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THE (p, q, r)-GENERATIONS OF THE ALTERNATING GROUP A_{11}

AYOUB B. M. BASHEER[1] , MALEBOGO J. MOTALANE[1*] AND THEKISO T. SERETLO[1]

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Abstract. A finite group G is called (l, m, n)-*generated*, if it is a quotient group of the triangle group $T(l, m, n) = \langle x, y, z | x^l = y^m = z^n = xyz = 1 \rangle$. Moori posed the question of finding all the (p, q, r) triples, where p, q, and r are prime numbers, such that a non-abelian finite simple group G is a (p, q, r)-generated. In this paper, we establish all the (p, q, r)-generations of the alternating group A_{11}. The Groups, Algorithms and Programming and the Atlas of finite group representations are used in our computations.

1. Introduction

Generations of finite groups by suitable subsets are of great interest and have many applications to groups and their representations. For example, the computations of the genus of simple groups can be reduced to the generations of the relevant simple groups (see [34] for details). Also Di Martino, Pellegrini, and Zalesski [26] established a useful connection between the generation of groups by conjugates and the existence of elements representable by almost cyclic matrices. Their motivation was to study irreducible projective representations of sporadic simple groups. Recently more attention is given to the generation of finite groups by conjugate elements. Ward, in his PhD thesis [32], considered a generation of a simple group by conjugate involutions satisfying certain conditions. In this paper, we are interested in the generation of the alternating group A_{11} by two elements

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of prime orders not necessarily distinct such that the product is an element of a prime order.

A finite group G is said to be (l,m,n)-*generated*, if $G = \langle x, y\rangle$, with $o(x) = l$, $o(y) = m$ and $o(xy) = o(z) = n$. Here $[x] = lX$, $[y] = mY$, and $[z] = nZ$, where $[x]$ is the conjugacy class of X in G containing elements of order l. The same applies to $[y]$ and $[z]$. In this, G is also a quotient group of the triangular group $T(l,m,n)$, and, by the definition of the triangular group, G is also a $(\sigma(l),\sigma(m),\sigma(n))$-generated group for any $\sigma \in S_3$. Therefore we may assume that $l \leq m \leq n$. In a series of papers [20–24, 27, 28], Moori and Ganief established all possible (p,q,r)-generations, where p, q, and r are distinct primes, of the sporadic groups J_1, J_2, J_3, HS, McL, Co_3, Co_2, and F_{22}. Ashrafi [3, 4] did the same for the sporadic simple groups He and HN. Also Darafsheh and Ashrafi established in [14–17], the (p,q,r)-generations of the sporadic simple groups Co_1, Ru, $O'N$, and Ly. The motivation for this study is outlined in these papers, and the reader is encouraged to consult these papers for background material as well as basic computational techniques.

In this paper, we intend to establish all the (p,q,r)-generations of the alternating group A_{11}. For more information on (p,q,r)-generations, the reader is referred to [1, 2]. We follow the methods used in the papers [5–11]. Note that, in general, if G is a $(2,2,n)$-generated group, then G is a dihedral group and therefore G is not simple. Also by [12], if G is a non-abelian (l,m,n)-generated group, then either $G \cong A_5$ or $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$. Thus for our purpose of establishing the (p,q,r)-generations of $G = A_{11}$, the only cases we need to consider are when $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. The result on the (p,q,r)-generations of A_{11} can be summarized in the following theorem.

Theorem 1.1. *With the notation being as in the Atlas* [13]*, the alternating group* A_{11} *is generated by all the triples* (p,q,r), *where* p, q, *and* r *are primes dividing* $|A_{11}|$, *except for the cases* $(p,q,r) \in \{(2,3,7),\ (2X,3Y,11Z),\ (2A,5B,5B),$ $(2X,5A,5Y),\ (2X,5Y,7A),\ (2X,5A,11Y),\ (2A,7A,11X),\ (3V,3W,5X),$ $(3V,3W,7A),\ (3A,3V,11X),\ (3B,3B,11X),\ (3V,5A,5X),\ (3A,5B,5B),$ $(3X,5Y,7A),\ (3C,5A,7A),(3X,5A,11Y),\ (3V,7A,7A),\ (3A,7A,11X),$ $(5A,5A,5X),\ (5A,5X,7A),\ (5A,5A,11X),\ (5A,7A,7A)\}$ *for all* $X,Y,Z \in \{A,B\}$ *and* $V,W \in \{A,B,C\}$.

2. Preliminaries

Let G be a finite group and let $C_1, C_2, \ldots, C_k$ (not necessarily distinct) for $k \geq 3$ be conjugacy classes of G with $g_1, g_2, \ldots, g_k$ being representatives for these classes, respectively.

For a fixed representative $g_k \in C_k$ and for $g_i \in C_i$, $1 \leq i \leq k-1$, denote by $\Delta_G = \Delta_G(C_1, C_2, \ldots, C_k)$ the number of distinct $(k-1)$-tuples $(g_1, g_2, \ldots, g_{k-1}) \in C_1 \times C_2 \times \cdots \times C_{k-1}$ such that $g_1 g_2 \cdots g_{k-1} = g_k$. This number is known as *class algebra constant* or *structure constant*. With $\mathrm{Irr}(G) = \{\chi_1, \chi_2, \ldots, \chi_r\}$, the number Δ_G is easily calculated from the character table of G through the formula

$$\Delta_G(C_1, C_2, \ldots, C_k) = \frac{\prod_{i=1}^{k-1} |C_i|}{|G|} \sum_{i=1}^{r} \frac{\chi_i(g_1)\chi_i(g_2)\cdots\chi_i(g_{k-1})\overline{\chi_i(g_k)}}{(\chi_i(1_G))^{k-2}}.$$

Also for a fixed $g_k \in C_k$, we denote by $\Delta_G^*(C_1, C_2, \ldots, C_k)$ the number of distinct $(k-1)$-tuples $(g_1, g_2, \ldots, g_{k-1})$ satisfying

$$g_1 g_2 \cdots g_{k-1} = g_k \quad \text{and} \quad G = \langle g_1, g_2, \ldots, g_{k-1} \rangle .$$

Definition 2.1. If $\Delta_G^*(C_1, C_2, \ldots, C_k) > 0$, then the group G is said to be $(C_1, C_2, \ldots, C_k)$**-generated**.

Remark 2.2. A group G is $(C_1, C_2, \ldots, C_k)$**-generated** if and only if $\Delta_G^*(C_1, C_2, \ldots, C_k) \geq 0$.

Furthermore if H is any subgroup of G containing a fixed element $h_k \in C_k$, we let $\Sigma_H(C_1, C_2, \ldots, C_k)$ be the total number of distinct tuples $(h_1, h_2, \ldots, h_{k-1})$ such that

$$h_1 h_2 \cdots h_{k-1} = h_k \quad \text{and} \quad \langle h_1, h_2, \ldots, h_{k-1} \rangle \leq H.$$

The value of $\Sigma_H(C_1, C_2, \ldots, C_k)$ can be obtained as a sum of the structure constants $\Delta_H(c_1, c_2, \ldots, c_k)$ of H-conjugacy classes $c_1, c_2, \ldots, c_k$ such that $c_i \subseteq H \cap C_i$.

Theorem 2.3. *Let G be a finite group and let H be a subgroup of G containing a fixed element g such that $\gcd(o(g), [N_G(H){:}H]) = 1$. Then the number $h(g, H)$ of conjugates of H containing g is $\chi_H(g)$, where $\chi_H(g)$ is the permutation character of G with action on the conjugates of H. In particular*

$$h(g, H) = \sum_{i=1}^{m} \frac{|C_G(g)|}{|C_{N_G(H)}(x_i)|},$$

where $x_1, x_2, \ldots, x_m$ are representatives of the $N_G(H)$-conjugacy classes fused to the G-class of g.

Proof. See [20] and [22, Theorem 2.1]. □

The above number $h(g, H)$ is useful in giving a lower bound for $\Delta_G^*(C_1, C_2, \ldots, C_k)$, namely, $\Delta_G^*(C_1, C_2, \ldots, C_k)$, where

$$\Delta_G^*(C_1, \ldots, C_k) \geq \Delta_G(C_1, \ldots, C_k) - \sum h(g_k, H)\Sigma_H(C_1, \ldots, C_k),$$

where g_k is a representative of the class C_k and the sum is taken over all the representatives H of G-conjugacy classes of maximal subgroups of G containing elements of all the classes $C_1, C_2, \ldots, C_k$. Since we have all the maximal subgroups of the sporadic simple groups except for $G = \mathbb{M}$ the Monster group, it is possible to build a small subroutine in GAP [25] to compute the values of $\Delta_G^* = \Delta_G(C_1, C_2, \ldots, C_k)$ for any collection of conjugacy classes and an alternating simple group.

The following results are in some cases useful in establishing nongeneration for finite groups.

Lemma 2.4. *Let G be a finite centerless group. If $\Delta^*_G(C_1, C_2, \ldots, C_k) < |C_G(g_k)|$, $g_k \in C_k$, then $\Delta^*_G(C_1, C_2, \ldots, C_k) = 0$, and therefore G is not $(C_1, C_2, \ldots, C_k)$-generated.*

Proof. See [6, Lemma 2.7]. □

Theorem 2.5 (Ree [29])**.** *Let G be a transitive permutation group generated by permutations $g_1, g_2, \ldots, g_s$ acting on a set of n elements such that $g_1 g_2 \cdots g_s = 1_G$. If the generator g_i has exactly c_i cycles for $1 \leq i \leq s$, then $\sum_{i=1}^{s} c_i \leq (s-2)n + 2$.*

For the alternating group $G = A_{11}$ and by the Atlas of finite group representations [33], we have G acting on 11 points, so that $n = 11$, and since our generation is triangular, we have $s = 3$. Hence if G is (l, m, n)-generated, then $\sum c_i \leq 13$.

Theorem 2.6 (Scott [30])**.** *Let $g_1, g_2, \ldots, g_s$ be elements generating a group G with $g_1 g_2 \cdots g_s = 1_G$ and let $\mathbb{V}$ be an irreducible module for G with $\dim \mathbb{V} = n \geq 2$. Let $C_{\mathbb{V}}(g_i)$ denote the fixed point space of $\langle g_i \rangle$ on $\mathbb{V}$ and let d_i be the codimension of $C_{\mathbb{V}}(g_i)$ in $\mathbb{V}$. Then $\sum_{i=1}^{s} d_i \geq 2n$.*

With χ being the ordinary irreducible character afforded by the irreducible module $\mathbb{V}$ and $\mathbf{1}_{\langle g_i \rangle}$ being the trivial character of the cyclic group $\langle g_i \rangle$, the codimension d_i of $C_{\mathbb{V}}(g_i)$ in $\mathbb{V}$ can be computed using the following formula (see [18]):

$$\begin{aligned} d_i &= \dim(\mathbb{V}) - \dim(C_{\mathbb{V}}(g_i)) = \dim(\mathbb{V}) - \left\langle \chi\downarrow^{G}_{\langle g_i \rangle}, \mathbf{1}_{\langle g_i \rangle} \right\rangle \\ &= \chi(1_G) - \frac{1}{|\langle g_i \rangle|} \sum_{j=0}^{o(g_i)-1} \chi(g_i^j). \end{aligned}$$

Theorem 2.7 ([6, Lemma 2.5])**.** *Let G be a $(2X, sY, tZ)$-generated simple group; then G is $(sY, sY, (tZ)^2)$-generated.*

Theorem 2.8 ([24, Theorem 1.2])**.** *Let G be a finite group and let l, m, and n be integers that are pairwise coprime. Then for any integer t coprime to n, we have*

$$\Delta_G(lx, mY, nZ) = \Delta_G(lX, mY, (nZ)^t).$$

Moreover, G is (lX, mY, nZ)-generated if and only if G is $(lX, mY, (nZ)^t)$-generated.We see that $(11A)^{-1} = 11B$ in A_{11}. As an application of the above theorem, the group A_{11} is $(p, q, 11A)$-generated if and only if it is $(p, q, 11B)$-generated. Therefore, it is sufficient to consider only the $(p, q, 11A)$-generations of A_{11}.

