infinite-dimensional spaces. A lot of works have been done for the modification of the normal Manns iteration so that the strong convergence is guaranteed (see, e.g., [26, 28, 31]). In 2017, Fan and Yao [9], motivated by the fact that the Krasnoselskii–Mann algorithm method is remarkably useful for finding fixed points of nonexpansive mapping, extended and improved many existing results in current literature.

However, we observe that in [4], the recursion formula studied is not simpler.

In this paper, motivated by the works of [4,9] and ongoing results, we prove the strong convergence theorems for finding a common element of the set of common fixed points of multivalued demicontractive mapping and the set of solutions of an equilibrium problem in a real Hilbert space. Our contribution lies in the fact that our iterative method solves the fixed point problem for set-valued mappings and equilibrium problem at same time. Finally, some applications are given to validate our new findings.

2. Main results

The demiclosedness of a nonlinear operator T usually plays an important role in dealing with the convergence of fixed point iterative algorithms.

Definition 2.1. Let H be a real Hilbert space and let $T : D(T) \subset H \to 2^H$ be a multivalued mapping. Then $I - T$ is said to be demiclosed at 0 if for any sequence $\{x_n\} \subset D(T)$ such that $\{x_n\}$ converges weakly to p and $d(x_n, Tx_n)$ converges to zero, then $p \in Tp$.

Now we state our main result.

Theorem 2.2. *Let H be a real Hilbert space and let K be a nonempty, closed convex cone of H. Let f be a bifunction from $K \times K \to \mathbb{R}$ satisfying (A1)–(A4) and let $S : K \to CB(K)$ be a multivalued β-demicontractive mapping such that $F := EP(f) \cap Fix(S) \neq \emptyset$ and $Sp = \{p\}$, for all $p \in F$. Assume that $I - S$ is demiclosed at the origin. Let $\{x_n\}$ and $\{u_n\}$ be sequences defined iteratively from arbitrary $x_0 \in K$ by*

$$\begin{cases} f(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \text{for all } y \in K, \\ y_n = \theta_n u_n + (1-\theta_n)w_n, \ \ w_n \in Su_n, \\ x_{n+1} = \alpha_n(\lambda_n x_n) + (1-\alpha_n)y_n, \ \ n \geq 0, \end{cases} \tag{2.1}$$

where $\{\alpha_n\} \subset (0,1)$, $\{\theta_n\}$, $\{\lambda_n\} \subset (0,1)$, and $\{r_n\} \subset]0,\infty[$ satisfy

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty}(1-\lambda_n)\alpha_n = \infty$ *and* $\lim_{n\to\infty} \lambda_n = 1$;

(ii) $\theta_n \in]\beta, 1[$ *and* $\lim_{n\to\infty} \inf(1-\theta_n)(\theta_n - \beta) > 0$,

(iii) $\lim_{n\to\infty} \inf r_n > 0$.

Then, $\{x_n\}$ *and* $\{u_n\}$ *defined by* (2.1) *converge strongly to* $x^* \in F$*, where* $x^* = P_F(0)$.

Proof. By using properties of K, we have $t(\lambda x)+(1-t)y \in K$, for all $\lambda, t \in (0,1)$, and $x, y \in K$. Therefore, the sequence $\{x_n\}$ generated by (2.1) is well defined. Now, we prove that the sequence $\{x_n\}$ is bounded. Let $p \in F$. Then from $u_n = T_{r_n}x_n$, we have

$$\|u_n - p\| = \|T_{r_n}x_n - T_{r_n}p\| \le \|x_n - p\|, \quad \text{for all } n \ge 0.$$

From (2.1) and the fact that $Sp = \{p\}$, we have

$$\begin{aligned}
\|y_n - p\|^2 &= \Big\|\theta_n(u_n - p) + (1-\theta_n)(w_n - p)\Big\|^2 \\
&= \theta_n\|u_n - p\|^2 + (1-\theta_n)\|w_n - p\|^2 - \theta_n(1-\theta_n)\|w_n - u_n\|^2.
\end{aligned}$$