3. The alternating group A_{11}

In this section, we apply the results discussed in Section 2, to the group A_{11}. We determine all the (p, q, r)-generations of A_{11}, where p, q, and r are primes dividing the order of A_{11}.

The group A_{11} is a simple group of order $19958400 = 2^7 \times 3^4 \times 5^2 \times 7 \times 11$. By the Atlas of finite groups [13], the group A_{11} has exactly 31 conjugacy classes of its elements and 7 conjugacy classes of its maximal subgroups. Representatives of these classes of maximal subgroups can be taken as follows:

$$H_1 = A_{10}, \quad H_2 = S_9, \quad H_3 = (A_8 \times 3){:}2,$$
$$H_4 = (A_7 \times A_4){:}2, \quad H_5 = (A_6 \times A_5){:}2, \quad H_6 = M_{11},$$
$$H_7 = M_{11}.$$

Throughout this paper, by G, we always mean the alternating group A_{11}, unless stated otherwise. From the electronic Atlas of finite group representations [33], we see that G can be generated in terms of permutations on 11 points. Generators g_1 and g_2 can be taken as follows:

$$\begin{aligned} g_1 &= (1,2,3), \\ g_2 &= (3,4,5,6,7,8,9,10,11), \end{aligned}$$

with $o(g_1) = 3$, $o(g_2) = 9$, and $o(g_1g_2) = 11$.

In Table 1, we list the values of the cyclic structure for each conjugacy of G containing elements of prime order together with the values of both c_i and d_i obtained from Ree and Scotts theorems, respectively.

Table 1. Cycle structures of prime order conjugacy classes of G

nX	Cycle Structure	c_i	d_i
$2A$	$1^7 2^2$	9	2
$2B$	$1^3 2^4$	7	4
$3A$	$1^8 3^1$	9	2
$3B$	$1^5 3^2$	7	4
$3C$	$1^2 3^3$	5	6
$5A$	$1^6 5^1$	7	4
$5B$	$1^1 5^2$	3	8
$7A$	$1^4 7^1$	5	6
$11A$	11^1	1	10
$11B$	11^1	1	10

In Table 2, we list the representatives of classes of the maximal subgroups together with the orbits lengths of G on these groups and the permutation characters.

Table 2. Maximal subgroups of G

Maximal Subgroup	Order	Orbit Lengths	Character
H_1	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	[1,10]	$1a + 10a$
H_2	$2^7 \cdot 3^4 \cdot 5 \cdot 7$	[2,9]	$1a + 10a + 44a$
H_3	$2^7 \cdot 3^3 \cdot 5 \cdot 7$	[3,8]	$1a + 10a + 44a + 110a$
H_4	$2^6 \cdot 3^3 \cdot 5 \cdot 7$	[7,4]	$1a + 10a + 44a + 110a + 165a$
H_5	$2^6 \cdot 3^3 \cdot 5^2$	[5,6]	$1a + 10a + 44a + 110a$ +132a + 165a
H_6	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	[11]	$1a + 132a + 462a + 825a + 1100a$
H_7	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	[11]	$1a + 132a + 462a + 825a + 1100a$

Table 3 gives the partial fusion maps of classes of maximal subgroups into the classes of G. These will be used in our computations.

TABLE 3. The partial fusion maps into G

H_1-class	2a	2b	3a	3b	3c	5a	5b	7a						
$\to G$	2A	2B	3A	3B	3C	5A	5B	7A						
h						6	1	4						
H_2-class	2a	2b	2c	2d	3a	3b	3c	5a	7a					
$\to G$	2A	2A	2B	2B	3C	3A	3B	5A	7A					
h								15	6					
H_3-class	2a	2b	2c	2d	3a	3b	3c	3d	3e	5a	7a			
$\to G$	2B	2B	2A	2A	3A	3B	3C	3A	3B	5A	7A			
h										20	4			
H_4-class	2a	2b	2c	2d	2e	3a	3b	3c	3d	3e	5a	7a		
$\to G$	2A	2A	2A	2B	2B	3A	3A	3B	3B	3C	5A	7A		
h											15	1		
H_5-class	2a	2b	2c	2d	2e	3a	3b	3c	3d	3e	5a	5b	5c	5d
$\to G$	2A	2A	2B	2A	2B	3A	3B	3A	3B	3C	5A	5A	5B	5B
h											1	6	1	1
H_6-class	2a	3a	5a	11a	11b									
$\to G$	2B	3C	5B	11A	11B									
h			5	1	1									
H_7-class	2a	3a	5a	11a	11b									
$\to G$	2B	3C	5B	11A	11B									
h			5	1	1									

4. $(2,q,r)$-GENERATIONS

Let pX, $p \in \{2,3,5,7,11\}$, be a conjugacy class of G and c_i be the number of disjoint cycles in a representative of pX. The group G is not $(2Y,2Z,pX)$-generated, for if G is $(2Y,2Z,pX)$-generated, then G is a dihedral group and thus is not simple for all $Y,Z \in \{A,B\}$. Also we know that if G is (l,m,n)-generated with $\frac{1}{l}+\frac{1}{m}+\frac{1}{n} \geq 1$ and G is simple, then $G \cong A_5$, but $G \cong A_{11}$ and $A_{11} \ncong A_5$. Hence if G is (p,q,r)-generated, then we must have $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} < 1$. Now the $(2,q,r)$-generations of G comprise the cases $(2,3,r)$-, $(2,5,r)$-, $(2,7,r)$-, and $(2,11,r)$-generations.

4.1. $(2,3,r)$**-generations.** By using the condition $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} < 1$, the group G is $(2,3,r)$-generated if and only if $r \in \{7,11\}$. Thus we have to consider the cases $(2X,3Y,7A)$ and $(2X,3Y,11Z)$ for all $X,Z \in \{A,B\}$, $Y \in \{A,B,C\}$.

Proposition 4.1. *The group G is not $(2X,3Y,7A)$-generated, where $X \in \{A,B\}$, $Y \in \{A,B,C\}$.*

Proof. If the group G is $(2X,3Y,7A)$-generated then we must have $c_{2X}+c_{3Y}+c_{7A} \leq 13$ where $X \in \{A,B\}$ and $Y \in \{A,B,C\}$. By Table 1, we see that $c_{2X} \in \{7,9\}$ and $c_{3Y} \in \{5,7,9\}$, which follows that

$$c_{2X}+c_{3Y}+c_{7A} = 9 + c_{3X} + 5 > 13$$

for $X \in \{A,B\}$ and $Y \in \{A,B,C\}$. Now using Ree's theorem, it follows that G is not $(2X,3Y,7A)$-generated, where $X \in \{A,B\}$ and $Y \in \{A,B,C\}$. □

Proposition 4.2. *The group G is*

(i) *neither $(2X,3Y,11Z)$- nor $(2A,3C,11Z)$-generated for all $X,Y,Z \in \{A,B\}$,*
(ii) *$(2B,3C,11X)$-generated for $X \in \{A,B\}$.*

Proof. (i) Since by Table 4, we have $\Delta_G(2A,3A,11X) = \Delta_G(2A,3B,11X) = \Delta_G(2A,3C,11X) = \Delta_G(2B,3A,11X) = \Delta_G(2B,3B,11X) = 0$, hence Lemma 2.4 implies that the group G is neither $(2X,3Y,11Z)$- nor $(2A,3C,11Z)$-generated for all $X,Y,Z \in \{A,B\}$.

(ii) From Table 3 we see H_6 (or H_7) (two nonconjugate copies) is the only maximal subgroup containing elements of orders 2, 3 and 11. The intersection of H_6 from one conjugacy class with H_7 from a different conjugacy class has no element of order 11. We obtain that $\sum_{H_6}(2a,3a,11x) = 11$ and $h(11X,H_6) = 1$ (see [19,35]). Since by Table 4 we have $\Delta_G(2B,3C,11X) = 110$, we then obtain that $\Delta_G^*(2B,5B,11X) = \Delta_G(2B,3C,11X) - \sum_{H_6}(2a,3a,11x) - \sum_{H_6}(2a,3a,11x) = 110 - 11 - 11 = 88 > 11 = |C_G(11X)|$ for $X \in \{A,B\}$. This proves that the group G is $(2B,3C,11X)$-generated for $X \in \{A,B\}$, proving (ii). □

4.2. $(2,5,r)$**-generations.** Using the condition $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, the group G is $(2,5,r)$-generated if and only if $r \in \{5,7,11\}$. Thus we have to consider the cases $(2X,5Y,5Z)$, $(2X,5Y,7A)$ and $(2X,5Y,11Z)$ for all $X,Y,Z \in \{A,B\}$.

Proposition 4.3. *The group G is*

(i) *neither $(2A,5B,5B)$- nor $(2X,5A,5Y)$-generated for all $X,Y \in \{A,B\}$,*
(ii) *$(2B,5B,5B)$-generated.*

Proof. (i) If G is $(2A,5B,5B)$-generated group, then we must have $c_{2A} + c_{5B} + c_{5B} \le 13$. For $r \in \{5A,5B\}$, then by Table 1, we have $c_r \in \{3,7\}$ and it follows that

$$c_{2A} + c_{5A} + c_r = 9 + 7 + c_r > 13,$$
$$c_{2A} + c_{5B} + c_r = 9 + 3 + c_r > 13,$$
$$c_{2B} + c_{5A} + c_r = 7 + 7 + c_r > 13.$$

Now using Ree's theorem [29], it follows that G is not $(2A,5B,5B)$-generated. The same applies to $(2X,5A,5Y)$ for all $X,Y \in \{A,B\}$. Thus G is neither $(2A,5B,5B)$- nor $(2X,5A,5Y)$-generated for all $X,Y \in \{A,B\}$, proving (i).

(ii) From Table 3, we can see that all the maximal subgroups of G have elements of order 5. Let T be the set of all maximal subgroups of G. We are looking at various intersections of these maximal subgroups to determine their contributions in the calculations of $\Delta_G^*(2B,5B,5B)$. We have the following:

- The intersection of any 6 or 7 maximal subgroups of T does not contain elements of order 5.
- Although some of the intersections of any 3, 4, or 5 maximal subgroups of T contains elements of order 5, they will be no contributions because none of their elements of order 5 fuses to $5B$.
- The intersection of any two maximal subgroups of T having contributions here is only $H_1 \cap H_6 \cong A_6 \cdot 2$ (or $H_1 \cap H_7 \cong A_6 \cdot 2$) and $(H_5) \cap H_6 \cong 5{:}4$

(or $H_5 \cap H_7 \cong 5{:}4$), because their elements of orders 2 and 5 fuse to $2B$ and $5B$, respectively.

We see that H_1, H_5, H_6 (or H_7), $H_1 \cap H_6$, and $H_5 \cap H_6$ (or $H_5 \cap H_7$) are the only subgroups having their elements of orders 2 and 5 fusing to $2B$ and $5B$, respectively. By Table 4, we have $\Delta_G(2B, 5B, 5B) = 825$ and we also obtain that $\sum_{H_1}(2b, 5b, 5b) = 225$, $\sum_{H_5}(2c, 5x, 5y) = \Delta_{H_5}(2c, 5c, 5c) + \Delta_{H_5}(2c, 5c, 5d) + \Delta_{H_5}(2c, 5d, 5d) = 50+50+50 = 150$, $\sum_{H_6}(2a, 5a, 5a) = 45$, $\sum_{H_1\cap H_6}(2a, 5a, 5a) = 53$ and $\Delta_{H_5\cap H_6}(2a, 5a, 5a) = 3$. We find that $h(5B, H_1) = 1 = h(5B, H_5)$, and $h(5B, H_6) = h(5B, H_1 \cap H_6) = h(5B, H_5 \cap H_6) = 5$. It then follows that

$$\begin{aligned}\Delta^*_G(2B, 5B, 5B) =& \Delta_G(2B, 5B, 5B) - \sum_{H_1}(2b, 5b, 5b) - \sum_{H_5}(2c, 5x, 5y)\\ & - 5\cdot\sum_{H_6}(2a, 5a, 5a) - 5\cdot\sum_{H_7}(2a, 5a, 5a)\\ & + 5\cdot\sum_{H_1\cap H_6}(2a, 5a, 5a) + 5\cdot\sum_{H_1\cap H_7}(2a, 5a, 5a)\\ & + 5\cdot\sum_{H_5\cap H_6}(2a, 5a, 5a) + 5\cdot\sum_{H_5\cap H_7}(2a, 5a, 5a)\\ =& 825 - 1(225) - 1(150) - 5(45) - 5(45)\\ & + 5(53) + 5(53) + 5(3) + 5(3) = 560 > 0.\end{aligned}$$

Hence the group G is $(2B, 5B, 5B)$-generated, proving (ii). □

Proposition 4.4. *The group G is not $(2X, 5Y, 7A)$-generated for all $X, Y \in \{A, B\}$.*

Proof. If G is a $(2X, 5Y, 7A)$-generated group, then we must have $c_{2X}+c_{5Y}+c_{7A} \leq 13$ for all $X, Y \in \{A, B\}$. From Table 1, we see that

$$\begin{aligned} c_{2A} + c_{5A} + c_{7A} &= 9 + 7 + 5 > 13,\\ c_{2A} + c_{5B} + c_{7A} &= 9 + 3 + 5 > 13,\\ c_{2B} + c_{5A} + c_{7A} &= 7 + 7 + 5 > 13,\\ c_{2B} + c_{5B} + c_{7A} &= 7 + 3 + 5 > 13.\end{aligned}$$

It follows by Ree's theorem that G is not $(2X, 5Y, 7A)$-generated for all $X, Y \in \{A, B\}$. □

Proposition 4.5. *The group G is*

(i) *not $(2X, 5A, 11Y)$-generated for all $X, Y \in \{A, B\}$,*
(ii) *$(2X, 5B, 11Y)$-generated for all $X, Y \in \{A, B\}$.*

Proof. (i) By Table 4, we see that $\Delta_G(2X, 5A, 11Y) = 0$ and by Lemma 2.4, G is not $(2X, 5A, 11Y)$-generated for all $X, Y \in \{A, B\}$ and (i) is complete.

(ii) From Table 3, we see that H_6 (or H_7) (two nonconjugate copies) is the only maximal subgroup containing elements of orders 2, 5, and 11. The intersection of H_6 from one conjugacy class with H_7 from a different conjugacy class has no element of order 11. No element of order 2 from this maximal subgroup fuses to the class $2A$ of G. By Table 4, we then obtain that $\Delta^*_G(2A,5B,11X) = \Delta_G(2A,5B,11X) = 44 > 11 = |C_G(11X)|$ for $X \in \{A,B\}$. This proves that the group G is $(2A,5B,11X)$-generated for $X \in \{A,B\}$. Also, we obtain that $\sum_{H_6}(2a,5a,11x) = 33$ and we find that $h(11X,H_6) = 1$ (or $h(11X,H_7) = 1$). Since by Table 4, we have $\Delta_G(2B,5B,11X) = 660$, we then obtain that $\Delta^*_G(2B,5B,11X) = \Delta_G(2B,5B,11X) - \sum_{H_6}(2a,5a,11x) - \sum_{H_7}(2a,5a,11x) = 660 - 33 - 33 = 594 > 11 = |C_G(11X)|$ for $X \in \{A,B\}$. This proves that the group G is $(2B,5B,11X)$-generated for $X \in \{A,B\}$. □

4.3. $(2,7,r)$**-generations.** Here we have to check the generation of G through the triples $(2A,7A,7A)$-, $(2A,7A,11A)$-, $(2A,7A,11B)$-, $(2B,7A,7A)$-, $(2B,7A,11A)$-, and $(2B,7A,11B)$-generation.

Proposition 4.6. *The group G is*

(i) *$(2A,7A,7A)$-generated,*
(ii) *not $(2B,7A,7A)$-generated.*

Proof. (i) From Table 3, we find only four maximal subgroups of G each have an element of order 7, namely, H_1, H_2, H_3, and H_4. We have $H_1 \cap H_2 \cong A_9$, $H_1 \cap H_3 \cong S_8 \cong A_8{:}2$, $H_1 \cap H_4 \cong C_3{:}S_7$, $H_2 \cap H_3 \cong S_8$, $H_2 \cap H_4 \cong C_2{:}S_7$, $H_3 \cap H_4 \cong A_7{:}S_3$, $H_1 \cap H_2 \cap H_3 \cong A_8$, $H_1 \cap H_2 \cap H_4 \cong S_7$, $H_1 \cap H_3 \cap H_4 \cong S_7$, $H_2 \cap H_3 \cap H_4 \cong S_7$, and $H_1 \cap H_2 \cap H_3 \cap H_4 \cong A_7$. By Table 4 we have $\Delta_G(2A,7A,7A) = 175$. We obtain that

$$\begin{aligned}
\sum_{H_1}(2a,7a,7a) =& 140,\\
\sum_{H_2}(2x,7a,7a) =& \Delta_{H_2}(2a,7a,7a) + \Delta_{H_2}(2c,7a,7a) = 105 + 0 = 105,\\
\sum_{H_3}(2x,7a,7a) =& \Delta_{H_3}(2b,7a,7a) + \Delta_{H_3}(2c,7a,7a) = 70 + 0 = 70,\\
\sum_{H_4}(2x,7a,7a) =& \Delta_{H_4}(2a,7a,7a) + \Delta_{H_4}(2b,7a,7a) + \Delta_{H_4}(2d,7a,7a)\\
=& 0 + 35 + 0 = 35,\\
\sum_{H_1\cap H_2}(2b,7a,7a) =& 105,\\
\sum_{H_1\cap H_3}(2x,7a,7a) =& \Delta_{H_1\cap H_3}(2a,7a,7a) + \Delta_{H_1\cap H_3}(2c,7a,7a)\\
=& 35 + 0 = 35,\\
\sum_{H_1\cap H_4}(2x,7a,7a) =& \Delta_{H_1\cap H_4}(2a,7a,7a) + \Delta_{H_1\cap H_4}(2c,7a,7a)\\
=& 35 + 0 = 35,\\
\sum_{H_2\cap H_3}(2x,7a,7a) =& \Delta_{H_2\cap H_3}(2a,7a,7a) + \Delta_{H_2\cap H_3}(2b,7a,7a) = 70 + 0 = 70,
\end{aligned}$$

$$\begin{aligned}
\sum_{H_2\cap H_4}(2x,7a,7a) =& \Delta_{H_2\cap H_4}(2a,7a,7a)+\Delta_{H_2\cap H_4}(2b,7a,7a)\\
&+\Delta_{H_2\cap H_4}(2c,7a,7a)+\Delta_{H_2\cap H_4}(2d,7a,7a)\\
=&0+0+0+35=35,\\
\sum_{H_3\cap H_4}(2x,7a,7a) =& \Delta_{H_3\cap H_4}(2a,7a,7a)+\Delta_{H_3\cap H_4}(2c,7a,7a)\\
=&0+35=35,\\
\sum_{H_1\cap H_2\cap H_3}(2b,7x,7y) =& \Delta_{H_1\cap H_2\cap H_3}(2b,7a,7a)+\Delta_{H_1\cap H_2\cap H_3}(2b,7a,7b)\\
&+\Delta_{H_1\cap H_2\cap H_3}(2b,7b,7b)=28+42+28=98,\\
\sum_{H_1\cap H_2\cap H_4}(2x,7a,7a) =& \Delta_{H_1\cap H_2\cap H_4}(2b,7a,7a)+\Delta_{H_1\cap H_2\cap H_4}(2c,7a,7a)\\
=&35+0=35,\\
\sum_{H_1\cap H_3\cap H_4}(2x,7a,7a) =& \Delta_{H_1\cap H_3\cap H_4}(2a,7a,7a)+\Delta_{H_1\cap H_3\cap H_4}(2c,7a,7a)\\
=&0+35=35,\\
\sum_{H_2\cap H_3\cap H_4}(2x,7a,7a) =& \Delta_{H_2\cap H_3\cap H_4}(2a,7a,7a)+\Delta_{H_2\cap H_3\cap H_4}(2b,7a,7a)\\
=&35+0=35,\\
\sum_{H_1\cap H_2\cap H_3\cap H_4}(2a,7x,7x) =& \Delta_{H_1\cap H_2\cap H_3\cap H_4}(2a,7a,7a)+\Delta_{H_1\cap H_2\cap H_3\cap H_4}(2a,7a,7b)\\
&+\Delta_{H_1\cap H_2\cap H_3\cap H_4}(2a,7b,7b)\\
=&7+28+7=42.
\end{aligned}$$

We find that

$$\begin{aligned}
h(7A,H_1) =& h(7A,H_3)=h(7A,H_1\cap H_4)=h(7A,H_3\cap H_4)=4,\\
h(7A,H_2) =& h(7A,H_2\cap H_4)=6,\quad h(7A,H_4)=1,\\
h(7A,H_1\cap H_2) =& h(7A,H_1\cap H_3)=h(7A,H_2\cap H_3)=h(7A,H_1\cap H_2\cap H_3)\\
=&h(7A,H_1\cap H_2\cap H_4)=h(7A,H_1\cap H_3\cap H_4)\\
=&h(7A,H_2\cap H_3\cap H_4)=h(7A,H_1\cap H_2\cap H_3\cap H_4)=12.
\end{aligned}$$