Using the fact that S is β-demi-contractive and $Sp = \{p\}$, we obtain

$$\begin{aligned}
\|y_n - p\|^2 &\le \theta_n\|u_n - p\|^2 + (1-\theta_n)D(Su_n, Sp)^2 - (1-\theta_n)\theta_n\|u_n - w_n\|^2 \\
&\le \theta_n\|u_n - p\|^2 + (1-\theta_n)\Big(\|u_n - p\|^2 + \beta d(u_n, Su_n)^2\Big) \\
&\quad -(1-\theta_n)\theta_n\|u_n - w_n\|^2.
\end{aligned}$$

Hence,

$$\|y_n - p\| \le \|u_n - p\|^2 - (1-\theta_n)(\theta_n - \beta)\|u_n - w_n\|^2. \tag{2.2}$$

Since $\theta_n \in]\beta, 1[$, we obtain

$$\|y_n - p\Big\| \le \|u_n - p\Big\|.$$

Therefore

$$\|y_n - p\| \le \|u_n - p\| \le \|x_n - p\|. \tag{2.3}$$

Using (2.1) and inequality (2.3), we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n(\lambda_n x_n) + (1-\alpha_n)y_n - p\| \\
&\le \alpha_n\lambda_n\|x_n - p\| + (1-\lambda_n)\alpha_n\|p\| + (1-\alpha_n)\|y_n - p\| \\
&\le \alpha_n\lambda_n\|x_n - p\| + (1-\lambda_n)\alpha_n\|p\| + (1-\alpha_n)\|x_n - p\| \\
&= [1-(1-\lambda_n)\alpha_n]\|x_n - p\| + (1-\lambda_n)\alpha_n\|p\|.
\end{aligned}$$

Therefore

$$\|x_{n+1} - p\| \le \max\{\|x_n - p\|, \|p\|\}.$$

Hence, $\{x_n\}$ is bounded and so $\{y_n\}$. From (2.1), (2.2), and convexity of $\|.\|^2$, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n(\lambda_n x_n) + (1-\alpha_n)y_n - p\|^2 \\
&\le \alpha_n\|\lambda_n x_n - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 \\
&\le \alpha_n\|\lambda_n x_n - p\|^2 + (1-\alpha_n)\Big[\|u_n - p\|^2
\end{aligned}$$

$$
\begin{aligned}
&\quad -(1-\theta_n)(\theta_n-\beta)\|u_n-w_n\|^2] \\
&\leq \alpha_n\|\lambda_n x_n - p\|^2 + \|u_n-p\|^2 - (1-\theta_n)(\theta_n-\beta)\|u_n-w_n\|^2 \\
&\leq \|x_n-p\|^2 + \alpha_n\|\lambda_n x_n - p\|^2 - (1-\theta_n)(\theta_n-\beta)\|u_n-w_n\|^2.
\end{aligned}
$$

Therefore,

$$(1-\theta_n)(\theta_n-\beta)\|u_n-w_n\|^2 \leq \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + \alpha_n\|\lambda_n x_n - p\|^2.$$

Since $\{x_n\}$ is bounded, then there exists a constant $B>0$ such that

$$\|\lambda_n x_n - p\|^2 \leq B, \text{ for all } n \geq 0.$$

Hence,

$$(1-\theta_n)(\theta_n-\beta)\|u_n-w_n\|^2 \leq \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + \alpha_n B. \tag{2.4}$$

Now we prove that $\{x_n\}$ converges strongly to x^*. We divide the proof into two cases.

Case 1. Assume that the sequence $\{\|x_n-p\|\}$ is monotonically decreasing. Then $\{\|x_n-p\|\}$ is convergent. Clearly, we have

$$\lim_{n\to\infty}\|x_n-p\|^2 - \|x_{n+1}-p\|^2 = 0. \tag{2.5}$$

It then implies from (2.4) that

$$\lim_{n\to\infty}(1-\theta_n)(\theta_n-\beta)\|u_n-w_n\|^2 = 0.$$

Using the fact that $\lim_{n\to\infty}\inf(1-\theta_n)(\theta_n-\beta)$, we have

$$\lim_{n\to\infty}\|u_n-w_n\| = 0.$$

Hence,

$$\lim_{n\to\infty} d(u_n, Su_n) = 0. \tag{2.6}$$

Let $p \in F$. Then

$$
\begin{aligned}
\|u_n-p\|^2 &= \|T_{r_n}x_n - T_{r_n}p\|^2 \\
&\leq \langle T_{r_n}x_n - T_{r_n}p, x_n-p\rangle \\
&\leq \langle u_n-p, x_n-p\rangle \\
&= \frac{1}{2}(\|u_n-p\|^2 + \|x_n-p\|^2 - \|x_n-u_n\|^2),
\end{aligned}
$$

and hence

$$\|u_n-p\|^2 \leq \|x_n-p\|^2 - \|x_n-u_n\|^2.$$

Therefore, from (2.1), we obtain

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_n(\lambda_n x_n) + (1-\alpha_n)y_n - p\|^2 \\
&\leq \|\alpha_n((\lambda_n x_n)-p) + (1-\alpha_n)(y_n-p)\|^2 \\
&\leq (1-\alpha_n)^2\|y_n-p\|^2 + 2\alpha_n\langle(\lambda_n x_n)-p, x_{n+1}-p\rangle \\
&\leq (1-\alpha_n)^2\|u_n-p\|^2 + 2\alpha_n\lambda_n\langle x_n-p, x_{n+1}-p\rangle \\
&\quad +2(1-\lambda_n)\alpha_n\langle p, x_{n+1}-p\rangle \\
&\leq (1-\alpha_n)^2(\|x_n-p\|^2 - \|x_n-u_n\|^2) \\
&\quad +2\alpha_n\lambda_n\|x_n-p\|\|x_{n+1}-p\| + 2\alpha_n(1-\lambda_n)\|p\|\|x_{n+1}-p\|
\end{aligned}
$$

$$\begin{aligned} &\leq (1-2\alpha_n+\alpha_n^2)\|x_n-p\|^2-(1-\alpha_n)^2\|x_n-u_n\|^2 \\ &\quad +2\alpha_n\lambda_n\|x_n-p\|\|x_{n+1}-p\|+2\alpha_n(1-\lambda_n)\|p\|\|x_{n+1}-p\| \\ &\leq \|x_n-p\|^2+\alpha_n\|x_n-p\|^2-(1-\alpha_n)^2\|x_n-u_n\|^2 \\ &\quad +2\alpha_n\lambda_n\|x_n-p\|\|x_{n+1}-p\|+2\alpha_n(1-\lambda_n)\|p\|\|x_{n+1}-p\|, \end{aligned}$$

and hence

$$\begin{aligned} (1-\alpha_n)^2\|x_n-u_n\|^2 \leq\ & \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n\|x_n-p\|^2 \\ & +2\alpha_n\lambda_n\|x_n-p\|\|x_{n+1}-p\| \\ & +2\alpha_n(1-\lambda_n)\|p\|\|x_{n+1}-p\|. \end{aligned}$$

Thanks inequality (2.5) and $\alpha_n \to 0$ as $n \to \infty$, we have

$$\|x_n-u_n\| \to 0 \text{ as, } n \to \infty.$$

Next, we prove that $\limsup_{n\to+\infty}\langle x^*, x^*-x_n\rangle \leq 0$. We choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to+\infty}\langle x^*, x^*-x_n\rangle = \lim_{k\to+\infty}\langle x^*, x^*-x_{n_k}\rangle.$$