We then obtain that

$$\begin{aligned}
\Delta_G^*(2A,7A,7A) =& \Delta_G(2A,7A,7A)-4\cdot\sum_{H_1}(2a,7a,7a)-6\cdot\sum_{H_2}(2x,7a,7a)\\
&-4\cdot\sum_{H_3}(2x,7a,7a)-\sum_{H_4}(2x,7a,7a)\\
&+12\cdot\sum_{H_1\cap H_2}(2b,7a,7a)+12\cdot\sum_{H_1\cap H_3}(2x,7a,7a)
\end{aligned}$$

$$\begin{aligned}
&+12\cdot\sum_{H_1\cap H_4}(2x,7a,7a)+12\cdot\sum_{H_2\cap H_3}(2x,7a,7a)\\
&+12\cdot\sum_{H_2\cap H_4}(2x,7a,7a)+12\cdot\sum_{H_3\cap H_4}(2x,7a,7a)\\
&-12\cdot\sum_{H_1\cap H_2\cap H_3}(2b,7a,7a)-12\cdot\sum_{H_1\cap H_2\cap H_4}(2x,7a,7a)\\
&-12\cdot\sum_{H_1\cap H_3\cap H_4}(2x,7a,7a)-12\cdot\sum_{H_2\cap H_3\cap H_4}(2x,7a,7a)\\
&+12\cdot\sum_{H_1\cap H_2\cap H_3\cap H_4}(2a,7x,7y)\\
=&175-4(140)-6(105)-4(70)-1(35)+12(105)+12(70)\\
&+4(35)+12(70)+6(35)+4(35)-12(98)-12(35)-12(35)\\
&+12(42)=588>0,
\end{aligned}$$

proving that G is $(2A,7A,7A)$-generated.

(ii) We compute the structure constant $\Delta_{A_{11}}=\Delta_{A_{11}}(2B,7A,7A)=644$. The only maximal subgroups of A_{11} that can potentially contribute to the structure constant $\Delta_{A_{11}}$ are isomorphic to A_{10}, S_9, $(A_8\times 3)$:2, and $(A_7\times A_4)$:2. We calculate now contribution from each these maximal subgroups to $\Delta_{A_{11}}$.

First, we consider the group $\Sigma_{(A_7\times A_4):2}$. The $2B$-class of A_{11} does not meet the group A_7. We have $\Sigma_{A_7}=0$. Further, as $\Sigma_{(A_7\times A_4):2}=\Sigma_{A_7}$, we have $\Sigma^*_{(A_7\times A_4):2}=0$. This means that the maximal subgroup $(A_7\times A_4)$:2 does not contribute to $\Delta_{A_{11}}$.

For the group $(A_8\times 3)$:2, we calculate $\Sigma_{(A_8\times 3):2}=\Sigma_{A_8}=35$. Up to isomorphism, A_7 and $2^3{:}L_3(2)$ (two nonconjugate copies) are the only maximal subgroups of A_8. From the above case, we know that $\Sigma_{A_7}=0$. Next consider the subchain of groups $2^3{:}7<(2^3{:}7){:}3<2^3{:}L_3(2)$. We compute that $\Sigma^*_{2^3:7}=\Sigma_{2^3:7}=7$ and $\Sigma_{(2^3:7):3}=7=\Sigma_{2^3:L_3(2)}$. As $|N_{(2^3:7):3}(2^3{:}7)|=(2^3{:}7){:}3=N_{2^3:L_3(2)}(2^3{:}7)$, we obtain that a fixed $z\in 7A$ is contained in a unique copy of each of $(2^3{:}7){:}3$-conjugate of $2^3{:}7$ and $2^3{:}L_3(2)$ groups. Thus we obtain $\Sigma^*_{(2^3:7):3}=\Sigma_{(2^3:7):3}-\Sigma_{2^3:7}=7-7=0$ and $\Sigma^*_{(2^3:L_3(2))}=\Sigma_{2^3:L_3(2)}-\Sigma_{2^3:7}=7-7=0$. Observe that, the only contribution toward Σ_{A_8} so far is coming from a unique conjugate of $2^3{:}7$. As there are two nonconjugate copies of $2^3{:}L_3(2)$, we compute

$$\Sigma^*_{A_8}=\Sigma_{A_8}-2\,\Sigma_{2^3:7}=35-2(7)=21.$$

Next, we treat the maximal group S_9. We compute $\Sigma_{S_9}=\Sigma_{A_9.2}=\Sigma_{A_9}=154$. From the list of maximal subgroups of A_9, observe that the $(2B,7A,7A)$-generated proper subgroups of A_{11} are contained in the subgroups isomorphic to S_7, A_8, or $2^3{:}L_2(8)$ (two nonconjugate copies). From above, we have $\Sigma_{S_7}=\Sigma_{A_7.2}=0$ as $A_7\cap 2B=\emptyset$. Also $\Sigma^*_{A_8}=21$. We investigate contribution from $L_2(8)$:3 to $\Delta_{A_{11}}$. We calculate $\Sigma_{L_2(8):3}=\Sigma_{L_2(8)}=28$ and $\Sigma^*_{2^3:7}=7$. Since $2^3{:}7<L_2(8)$ and a fixed element $z\in 7A$ lies in two $L_2(8)$-conjugates of $2^3{:}7$, we

have

$$\Sigma^*_{L_2(8)} = \Sigma_{L_2(8)} - 2\Sigma_{2^3:7} = 28 - 2(7) = 14.$$

We now collect the total contribution coming from A_9 to $\Delta_{A_{11}}$. Note that a fixed element z of order 7 (in A_{11}) lies in two, two and four A_9-conjugates of groups A_8, $L_2(8)$, and 2^3:7, respectively. We obtain

$$\begin{aligned}\Sigma^*_{A_9} &= \Sigma_{A_9} - 2\,\Sigma^*_{A_8} - 2\,\Sigma^*_{L_2(8)} - 4\,\Sigma^*_{2^3:7}\\ &= 154 - 2(28) - 2(14) - 4(7) = 56.\end{aligned}$$

Finally, it remains to compute contribution from the group A_{10}. We calculate $\Sigma(A_{10}) = 644$. From the list of maximal subgroups of A_{10}, the groups that may contain $(2B, 7A, 7A)$-generated proper subgroups, up to isomorphism, are A_9, S_8, and $(A_7{\times}3)$:2. In fact, we have already contributions from these groups as $\Sigma_{S_8} = \Sigma_{A_8.2} = \Sigma_{A_8}$, $\Sigma^*_{A_9} = 56$, and $\Sigma_{(A_7\times3):2} = \Sigma_{A_7} = 0$. As, $N_{A_{10}}(A_8) = S_8$, $N_{A_{10}}(L_2(8)) = 3{:}L_2(8)$, $N_{A_{10}}(2^3{:}7) = (2^3{:}7):3$, and A_9 is self normalized in A_{10} being maximal in A_{10}. A fixed element $z \in 7B$ is contained in three, three, six and six A_{10}-conjugates of groups A_9, A_8, $L_2(8)$, and 2^3:7, respectively. We calculate that

$$\begin{aligned}\Sigma^*_{A_{10}} &= \Sigma_{A_{10}} - 3\,\Sigma^*_{A_9} - 3\,\Sigma^*_{A_8} - 6\,\Sigma^*_{L_2(8)} - 6\,\Sigma_{2^3:7}\\ &= 357 - 3(56) - 3(21) - 6(14) - 6(7) = 0.\end{aligned}$$

To summarize, the only proper $(2B, 7A, 7A)$-subgroups of A_{11} are A_9, A_8, 2^3:7, and $L_2(8)$. As the respective numbers of A_{11}-conjugates of these subgroups containing a fixed element $z \in 7A$ are six, four, six and twelve, we obtain

$$\begin{aligned}\Delta^*_{A_{11}} &\leq \Delta_{A_{11}} - 6\,\Sigma^*_{A_9} - 4\,\Sigma^*_{A_8} - 6\,\Sigma_{2^3:7} - 12\,\Sigma^*_{L_2(8)}\\ &= 644 - 6(56) - 4(21) - 6(7) - 12(14)\\ &= 14 < 84 = |C_{A_{11}}(7A)|,\end{aligned}$$

which establishes that A_{11} is not $(2B, 7A, 7A)$-generated. □

Proposition 4.7. *The group G is*

(i) *not $(2A, 7A, 11X)$-generated for $X \in \{A, B\}$,*
(ii) *$(2B, 7A, 11X)$-generated for $X \in \{A, B\}$.*

Proof. (i) Table 4 gives that $\Delta_G(2A, 7A, 11X) = 0$ for $X \in \{A, B\}$ and thus the result holds.

(ii) By Table 3, we see that none of the maximal subgroups of G have elements of orders 7 and 11. By Table 4, we obtain that $\Delta^*_G(2B, 7A, 11X) = \Delta_G(2B, 7A, 11X) = 55 > 0$ for $X \in \{A, B\}$. □

4.4. $(2, 11, r)$**-generations.** Also here we have to check for the generation of G through the triples $(2A, 11A, 11A)$-, $(2A, 11A, 11B)$- $(2A, 11B, 11B)$-, $(2B, 11A, 11A)$-, $(2B, 11A, 11B)$- and $(2B, 11B, 11B)$-generation. We handle all these cases in the following proposition.

Proposition 4.8. *The group G is $(2X, 11Y, 11Z)$-generated for all $X, Y, Z \in \{A, B\}$.*

Proof. As in Proposition 4.5, the two noncojugate copies of H_6 (or H_7) contain elements of orders 2 and 11. No maximal subgroup of G meets the classes $2A$, $11A$, and $11B$ of G. Thus $\Delta^*_G(2A, 11A, 11B) = \Delta_G(2A, 11A, 11B) = 220 > 0$, $\Delta^*_G(2A, 11A, 11A) = \Delta_G(2A, 11A, 11A) = 110 > 0$, and $\Delta^*_G(2A, 11B, 11B) = \Delta_G(2A, 11B, 11B) = 110 > 0$. This proves that G is $(2A, 11A, 11B)$-, $(2A, 11A, 11A)$-, and $(2A, 11B, 11B)$-generated.