Since H is reflexive and $\{u_{n_k}\}$ is bounded, there exists a subsequence $\{u_{n_{k_j}}\}$ of $\{u_{n_k}\}$ that converges weakly to $a \in K$. From (2.6) and the fact that $I-S$ is demiclosed, we obtain $a \in Fix(S)$. Without loss of generality, we can assume that $u_{n_k} \rightharpoonup a$. Let us show $a \in EP(f)$. By the same argument as in the proof of [1] and $(A2)$ that

$$\frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq f(y,u_n)$$

and hence

$$\langle y-u_{n_k}, \frac{u_{n_k}-x_{n_k}}{r_{n_k}}\rangle \geq f(y,u_{n_k}).$$

Since $\frac{u_{n_k}-x_{n_k}}{r_{n_k}} \to 0$ and $u_{n_k} \rightharpoonup a$, it follows from $(A4)$ that $f(y,a) \leq 0$ for all $y \in K$. For t with $0<t<1$ and $y \in K$, let $y_t = ty+(1-t)a$. Since $y \in K$ and $a \in K$, we have $y_t \in K$ and hence $f(y_t,a) \leq 0$. Therefore, from $(A1)$ and $(A4)$, we have

$$0 = f(y_t,y_t) \leq tf(y_t,y)+(1-t)f(y_t,a) \leq tf(y_t,y)$$

and hence $0 \leq f(y_t,y)$. From $(A3)$, we have $f(a,y) \geq 0$ for all $y \in K$ and hence $a \in EP(f)$. Therefore, $a \in Fix(S) \cap EP(f) = F$.

On other hand, using the fact that $x^* = P_F(0)$ (properties of metric projection), we then have

$$\begin{aligned} \limsup_{n\to+\infty}\langle x^*, x^*-x_n\rangle &= \lim_{k\to+\infty}\langle x^*, x^*-x_{n_k}\rangle \\ &= \langle x^*, x^*-a)\rangle \leq 0. \end{aligned}$$

Finally, we show that $x_n \to x^*$. From (2.1), we have

$$\|x_{n+1}-x^*\|^2 = \langle x_{n+1}-x^*, x_{n+1}-x^*\rangle = \alpha_n\lambda_n\langle x_n-x^*, x_{n+1}-x^*\rangle$$

$$
\begin{aligned}
& +(1-\lambda_n)\alpha_n\langle x^*, x^*-x_{n+1}\rangle + (1-\alpha_n)\langle y_n - x^*, x_{n+1}-x^*\rangle \\
\le\ & \alpha_n\lambda_n\langle x_n - x^*, x_{n+1}-x^*\rangle + (1-\lambda_n)\alpha_n\langle x^*, x^*-x_{n+1}\rangle \\
& +(1-\alpha_n)\|y_n - x^*\|\|x_{n+1}-x^*\| \\
\le\ & \alpha_n\lambda_n\|x_n - x^*\|\|x_{n+1}-x^*\| + (1-\lambda_n)\alpha_n\langle x^*, x^*-x_{n+1}\rangle \\
& +(1-\alpha_n)\|x_n - x^*\|\|x_{n+1}-x^*\| \\
\le\ & [1-(1-\lambda_n)\alpha_n]\|x_n - x^*\|\|x_{n+1}-x^*\| \\
& +(1-\lambda_n)\alpha_n\langle x^*, x^*-x_{n+1}\rangle \\
\le\ & \frac{1-(1-\lambda_n)\alpha_n}{2}(\|x_n - x^*\|^2 + \|x_{n+1}-x^*\|^2) \\
& +(1-\lambda_n)\alpha_n\langle x^*, x^*-x_{n+1}\rangle,
\end{aligned}
$$

which implies that

$$\|x_{n+1}-x^*\|^2 \le [1-(1-\lambda_n)\alpha_n]\|x_n - x^*\| + 2(1-\lambda_n)\alpha_n\langle x^*, x^*-x_{n+1}\rangle.$$

Moreover, thanks to [30], we then have $x_n \to x^*$.