We obtain that $\sum_{H_6}(2a, 11x, 11y) = \sum_{H_7}(2a, 11x, 11y) = 11$ for all $x, y \in \{a, b\}$, and we have $h(11X, H_6) = 1$ or $(h(11X, H_7) = 1)$ for $X \in \{A, B\}$. By Table 4, we have $\Delta_G(2B, 11A, 11B) = 1320$ and $\Delta_G(2B, 11A, 11A) = 2145 = \Delta_G(2B, 11B, 11B)$. It renders that $\Delta^*_G(2B, 11A, 11B) = 1320-11-11 = 1298 > 0$ and $\Delta^*_G(2B, 11A, 11A) = \Delta^*_G(2B, 11B, 11B) = 2145-11-11 = 2123 > 0$, proving that G is $(2B, 11A, 11B)$-, $(2B, 11A, 11A)$-, and $(2B, 11B, 11B)$-generated. □

5. $(3, q, r)$-GENERATIONS

In this section, we handle all the possible $(3, q, r)$-generations, namely $(3X, 3Y, 5A)$-, $(3X, 3Y, 5B)$-, $(3X, 3Y, 7A)$-, $(3X, 3Y, 11A)$-, $(3X, 3Y, 11B)$-, $(3X, 5A, 5A)$-, $(3X, 5A, 5B)$-, $(3X, 5A, 7A)$-, $(3X, 5A, 11A)$-, $(3X, 5A, 11B)$-, $(3X, 5B, 5B)$-, $(3X, 5B, 7A)$-, $(3X, 5B, 11A)$-, $(3X, 5B, 11B)$-, $(3X, 7A, 7A)$-, $(3X, 7A, 11A)$-, $(3X, 7A, 11B)$-, $(3X, 11A, 11A)$-, $(3X, 11A, 11B)$-, and $(3X, 11B, 11B)$-generations.

5.1. $(3, 3, r)$-generations.

Proposition 5.1. *The group G is neither $(3X, 3Y, 5Z)$- nor $(3X, 3Y, 7A)$-generated group for all $X, Y \in \{A, B, C\}$ and $Z \in \{A, B\}$.*

Proof. The group G acts on a 10-dimensional irreducible complex module $\mathbb{V}$. Applying Scott's theorem to the module $\mathbb{V}$ and using the Atlas of finite groups, we get

$$\begin{aligned}
d_{3A} = \dim(\mathbb{V}/C_{\mathbb{V}}(3A)) &= \frac{2(10-7)}{3} = 2,\\
d_{3B} = \dim(\mathbb{V}/C_{\mathbb{V}}(3B)) &= \frac{2(10-4)}{3} = 4,\\
d_{3C} = \dim(\mathbb{V}/C_{\mathbb{V}}(3C)) &= \frac{2(10-1)}{3} = 6,\\
d_{5A} = \dim(\mathbb{V}/C_{\mathbb{V}}(5A)) &= \frac{4(10-5)}{5} = 4,\\
d_{5B} = \dim(\mathbb{V}/C_{\mathbb{V}}(5B)) &= \frac{4(10-0)}{5} = 8,\\
d_{7A} = \dim(\mathbb{V}/C_{\mathbb{V}}(7A)) &= \frac{6(10-3)}{7} = 6.
\end{aligned}$$

For the cases $(3A, 3A, nX)$, we get $d_{3A}+d_{3A}+d_{nX} = 2\times2+d_{nX} < 2\times10$ and hence by Scott's theorem, G is not $(3A, 3A, nX)$-generated for all $nX \in \{5A, 5B, 7A\}$. We get nongenerations when Scott's theorem is applied to the following cases $(3A, 3B, nX)$, $(3A, 3C, nX)$, $(3B, 3B, nX)$, $(3B, 3C, nX)$, and $(3C, 3C, nX)$ for all $nX \in \{5A, 5B, 7A\}$. □

Proposition 5.2. *The group G is*

(i) *neither* $(3A, 3X, 11Y)$- *nor* $(3B, 3B, 11Y)$-*generated for* $X \in \{A, B, C\}$ *and* $Y \in \{A, B\}$,

(ii) $(3B, 3C, 11X)$- *and* $(3C, 3C, 11X)$-*generated for* $X \in \{A, B\}$.

Proof. (i) By Table 5, we see that $\Delta_G(3A, 3X, 11Y) = 0 = \Delta_G(3B, 3B, 11Y)$ for $X \in \{A, B, C\}$ and $Y \in \{A, B\}$. Hence, G is neither $(3A, 3X, 11Y)$- nor $(3B, 3B, 11Y)$-generated for $X \in \{A, B, C\}$ and $Y \in \{A, B\}$.

(ii) No maximal subgroup of G meets the classes $3B$, $3C$, and $11A$ or $11B$ of G. By Table 5, we then obtain that $\Delta_G^*(3B, 3C, 11X) = \Delta_G(3B, 3C, 11X) = 66 > 0$, proving that G is $(3B, 3C, 11X)$-generated for $X \in \{A, B\}$. Now we prove that G is $(3C, 3C, 11X)$-generated for $X \in \{A, B\}$. By Proposition 4.2, we prove that G is $(2B, 3C, 11X)$-generated for $X \in \{A, B\}$. It follows by Theorem 2.7 that G is $(3C, 3C, (11A)^2)$- and $(3C, 3C, (11B)^2)$-generated. By GAP, we see that $(11A)^2 = 11B$ and $(11B)^2 = 11A$, and thus G is $(3C, 3C, 11X)$-generated for $X \in \{A, B\}$. □

5.2. $(3, 5, r)$**-generations.**

Proposition 5.3. *The group G is*

(i) *neither* $(3X, 5A, 5Y)$- *nor* $(3A, 5B, 5B)$ *generated for* $X \in \{A, B, C\}$ *and* $Y \in \{A, B\}$, *while G is,*

(ii) $(3X, 5B, 5B)$-*generated for* $X \in \{B, C\}$.

Proof. (i) If G is a $(3X, 5A, 5Y)$-generated group, then we must have $c_{3X} + c_{5A} + c_{5Y} \leq 13$ where $X \in \{A, B, C\}$ and $Y \in \{A, B\}$. Since by Table 1, we have $c_{3X} \in \{5, 7, 9\}$ for $X \in \{A, B, C\}$, we then obtain by the same Table 1 that

$$c_{3X} + c_{5A} + c_{5A} = c_{3X} + 7 + 7 > 13,$$
$$c_{3X} + c_{5A} + c_{5B} = c_{3X} + 7 + 3 > 13.$$

Now using Ree's theorem, it follows that G is not $(3X, 5A, 5Y)$-generated for $X \in \{A, B, C\}$ and $Y \in \{A, B\}$. Again by Table 1, we have $c_{3A} + c_{5B} + c_{5B} = 9 + 3 + 3 > 13$ and by Ree's theorem, the group G is not $(3A, 5B, 5B)$-generated.

(ii) We show that G is $(3X, 5B, 5B)$-generated for $X \in \{B, C\}$. We firstly consider the triple $(3B, 5B, 5B)$. By Table 5, we have $\Delta_G(3B, 5B, 5B) = 1080$. We notice that the elements of order 3 for both H_6 and $H_1 \cap H_6$ do not fuse to $3B$, and also $H_5 \cap H_6$ does not have elements of order 3. Therefore there is no any contributions here. We obtain that $\sum_{H_1}(3b, 5b, 5b) = 650$ and $\sum_{H_5}(3b, 5x, 5y) = \Delta_{H_5}(3b, 5c, 5c) + \Delta_{H_5}(3b, 5c, 5d) + \Delta_{H_5}(3b, 5d, 5d) = 5 + 10 + 5 = 20$. We then obtain that

$$\begin{aligned} \Delta_G^*(3B, 5B, 5B) &= \Delta_G(3B, 5B, 5B) - \sum_{H_1}(3b, 5b, 5b) - \sum_{H_5}(3b, 5x, 5y) \\ &= 1080 - 1(650) - 1(20) = 410 > 0. \end{aligned}$$

We turn to the other case, namely, the triple $(3C, 5B, 5B)$. In order to show that $(3C, 5B, 5B)$ is a generating triple of A_{11}, we consider its 10-dimensional irreducible representation over $\mathbb{F}_2$ (see [33]). The group $A_{11} = \langle a, b \rangle$ is generated by its standard generators a and b, where a and b are 10×10 matrices over $\mathbb{F}_2$ with orders 3 and 9, respectively such that a is in class $3A$ and ab has order 11. Then via GAP, we produce $c = ab^3a^{-1}b^2(ba)^3b^3aba^{-1}$ and $d = ab^{-1}ab^2a^{-1}b^4ab^{-1}a^{-1}b^2ab^2$ such that c and d are in $5B$ and $cd \in 12A$. Set $y = c$ and $x = dc^{-1}$; then we see that $P = \langle x, y\rangle$ and such that $x \in 3C$, $y \in 5B$ and $xy \in 5B$. Moreover, there are elements of order 5, 7, and 11 in P. As A_{11} has no proper subgroup divisible by $5\times7\times11$, we have $A_{11} = \langle x, y\rangle = P$, as claimed. Hence (ii) follows. □

Proposition 5.4. *The group G is*

(i) *neither* $(3X, 5Y, 7A)$- *nor* $(3C, 5A, 7A)$-*generated for all* $X, Y \in \{A, B\}$,
(ii) $(3C, 5B, 7A)$-*generated.*

Proof. (i) Since by Table 5, we have $\Delta_G(3A, 5A, 7A) = 7 < 84 = |C_{A11}(7A)|$ and $\Delta_G(3A, 5B, 7A) = 0$, it follows that G is not $(3A, 5X, 7A)$-generated for $X \in \{A, B\}$. By Proposition 5.1, we see that $d_{3B} = 4$, $d_{3C} = 6$, $d_{5A} = 4$, $d_{5B} = 8$, and $d_{7A} = 6$. Thus $d_{3B} + d_{5X} + d_{7A} = 4 + d_{5X} + 6 < 20$ and $d_{3C} + d_{5A} + d_{7A} = 6+4+6 < 20$ for $X \in \{A, B\}$. By Scott's theorem the group G is not $(3B, 5X, 7A)$- and $(3C, 5A, 7A)$-generated for $X \in \{A, B\}$.

(ii) By Table 5, we have $\Delta_G(3C, 5B, 7A) = 5376$. The only maximal subgroup meeting the classes $3C$, $5B$, and $7A$ of G is H_1. We obtain that $\sum_{H_1}(3c, 5b, 7a) = 882$ and $h(7A, H_1) = 4$. It then follows that $\Delta^*_G(3C, 5B, 7A) = 5376 - 4(882) = 1848 > 0$, proving (ii). □

Proposition 5.5. *The group G is*

(i) *not* $(3X, 5A, 11Y)$-*generated for all* $X, Y \in \{A, B\}$,
(ii) $(3C, 5A, 11Y)$- *and* $(3X, 5B, 11Y)$-*generated for* $X \in \{A, B, C\}$ *and* $Y \in \{A, B\}$.

Proof. (i) Since $c_{3X} \in \{7, 9\}$, by Table 1, it follows that $c_{3X} + c_{5A} + c_{11Y} = c_{3X} + 7 + 1 > 13$ for all $X, Y \in \{A, B\}$ and the result follows.

(ii) From Table 3, we see that two nonconjugate copies of H_6 (or H_7) are the only two maximal subgroups containing elements of orders 3, 5, and 11. The intersection of H_6 from one conjugacy class with H_7 from a different conjugacy class has no element of order 11. By Table 5, we have $\Delta_G(3C, 5A, 11X) = 22$ for $X \in \{A, B\}$. No element of order 5 from these two maximal subgroups fuses to the class $5A$ of G. We then obtain that $\Delta^*_G(3C, 5A, 11X) = \Delta_G(3C, 5A, 11X) = 22 > 11 = |C_G(11X)|$, proving that G is $(3C, 5A, 11X)$-generated for $X \in \{A, B\}$. Similarly, we have $\Delta_G(3A, 5B, 11X) = 11$ for $X \in \{A, B\}$. From Table 3, we see that no maximal subgroup of G meets the classes $3A$, $5B$, and $11A$ or $11B$ of G. It follows that $\Delta^*_G(3A, 5B, 11X) = \Delta_G(3A, 5B, 11X) = 11 > 0$, proving that G is $(3A, 5B, 11X)$-generated for $X \in \{A, B\}$. By the same Table 5, we have $\Delta_G(3B, 5B, 11X) = 704$ for $X \in \{A, B\}$. No element of order 3 from these two maximal subgroups fuses to the class $3B$ of G. Therefore we get $\Delta^*_G(3B, 5B, 11X) = \Delta_G(3B, 5B, 11X) = 704 > 11 = |C_G(11X)|$, proving that G is $(3B, 5B, 11X)$-generated for $X \in \{A, B\}$. For the other argument, the computations show that $\sum_{H_6}(3a, 5a, 11x) = \Delta_{H_6}(3a, 5a, 11a) + \Delta_{H_6}(3a, 5a, 11b) = 99+99 = 198$ and $h(11X, H_6) = 1$. Similarly $\sum_{H_7}(3a, 5a, 11x) = 198$. Since by Table 5, we have $\Delta_G(3C, 5B, 11X) = 4928$, and we obtain that $\Delta^*_G(3C, 5B, 11X) =$

$\Delta_G(3C, 5B, 11X) - \sum_{H_6}(3a, 5a, 11x) - \sum_{H_7}(3a, 5a, 11x) = 4928 - 198 - 198 = 4532 > 11 = |C_G(11X)|$ for $X \in \{A, B\}$. This proves that G is $(3C, 5B, 11X)$-generated for $X \in \{A, B\}$. $\square$

5.3. $(3, 7, r)$**- and** $(3, 11, r)$**-generations.** In this subsection we discuss the cases $(3, 7, r)$- and $(3, 11, r)$-generations. This comprises of 18 cases: $(3A, 7A, 7A)$-, $(3A, 7A, 11A)$-, $(3A, 7A, 11B)$-, $(3B, 7A, 7A)$-, $(3B, 7A, 11A)$-, $(3B, 7A, 11B)$-, $(3C, 7A, 7A)$-, $(3C, 7A, 11A)$-, $(3C, 7A, 11B)$-, $(3A, 11A, 11A)$-, $(3A, 11A, 11B)$-, $(3A, 11B, 11B)$-, $(3B, 11A, 11A)$-, $(3B, 11A, 11B)$-, $(3B, 11B, 11B)$-, $(3C, 11A, 11A)$-, $(3C, 11A, 11B)$- and $(3A, 11B, 11B)$-generation.

Proposition 5.6. *The group G is not $(3X, 7A, 7A)$-generated for $X \in \{A, B, C\}$.*

Proof. This is a direct application of Ree's theorem. Since by Table 1, we see that $c_{3X} \in \{5, 7, 9\}$, it then follows that $c_{3X} + c_{7A} + c_{7A} = c_{3X} + 5 + 5 > 13$, which implies that G is not $(3X, 7A, 7A)$-generated for $X \in \{A, B, C\}$. $\square$

Proposition 5.7. *The group G is*

(i) *not $(3A, 7A, 11X)$-generated for $X \in \{A, B\}$,*
(ii) *$(3Y, 7A, 11X)$-generated for $X \in \{A, B\}$ and $Y \in \{B, C\}$.*

Proof. (i) Since by Table 5, we have $\Delta_G(3A, 7A, 11X) = 0$, it follows that G is not $(3A, 7A, 11X)$-generated for $X \in \{A, B\}$.

(ii) No maximal subgroup of G contains both elements of orders 3, 7, and 11. Therefore $\Delta^*_G(3B, 7A, 11X) = \Delta_G(3B, 7A, 11X) = 33 > 0$ and $\Delta^*_G(3C, 7A, 11X) = \Delta_G(3C, 7A, 11X) = 990 > 0$. $\square$

Proposition 5.8. *The group G is $(3X, 11Y, 11Z)$-generated for $X \in \{A, B, C\}$ and $Y, Z \in \{A, B\}$.*

Proof. By Table 3, we see that no elements of the maximal subgroups of G meet the classes $3A$, $11A$, and $11B$ of G. Then by Table 5, we have $\Delta^*_G(3A, 11A, 11A) = \Delta_G(3A, 11A, 11A) = 110 > 0$, $\Delta^*_G(3A, 11B, 11B) = \Delta_G(3A, 11B, 11B) = 110 > 0$, and $\Delta^*_G(3A, 11A, 11B) = \Delta_G(3A, 11A, 11B) = 55 > 0$, proving that G is $(3A, 11X, 11Y)$-generated for all $X, Y \in \{A, B\}$. Again, by Table 3, we see that no elements of the maximal subgroups of G meet the classes $3B$, $11A$, and $11B$ of G. Then by Table 5, we have $\Delta^*(3B, 11X, 11X) = \Delta_G(3B, 11X, 11X) = 3212 > 0$ and $\Delta^*_G(3B, 11A, 11B) = \Delta_G(3B, 11A, 11B) = 2332 > 0$, proving that G is $(3B, 11X, 11Y)$-generated for all $X, Y \in \{A, B\}$. By Table 5, we have $\Delta(3C, 11X, 11Y) = 12760$ for all $X, Y \in \{A, B\}$. Only two nonconjugate copies of H_6 (or H_7) meet the classes $3C$, $11A$ and $11B$ of G. We obtain that $\sum_{H_6}(3a, 11a, 11b) = 22$, $\sum_{H_6}(3a, 11a, 11a) = 77 = \sum_{H_6}(3a, 11b, 11b)$, and $h(11X, H_6) = 1$ for $X \in \{A, B\}$. Similarly $\sum_{H_7}(3a, 11a, 11b) = 22$, $\sum_{H_7}(3a, 11a, 11a) = 77 = \sum_{H_7}(3a, 11b, 11b)$, and $h(11X, H_7) = 1$ for $X \in \{A, B\}$. Therefore $\Delta^*_G(3C, 11A, 11B) = \Delta_G(3C, 11A, 11B) - \sum_{H_6}(3a, 11a, 11b) - \sum_{H_7}(3a, 11a, 11b) = 12760 - 77 - 77 = 12606 > 0$ and $\Delta^*_G(3C, 11A, 11A) = \Delta_G(3C, 11A, 11A) - \sum_{H_6}(3a, 11a, 11a) - \sum_{H_7}(3a, 11a, 11a) = 12760 - 22 - 22 = 12716 > 0$. Similarly $\Delta^*_G(3C, 11B, 11B) = 12716$. Hence G is $(3C, 11X, 11Y)$-generated for all $X, Y \in \{A, B\}$. $\square$

6. Other results

In this section, we handle all the remaining cases, namely, the $(5,q,r)$-, $(7,q,r)$-, and $(11,q,r)$-generations.

6.1. $(5,5,r)$**-generations.** We have to check for the generation of G through the triples $(5A,5A,5A)$-, $(5A,5A,5B)$-, $(5A,5A,7A)$-, $(5A,5A,11A)$-, $(5A,5A,11B)$-, $(5A,5B,5B)$-, $(5A,5B,7A)$-, $(5A,5B,11A)$-, $(5A,5B,11B)$-, $(5B,5B,5B)$-, $(5B,5B,7A)$-, $(5B,5B,11A)$-, and $(5A,5A,11B)$-generation.

Proposition 6.1. *The group G is*

(i) *not $(5A,5A,5X)$-generated for $X \in \{A,B\}$,*
(ii) *$(5X,5B,5B)$-generated for $X \in \{A,B\}$.*

Proof. (i) Since by Table 6, we have that $\Delta_G(5A,5A,5A) = 428 < 1800 = |C_G(5A)|$ and $\Delta_G(5A,5A,5B) = 2 < 25 = |C_G(5B)|$, it follows by Lemma 2.4 that G is not $(5A,5A,5X)$-generated for $X \in \{A,B\}$.

(ii) From Table 3, we can see that all the maximal subgroups of G have elements of order 5. Let T be the set of all maximal subgroups of G. We are looking at various intersections of these maximal subgroups to determine their contributions in the calculations of $\Delta^*_G(5A,5B,5B)$. We have the following:

- The intersection of any 6 or 7 maximal subgroups of T does not contain elements of order 5.
- Although some of the intersections of any 2, 3, 4, or 5 maximal subgroups of T contains elements of order 5, they will not contribute here because none of their elements of order 5 fuse to both $5A$ and $5B$.

Out of all the subgroups of G, only H_1 and H_5 have contributions in the calculations of $\Delta^*_G(5A,5B,5B)$ because they both meet $5A$ and $5B$ classes of G. The computations render $\sum_{H_1}(5a,5b,5b) = 316$ and $\sum_{H_5}(5x,5c,5y) = \Delta_{H_5}(5a,5c,5c) + \Delta_{H_5}(5a,5c,5d) + \Delta_{H_5}(5b,5c,5c) + \Delta_{H_5}(5b,5c,5d) = 6+2+31+22 = 61$. We find that $h(5B,H_1) = h(5B,H_5) = 1$. Since by Table 6, we have $\Delta_G(5A,5B,5B) = 456$, we have $\Delta^*_G(5A,5B,5B) = \Delta_G(5A,5B,5B) - \sum_{H_1}(5a,5b,5b) - \sum_{H_5}(5x,5c,5y) = 456 - 316 - 61 = 79 > 25 = |C_G(5B)|$. This proves that G is $(5A,5B,5B)$-generated. Now we prove that G is $(5B,5B,5B)$-generated. By Proposition 4.3, we prove that G is $(2B,5B,5B)$-generated. It follows by Theorem 2.7 that G is $(5B,5B,(5B)^2)$-generated. By GAP, we see that $(5B)^2 = 5B$ so that G becomes $(5B,5B,5B)$-generated as required. □

Proposition 6.2. *The group G is*

(i) *not $(5A,5X,7A)$-generated for $X \in \{A,B\}$,*
(ii) *$(5B,5B,7A)$-generated.*

Proof. (i) If G is $(5A,5X,7A)$-generated group, then we must have $c_{5A} + c_{5X} + c_{7A} \leq 13$, where $X \in \{A,B\}$. Since by Table 1, we have $c_{5X} \in \{3,7\}$, we then obtain by the same Table 1 that $c_{5A}+c_{5X}+c_{7A} = 7+c_{5X}+5 > 13$ for $X \in \{A,B\}$. By Ree's theorem, G is not $(5A,5X,7A)$-generated for $X \in \{A,B\}$.