Case 2. Assume that the sequence $\{\|x_n - x^*\|\}$ is not monotonically decreasing. Set $B_n = \|x_n - x^*\|$, and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \ge n_0$ (for some n_0 large enough) by $\tau(n) = \max\{k \in \mathbb{N} : k \le n,\ \ B_k \le B_{k+1}\}$.
We have τ is a nondecreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $B_{\tau(n)} \le B_{\tau(n)+1}$ for $n \ge n_0$. From (2.4), we have

$$(1-\alpha_{\tau(n)})(\theta_{\tau(n)}-\beta)(1-\theta_{\tau(n)})\|u_{\tau(n)} - w_{\tau(n)}\|^2 \le \alpha_{\tau(n)}B \to 0 \ \text{ as } \ n \to \infty.$$

Furthermore, we have

$$\|u_{\tau(n)} - w_{\tau(n)}\| \to 0 \ \text{ as } \ n \to \infty.$$

Hence,

$$\lim_{n\to\infty} d\Big(u_{\tau(n)}, Su_{\tau(n)}\Big) = 0.$$

By the same argument as in case 1, we can show that $x_{\tau(n)}$ converges weakly in H and $\limsup\limits_{n\to+\infty}\langle x^*, x^* - x_{\tau(n)}\rangle \le 0$. We have for all $n \ge n_0$,

$$
\begin{aligned}
0 \le \|x_{\tau(n)+1} - x^*\|^2 - \|x_{\tau(n)} - x^*\|^2 \ \ \le\ & \Big(1-\lambda_{\tau(n)}\Big)\alpha_{\tau(n)}[-\|x_{\tau(n)} - x^*\|^2 \\
& +2\langle x^*, x^* - x_{\tau(n)+1}\rangle],
\end{aligned}
$$

which implies that

$$\|x_{\tau(n)} - x^*\|^2 \le 2\langle x^*, x^* - x_{\tau(n)+1}\rangle.$$

Then, we have

$$\lim_{n\to\infty}\|x_{\tau(n)} - x^*\|^2 = 0.$$

Therefore,

$$\lim_{n\to\infty} B_{\tau(n)} = \lim_{n\to\infty} B_{\tau(n)+1} = 0.$$

Moreover, thanks to [16], we then have

$$0 \le B_n \le \max\{B_{\tau(n)},\ B_{\tau(n)+1}\} = B_{\tau(n)+1}.$$

Hence, $\lim_{n\to\infty} B_n = 0$, that is, $\{x_n\}$ converges strongly to x^*. □

Remark 2.3. Many already studied problems in the literature can be considered as special cases of this paper; see, for example, [1,4,9,12,24,28,29] and the references therein. Our results are applicable for finding a common solution of variational and fixed point problems involving set-valued operators in real Hilbert spaces (see, for example, [27] for more details).

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References

1. E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Student **63** (1994) 123–145.
2. L.E.J. Brouwer, *Uber Abbildung von Mannigfaltigkeiten*, Math. Ann. **71** (1912), no. 4, 598.
3. S. Chang, Y. Tang, L. Wang, Y. Xu, Y. Zhao, and G. Wang, *Convergence theorems for some multi-valued generalized nonexpansive mappings*, Fixed Point Theory Appl. **2014** (2014), no 33, 11 pp.
4. P. Cholamjiak, W. Cholamjiak and S. Suantai, *Viscosity approximation methods for nonexpansive multivalued nonself mapping and equilibrium problem*, Demonstr. Math. **47** (2014) 382–395.
5. P.L. Combettes and S.A. Hirstoaga, *Equilibrium programming using proximal-like algorithms*, Math. Program. **78** (1997) 29–41.
6. P.L. Combettes and S.A. Hirstoaga, *Equilibrium programming in Hilbert spaces*, J. Nonlinear Convex Anal. **6** (2005) 117–136.
7. D. Downing and W.A. Kirk, *Fixed point theorems for set-valued mappings in metric and Banach spaces*, Math. Japon. **22** (1977), no. 1, 99–112.
8. K. Fan, *A minimax inequality and applications. Inequalities, III,* in: Proc. Third Sympos. Univ. California, Los Angeles, Calif. 1969; dedicated to the memory of Theodore S. Motzkin), pp. 103–113. Academic Press, New York, 1972.
9. Q. Fan and Z. Yao, *Strong convergence theorems for a nonexpansive mapping and its applications for solving the split feasibility problem*, J. Nonlinear Sci. Appl. **10** (2017), no.4, 1470–1477.
10. J. Geanakoplos, *Nash and Walras equilibrium via Brouwer,* in: Symposium in Honor of Mordecai Kurz (Stanford, CA, 2002), Econom. Theory **21** (2003), no. 2-3, 585–603.
11. L. Gorniewicz, *Topological Fixed Point Theory of Multivalued Mappings*, Kluwer Academic Publishers, Dordrecht, 1999.
12. H. He, S. Liu and Y.J. Cho, *An explicit method for systems of equilibrium problems and fixed points of infinite family of nonexpansive mappings*, J. Comput. Appl. Math. **235** (2011) 4128–4139.
13. F.O. Isiogugu and M.O. Osilike, *Convergence theorems for new classes of multivalued hemicontractive-type mappings*, Fixed Point Theory Appl. **2014**, (2014), no. 93, 12 pp.
14. S. Kakutani, *A generalization of Brouwer's fixed point theorem*, Duke Math. J. **8** (1941) 457–459.
15. T.C. Lim and H.K. Xu, *Fixed point theorems for assymptotically nonexpansive mapping*, Nonliear Anal. **22** (1994), no. 11, 1345–1355.
16. P.E. Mainge, *Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization*, Set-Valued Anal. **16** (2008), no. 7-8, 899–912.
17. W.R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953) 506–510.
18. J.T. Markin, *Continuous dependence of fixed point sets*, Proc. Amer. Math. Soc. **38** (1973) 545–547.

19. A. Moudafi, *Viscosity approximation methods for fixed point problems*, J. Math. Anal. Appl. **241** (2000) 46–55.
20. S.B. Nadler, Jr. *Multivalued contraction mappings*, Pacific J. Math. **30** (1969) 475–488.
21. J.F. Nash, *Equilibrium points in n-person games*, Proc. Natl. Acad. Sci. U.S.A. **36** (1950) 48–49.
22. J.F. Nash, *Non-cooperative games*, Ann. of Math. **54** (1951) 286–295.
23. B. Panyanak, *Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces*, Comput. Math. Appl. **54** (2007) 872–877.
24. N. Petrot, K. Wattanawitoon and P. Kumam, *A hybrid projection method for generalized mixed equilibrium problems and fixed point problems in Banach spaces*, Nonlinear Anal. Hybrid Syst. **4** (2010), no. 4, 631–643.
25. X. Qin, S.Y. Cho and S.M. Kang, *Convergence of an iterative algorithm for systems of variational inequalities and nonexpansive mappings with applications*, J. Comput. Appl. Math. **233** (2009), no. 2, 231–240.
26. X. Qin, Y.J. Cho, S.M. Kang and H. Zhou, *Convergence of a modified Halpern-type iteration algorithm for quasi-ϕ-nonexpansive mappings,* Appl. Math. Lett. **22** (2009), no. 7, 1051–1055.
27. S. Shoham, *Iterative Methods for Solving Optimization Problems*, Technion-Isreal Institute of Technology, Haifa, 2012.
28. T.M.M. Sow, N. Djitté and C.E. Chidume, *A path convergence theorem and construction of fixed points for nonexpansive mappings in certain Banach spaces*, Carpathian J. Math. **32** (2016), no. 2, 241–250.
29. S. Takahashi and W. Takahashi, *Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces*, J. Math. Anal. Appl. **331** (2007) 506–515.
30. H.K. Xu, *Iterative algorithms for nonlinear operators*, J. Lond. Math. Soc. **66** (2002), no. 1, 240–256.
31. Y. Yao and Y.Je. Cho *Strong convergence of modified Krasnoselskii–Mann iterative algorithm for non-expansive mappings in Hilbert spaces*, Math. Model. Anal. **15** (2010) 265–274.

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