(ii) As in Proposition 4.6, only four maximal subgroups of G have an element of order 7, namely, H_1, H_2, H_3, and H_4. By Table 6, we have that

$\Delta_G(5B, 5B, 7A) = 32256$. Out of all subgroups having elements of order 7, only H_1 will have contributions here because it is the only maximal subgroup meeting the classes $5B$ and $7A$ of G. We obtain that $\sum_{H_1}(5b, 5b, 7a) = 3654$ and $h(7A, H_1) = 4$. We have $\Delta^*_G(5B, 5B, 7A) = \Delta_G(5B, 5B, 7A) - 4\cdot\sum_{H_1}(5b, 5b, 7a) = 32256 - 4(3654) = 17640 > 0$ and G is a $(5B, 5B, 7A)$-generated group. □

Proposition 6.3. *The group G is*

(i) *not $(5A, 5A, 11X)$-generated for $X \in \{A, B\}$,*
(ii) *$(5X, 5B, 11Y)$-generated for all $X, Y \in \{A, B\}$.*

Proof. (i) By Table 6, we have $\Delta_G(5A, 5A, 11X) = 0$.

(ii) As in Proposition 4.5, we see that two nonconjugate copies of H_6 (or H_7) contain elements of orders 5 and 11. None of these two maximal subgroups meet the classes $5A$ and $11A$ or $11B$ of G. It follows that $\Delta^*_G(5A, 5B, 11X) = \Delta_G(5A, 5B, 11X) = 440 > 0$, proving that G is $(5A, 5B, 11X)$-generated for $X \in \{A, B\}$. We now prove that G is $(5B, 5B, 11X)$-generated for $X \in \{A, B\}$. By Proposition 4.5, the group G is $(2X, 5B, 11Y)$-generated, it follows by Theorem 2.7 that G is $(5B, 5B, 11X)$-generated for all $X, Y \in \{A, B\}$. □

Proposition 6.4. *The group G is*

(i) *not $(5A, 7A, 7A)$-generated,*
(ii) *$(5B, 7A, 7A)$-generated.*

Proof. (i) The group G acts on a 10-dimensional irreducible complex module $\mathbb{V}$. By applying Scott's theorem [30] to the module $\mathbb{V}$ and using the Atlas of finite groups, we get $d_{5A} + d_{7A} + d_{7A} = 4 + 6 + 6 = 16 < 2 \times 10$, and hence by Scott's theorem, G is not $(5A, 7A, 7A)$-generated.

(ii) By Table 6, we have that $\Delta_G(5B, 7A, 7A) = 8736$. As in Proposition 6.2, only the maximal subgroup H_1 will have contributions here because it is the only one meeting of the classes $5B$ and $7A$ of G. We have $\sum_{H_1}(5b, 7a, 7a) = 1974$ and $h(7A, H_1) = 4$. We then obtain $\Delta^*_G(5B, 7A, 7A) = \Delta_G(5B, 7A, 7A) - 4 \cdot \sum_{H_1}(5b, 7a, 7a) = 8736 - 4(1974) = 840 > 0$, and hence G is a $(5B, 7A, 7A)$-generated group. □

Proposition 6.5. *The group G is a $(5X, 7A, 11Y)$-generated group for all $X, Y, Z \in \{A, B\}$.*

Proof. By Table 3, we see there is no maximal subgroup whose order is divisible by $(5 \times 7 \times 11)$. Since there is no contribution from any of the maximal subgroups, by Table 6, we have $\Delta^*_G(5A, 7A, 11X) = \Delta_G(5A, 7A, 11X) = 11 > 0$ and $\Delta^*_G(5B, 7A, 11X) = \Delta_G(5B, 7A, 11X) = 9504 > 0$ for $X \in \{A, B\}$. Hence G is $(5A, 7A, 11X)$- and $(5B, 7A, 11X)$-generated for $X \in \{A, B\}$. □

Proposition 6.6. *The group G is $(5X, 11Y, 11Z)$-generated for all $X, Y, Z \in \{A, B\}$.*

Proof. As in Proposition 4.5, H_6 (or H_7) (two nonconjugate copies) is the only maximal subgroup of G that contains elements of order 11. Since none of them meet the classes $5A$, $11A$, and $11B$ of G, by Table 6, we have $\Delta^*_G(5A, 11A, 11B) = \Delta_G(5A, 11A, 11B) = 1804$, $\Delta^*_G(5A, 11A, 11A) = \Delta_G(5A, 11A, 11A) = 1892$ and

$\Delta^*_G(5A, 11B, 11B) = \Delta_G(5A, 11B, 11B) = 1892$. Hence G is $(5A, 11A, 11B)$-, $(5A, 11A, 11A)$, and $(5A, 11B, 11B)$-generated.

By the same Table 6, we have $\Delta_G(5B, 11A, 11B) = 69696$ and $\Delta_G(5B, 11A, 11A) = \Delta_G(5B, 11B, 11B) = 76032$. We have $\sum_{H_6}(5a, 11a, 11b) = \sum_{H_7}(5a, 11a, 11b) = 99$, $h(11b, H_6) = h(11b, H_7) = 1$. We also have $\sum_{H_6}(5a, 11a, 11a) = \sum_{H_6}(5a, 11b, 11b) = 198$, $\sum_{H_7}(5a, 11a, 11a) = \sum_{H_7}(5a, 11b, 11b) = 198$, $h(11x, H_6) = h(11x, H_6) = 1$ for $x \in \{a, b\}$. It follows that $\Delta^*_G(5B, 11A, 11B) = \Delta_G(5B, 11A, 11B) - \sum_{H_6}(5a, 11a, 11b) - \sum_{H_7}(5a, 11a, 11b) = 69696 - 99 - 99 = 69498 > 0$ and $\Delta^*_G(5B, 11A, 11A) = \Delta_G(5B, 11A, 11A) - \sum_{H_6}(5a, 11a, 11a) - \sum_{H_7}(5a, 11a, 11a) = 76032 - 198 - 198 = 75636 > 0$. Similarly $\Delta^*_G(5B, 11A, 11A) = 75636$. Hence G is $(5B, 11A, 11B)$- and $(5B, 11X, 11X)$-generated for $X \in \{A, B\}$. □

6.2. $(7, 7, r)$**-generations.**

Proposition 6.7. *The group G is a $(7A, 7A, 7A)$-generated group.*

Proof. By Proposition 4.6, we prove that G is $(2A, 7A, 7A)$-generated. It follows by Theorem 2.7 that G is $(7A, 7A, (7A)^2)$- generated. Since there is only one class of element 7, it follows that $(7A)^2 = 7A$, so that G becomes $(7A, 7A, 7A)$-generated. □

Proposition 6.8. *The group G is a $(7A, 7A, 11X)$-generated group.*

Proof. By Proposition 4.7, we prove that G is $(2B, 7A, 11X)$-generated for $X \in \{A, B\}$. It follows by Theorem 2.7 that G is $(7A, 7A, (11A)^2)$- and $(7A, 7A, (11B)^2)$-generated. By GAP, we see that $(11A)^2 = 11B$ and $(11B)^2 = 11A$ so that G becomes $(7A, 7A, 11X)$-generated for $X \in \{A, B\}$. □

Proposition 6.9. *The group G is $(7A, 11X, 11Y)$-generated for all $X, Y \in \{A, B\}$.*

Proof. Since H_6 (or H_7) (two nonconjugate copies) is the only maximal subgroup of G whose order is divisible by 11 and H_6 (or H_7) does not have elements of order 7, then by Table 7, we have $\Delta^*_G(7A, 11X, 11Y) = \Delta_G(7A, 11X, 11Y) = 29700 > 0$, proving that G is $(7A, 11X, 11Y)$-generated for all $X, Y \in \{A, B\}$. □

6.3. $(11, 11, r)$**-generations.** We conclude our investigation on the (p, q, r)-generation of the alternating group G by considering the $(11, 11, 11)$-generations. Thus we will be looking at the cases $(11A, 11A, 11A)$-, $(11A, 11A, 11B)$-, $(11A, 11B, 11B)$-, and $(11B, 11B, 11B)$-generation.

Proposition 6.10. *The group G is a $(11A, 11A, 11A)$-, $(11A, 11A, 11B)$-, $(11A, 11B, 11B)$- and $(11B, 11B, 11B)$-generated group.*

Proof. The cases $(11A, 11A, 11A)$, $(11A, 11A, 11B)$ and $(11B, 11B, 11B)$ follow by Proposition 4.8 together with the applications of Theorem 2.7. Since by Proposition 4.2, H_6 (or H_7) (two nonconjugate copies) is the only maximal subgroup of G whose order is divisible by 11. we have $\sum_{H_6}(11a, 11b, 11b) = 35$ and $h(11b, H_6) = 1$. Similarly $\sum_{H_7}(11a, 11b, 11b) = 35$ and $h(11b, H_7) = 1$. Since by Table 7, we have $\Delta_G(11A, 11B, 11B) = 1476600$, we then obtain that

$\Delta^*_G(11A, 11B, 11B) = \Delta_G(11A, 11B, 11B) - \sum_{H_6}(11a, 11b, 11b) - \sum_{H_7}(11a, 11b, 11b) = 1476600 - 35 - 35 = 1476530 > 0$, proving that G is a $(11A, 11B, 11B)$-generated group. □

Tables : Structure constants of G

Table 4. The structure constants $\Delta_G(2X, qY, rZ)$

pX	2A	2B	3A	3B	3C	5A	5B	7A	11A	11B
$\Delta_G(2A, 2A, pX)$	44	6	84	9	0	5	0	0	0	0
$\Delta_G(2A, 2B, pX)$	105	24	0	0	0	0	0	0	0	0
$\Delta_G(2A, 3A, pX)$	28	0	24	0	0	5	0	0	0	0
$\Delta_G(2A, 3B, pX)$	168	0	0	39	0	30	0	14	0	0
$\Delta_G(2A, 3C, pX)$	0	0	0	0	45	0	0	0	0	0
$\Delta_G(2A, 5A, pX)$	56	0	168	18	0	95	0	14	0	0
$\Delta_G(2A, 5B, pX)$	0	0	0	0	0	0	90	0	44	44
$\Delta_G(2A, 7A, pX)$	0	0	0	180	0	300	0	175	0	0
$\Delta_G(2A, 11A, pX)$	0	0	0	0	0	0	100	0	110	220
$\Delta_G(2A, 11B, pX)$	0	0	0	0	0	0	100	0	220	110
$\Delta_G(2B, 2B, pX)$	420	84	1260	165	54	225	25	42	0	0
$\Delta_G(2B, 3A, pX)$	0	24	0	0	0	0	0	0	0	0
$\Delta_G(2B, 3B, pX)$	0	176	0	180	0	0	25	28	0	0
$\Delta_G(2B, 3C, pX)$	0	384	0	0	216	0	100	84	110	110
$\Delta_G(2B, 5A, pX)$	0	144	0	0	0	0	0	28	0	0
$\Delta_G(2B, 5B, pX)$	0	1152	0	1080	648	0	825	504	660	660
$\Delta_G(2B, 7A, pX)$	0	576	0	360	162	600	150	644	55	55
$\Delta_G(2B, 11A, pX)$	0	0	0	0	1620	0	1500	420	2145	1320
$\Delta_G(2B, 11B, pX)$	0	0	0	0	1620	0	1500	420	1320	2145
$\|C_G(pX)\|$	20160	1152	60480	1080	162	1800	25	84	11	11

Table 5. The structure constants $\Delta_G(3X, qY, rZ)$

pX	2A	2B	3A	3B	3C	5A	5B	7A	11A	11B
$\Delta_G(3A, 3A, pX)$	8	0	25	2	0	5	0	0	0	0
$\Delta_G(3A, 3B, pX)$	0	0	112	32	3	30	0	7	0	0
$\Delta_G(3A, 3C, pX)$	0	0	0	20	21	0	0	0	0	0
$\Delta_G(3A, 5A, pX)$	56	0	168	18	0	40	0	7	0	0
$\Delta_G(3A, 5B, pX)$	0	0	0	0	0	0	30	0	11	11
$\Delta_G(3A, 7A, pX)$	0	0	0	90	0	150	0	63	0	0
$\Delta_G(3A, 11A, pX)$	0	0	0	0	0	0	25	0	110	55
$\Delta_G(3A, 11B, pX)$	0	0	0	0	0	0	25	0	55	110
$\Delta_G(3B, 3B, pX)$	728	192	1792	440	42	380	25	168	0	0
$\Delta_G(3B, 3C, pX)$	0	0	1120	280	390	600	100	224	66	66
$\Delta_G(3B, 5A, pX)$	336	0	1008	228	54	540	0	140	0	0
$\Delta_G(3B, 5B, pX)$	0	1152	0	1080	648	0	1080	504	704	704
$\Delta_G(3B, 7A, pX)$	3360	384	5040	2160	432	3000	150	1428	33	33
$\Delta_G(3B, 11A, pX)$	0	0	0	0	972	0	1600	252	3212	2332
$\Delta_G(3B, 11B, pX)$	0	0	0	0	972	0	1600	252	2332	3212
$\Delta_G(3C, 3C, pX)$	5600	1536	7840	2600	1198	2000	900	840	660	660
$\Delta_G(3C, 5A, pX)$	0	0	0	360	180	0	100	168	22	22
$\Delta_G(3C, 5B, pX)$	0	4608	0	4320	5832	7200	440	5376	4928	4928
$\Delta_G(3C, 7A, pX)$	0	1152	0	2880	1620	3600	1600	3024	990	990
$\Delta_G(3C, 11A, pX)$	0	11520	0	6480	9720	3600	11200	7560	12760	12760
$\Delta_G(3C, 11B, pX)$	0	11520	0	6480	9720	3600	11200	7560	12760	12760
$\|C_G(pX)\|$	20160	1152	60480	1080	162	1800	25	84	11	11

TABLE 6. The structure constants $\Delta_G(5X, qY, rZ)$

pX	2A	2B	3A	3B	3C	5A	5B	7A	11A	11B
$\Delta_G(5A, 5A, pX)$	1064	0	1344	324	0	428	2	112	0	0
$\Delta_G(5A, 5B, pX)$	0	0	0	0	648	144	456	336	440	440
$\Delta_G(5A, 7A, pX)$	3360	384	5040	1800	324	2400	100	1092	11	11
$\Delta_G(5A, 11A, pX)$	0	0	0	0	324	0	1000	84	1892	1804
$\Delta_G(5A, 11B, pX)$	0	0	0	0	324	0	1000	84	1804	1892
$\Delta_G(5B, 5B, pX)$	72576	38016	72576	46656	28512	32832	33984	32256	31680	31680
$\Delta_G(5B, 7A, pX)$	0	6912	0	6480	10368	7200	9600	8736	9504	9504
$\Delta_G(5B, 11A, pX)$	80640	69120	60480	69120	72576	72000	72000	72576	76032	69696
$\Delta_G(5B, 11B, pX)$	80640	69120	60480	69120	72576	72000	72000	72576	69696	76032
$\lvert C_G(pX)\rvert$	20160	1152	60480	1080	162	1800	25	84	11	11

TABLE 7. The structure constants $\Delta_G(7A, qY, rZ)$ and $\Delta_G(11X, qY, rZ)$

pX	2A	2B	3A	3B	3C	5A	5B	7A	11A	11B
$\Delta_G(7A, 7A, pX)$	42000	8832	45360	18360	5832	23400	2600	11996	825	825
$\Delta_G(7A, 11A, pX)$	0	5760	0	3240	14580	1800	21600	6300	29700	29700
$\Delta_G(7A, 11B, pX)$	0	5760	0	3240	14580	1800	21600	6300	29700	29700
$\Delta_G(11A, 11A, pX)$	403200	138240	302400	228960	187920	295200	158400	226800	147600	162000
$\Delta_G(11A, 11B, pX)$	201600	224640	604800	315360	187920	309600	172800	226800	147600	147600
$\Delta_G(11B, 11B, pX)$	403200	138240	302400	228960	187920	295200	158400	226800	162000	147600
$\lvert C_G(pX)\rvert$	20160	1152	60480	1080	162	1800	25	84	11	11

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REFERENCES

1. F. Ali, M.A.F. Ibrahim and A. Woldar, *(3, q, r)-generations of Fischers sporadic group* Fi_{24}', J. Group Theory **22** (2019) 453–489.
2. F. Ali, M.A.F. Ibrahim and A. Woldar, *On (2, 3)-generations of Fischers largest sporadic simple group* Fi_{24}', C. R. Acad. Sci. Paris, Ser. I, **357** (2019) 401–412.
3. A.R. Ashrafi, *Generating pairs for the Held group* He, J. Appl. Math. Comput. **10** (2002) 167–174.
4. A.R. Ashrafi, *(p,q,r)-generations of the sporadic group HN*, Taiwanese J. Math. **10** (2006), no. 3, 613–629.
5. A.B.M. Basheer, *The ranks of the classes of* A_{10}, Bull. Iranian Math. Soc. **43** (2017), no. 7, 2125–2135.
6. A.B.M. Basheer and J. Moori, *On the ranks of finite simple groups*, Khayyam J. Math. **2** (2016), no. 1, 18–24.
7. A.B.M. Basheer and J. Moori, *A survey on some methods of generating finite simple groups,* in: Groups St. Andrews 2017 in Birmingham, PP. 106–118, London Math. Soc. Lecture Note Ser. 455, Cambridge University Press, Cambridge, 2019.
8. A.B.M. Basheer, J. Motalane and T.T. Seretlo, *The (p,q,r)-generations of the Mathieu group* M_{23}, submitted.
9. A.B.M. Basheer and T.T. Seretlo, *On two generation methods for the simple linear group* $PSL(3,5)$, Khayyam J. Math. **5** (2019), no. 1, 125–139.
10. A.B. M. Basheer and T.T. Seretlo, *The* (p,q,r)*-generations of the alternating group* A_{10}, Quaest. Math. **43** (2020), no. 3, 395–408.
11. A.B.M. Basheer and T. T. Seretlo, *(p,q,r)-generations of the Mathieu group* M_{22}, Southeast Asian Bull. Math. (to appear).

12. M.D.E. Conder, *Some results on quotients of triangle groups*, Bull. Aust. Math. Soc. **30** (1984) 167–174.
13. J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker and R.A. Wilson, *Atlas of finite groups. Maximal subgroups and ordinary characters for simple groups. With computational assistance from J. G. Thackray,* Oxford University Press, Eynsham, 1985.
14. M.R. Darafsheh, A.R. Ashrafi, *Generating pairs for the sporadic group Ru*, J. Appl. Math. Comput. **12** (2003), no. 1-2, 143–154.
15. M.R. Darafsheh, A.R. Ashrafi and G.A. Moghani, (p, q, r)*-generations of the Conway group* Co_1*, for odd* p, Kumamoto J. Math. **14** (2001) 1–20.
16. M.R. Darafsheh, A.R. Ashrafi and G.A. Moghani, *(p,q,r)-generations of the sporadic group O'N*, in: Groups St. Andrews 2001 in Oxford. Vol. I, pp. 101–109, London Math. Soc. Lecture Note Ser. 304, Cambridge University Press, Cambridge, 2003.
17. M.R. Darafsheh, A.R. Ashrafi and G.A. Moghani, *(p,q,r)-generations and nX-complementary generations of the sporadic group Ly*, Kumamoto J. Math. **16** (2003), 13–25.
18. S. Ganief, *2-Generations of the Sporadic Simple Groups*, PhD thesis, University of KwaZulu-Natal, Pietermaritzburg, 1997.
19. S. Ganief and J. Moori, $(2, 3, t)$*-generations for the Janko group* J_3, Comm. Algebra **23** (1995), no. 12, 4427–4437.
20. S. Ganief and J. Moori, *2-generations of the smallest Fischer group* Fi_{22}, Nova J. Math. Game Theory Algebra **6** (1997), no. 2-3, 127–145.
21. S. Ganeif and J. Moori, *(p,q,r)-generations and nX-complementary generations of the sporadic groups HS and McL.*, J. Algebra **188** (1997), no. 2, 531–546.
22. S. Ganief and J. Moori, *(p,q,r)-generations of the smallest Conway group* Co_3, J. Algebra **188** (1997), no. 2, 516–530.
23. S. Ganief and J. Moori, *Generating pairs for the Conway groups* Co_2 *and* Co_3, J. Group Theory **1** (1998), no. 3, 237–256.
24. S. Ganief and J. Moori, *2-generations of the fourth Janko group* J_4, J. Algebra **212** (1999), no. 1, 305–322.
25. The GAP Group, *GAP-Groups, Algorithms, and Programming, Version 4.10.2;* 2019. http://www.gap-system.org
26. L. Di Martino, M. Pellegrini and A. Zalesski, *On generators and representations of the sporadic simple groups*, Comm. Algebra **42** (2014), 880–908.
27. J. Moori, (p, q, r)*-generations for the Janko groups* J_1 *and* J_2, Nova J. Algebra Geom. **2** (1993), no. 3, 277–285.
28. J. Moori, $(2, 3, p)$*-generations for the Fischer group* F_{22}, Comm. Algebra, **2** (1994), no. 11, 4597–4610.
29. R. Ree, *A theorem on permutations*, J. Combbin. Theory Sr. A, **10** (1971) 174–175.
30. L.L. Scott, *Matrices and cohomolgy*, Ann. Math. **105** (1977), no. 3, 67–76.
31. R. Vincent and A.E. Zalesski, *Non-Hurwitz classical groups*, LMS J. Comput. Math. **10** (2007), 21–82.
32. J. Ward, *Generation of Simple Groups by Conjugate Involutions*, PhD Thesis, University of London, 2009.
33. R. Wilson, P. Walsh, J. Tripp, I. Suleiman, R. Parker, S. Norton, S. Nickerson, S. Linton, J. Bray and R. Abbott, *Atlas of finite group representations*, http://brauer.maths.qmul.ac.uk/Atlas/v3/
34. A.J. Wodlar, *Sporadic simple groups which are Hurwitz*, J. Algebra **144** (1991) 443–450.
35. A.J. Woldar, *3/2-generarion of the sporadic simple groups*, Comm. Algebra **22**, (1994), no. 2, 675–685.

[1]School of Mathematical and Computer Sciences, University of Limpopo (Turfloop), P Bag X1106, Sovenga 0727, South Africa

Email address: ayoubbasheer@gmail.com; john.motalane@ul.ac.za; thekisoseretlo@gmail.com, thekiso.seretlo@ul.ac.za