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THE CORRESPONDENCE OF FUSION FRAMES AND FRAMES IN HILBERT C^*-MODULES AND FINITE GABOR FUSION FRAMES

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Abstract. We show that fusion frames in a finite-dimensional Hilbert space H correspond to frames in a Hilbert C^*-module $\mathcal{B}(\mathbb{C}^n)$. Moreover, we show that every tight fusion frame (resp. Riesz fusion basis) in $\mathbb{C}^n$ corresponds to a tight frame (resp. a Riesz basis) in the Hilbert C^*-module $\mathcal{B}(\mathbb{C}^n)$. Then, we use this fact to characterize the dual of a Riesz fusion basis. Finally, we introduce Gabor fusion frames as a new notion.

1. Introduction

Fusion frames were originally called frames of subspaces introduced by Casazza and Kutyniok [10]. Fusion frames are a generalization of frames, which are useful for robust and stable representation of a signal. More Precisely, in the frame theory, a signal is represented by the magnitude of the projection of the signal on frame vectors. However, in the fusion frame theory, a signal is represented by a collection of vectors, in which their entries are equal to the inner product of the signal and orthogonal basis of subspaces of the fusion frame. Fusion frames have been applied in several different fields, such as sampling theory [12], data quantization [7], coding [6], image processing [8], time-frequency analysis [11], and speech recognition [5].

Many mathematicians have generalized the notion of a frame in a Hilbert space to a frame in a Hilbert C^*-module and achieved significant results. Standard frames in Hilbert C^*-modules over unital C^*-algebras were first defined by Frank

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and Larson in 1998 [14]. However, the case of Hilbert C^*-module over nonunital C^*-algebra was investigated in [19] as well as in [4]. The most significant advantage of frames in Hilbert C^*-module to frames in Hilbert space is the additional degree of freedom coming from C^*-algebra of coefficients. Therefore, similar procedures used for frames in Hilbert spaces are applied to deal with frames in Hilbert C^*-modules.

The Gabor transform as the set of all time-frequency shifts of a single vector in $\mathbb{C}^n$ has profound implications in signal processing. Such a frame is an equal norm tight frame that is maximally robust to erasures, a fact which allows the recovery of the original signal even if some parts of the information are lost; see [9]. In addition, the Gabor transform is used to determine the sinusoidal frequency and phase content of local sections of a signal as it changes over time.

In this article, we show that every fusion frame in $\mathbb{C}^n$ is a frame in Hilbert C^*-module $\mathcal{B}(\mathbb{C}^n)$ and vice versa. Thus, it is clear that they share similar properties and results in different interpretations. We also define a Gabor fusion frame using modular frames in Hilbert C^*-module $\mathcal{B}(\mathbb{C}^n)$. We can use similar procedures, using modular frames in Hilbert C^*-module $\mathcal{B}(\mathbb{C}^n)$, to define generalized wavelet fusion frames [16] and Galois wavelet fusion frames [15].

This article is structured as follows: In Section 2, we start with preliminaries about fusion frames and frames in Hilbert C^*-modules and some characteristics of them. Moreover, we present a brief summary of Gabor frames, which is useful in the construction of Gabor fusion frames. In section 3, we present the equivalence of fusion frames and frames in Hilbert C^*-modules. Then, we discuss some properties of these two notions and show that they share similar properties. We also argue the unique dual fusion sequence for a Riesz fusion basis by using the equivalence of these two notions. Finally, we devote Section 4 to introducing a new notion, Gabor fusion frame, using modular frames in Hilbert C^*-modules and explore some of its properties.

2. Preliminaries and notation

We recall that a fusion frame in a finite-dimensional Hilbert space $\mathbb{C}^n$ is a family of subspaces $\{W_i\}_{i=1}^N$ in $\mathbb{C}^n$ and a family of positive weights $\{\omega_i\}_{i=1}^N$ such that there exist two positive constants A and B with $A \leq B$ and for every $\mathbf{x} \in \mathbb{C}^n$

$$A\|\mathbf{x}\|^2 \leq \sum_{i=1}^{N} \omega_i^2 \|P_{W_i}\mathbf{x}\|^2 \leq B\|\mathbf{x}\|^2, \tag{2.1}$$

where P_{W_i} is the orthogonal projection onto W_i. In this article, we use fusion frames with weights equal to one. Therefore, in this case, (2.1) takes the form

$$A\|\mathbf{x}\|^2 \leq \sum_{i=1}^{N} \|P_{W_i}\mathbf{x}\|^2 \leq B\|\mathbf{x}\|^2. \tag{2.2}$$

The constants A and B are called the fusion frame bounds. Furthermore, the fusion frame is tight, whenever $A = B$. Note that the second inequality of (2.2) is always true as $\mathbf{x}$ belongs to the finite-dimensional space $\mathbb{C}^n$. A fusion frame $\{W_i\}_{i=1}^N$ is said to be an orthonormal fusion basis if $\mathbb{C}^n = \oplus_{i=1}^N W_i$, and it is a Riesz decomposition of $\mathbb{C}^n$ if for every $\mathbf{x} \in \mathbb{C}^n$, there exists a unique choice of $\mathbf{x}_i \in W_i$ such that $\mathbf{x} = \sum_{i=1}^N \mathbf{x}_i$. Moreover, a family of subspaces $\{W_i\}_{i=1}^N$ of $\mathbb{C}^n$ is called a Riesz fusion basis whenever it is complete for $\mathbb{C}^n$ and there exist two positive constants $0 < C \leq D < \infty$ such that for any arbitrary vector $\{\mathbf{x}_i\}_{i=1}^N \in \oplus_{i=1}^N W_i$ we have

$$C \sum_{i=1}^N \|\mathbf{x}_i\|^2 \leq \| \sum_{i=1}^N \mathbf{x}_i\|^2 \leq D \sum_{i=1}^N \|\mathbf{x}_i\|^2.$$

It is clear that any Riesz fusion basis is a fusion frame and that also a fusion frame is a Riesz basis if and only if it is a Riesz decomposition for $\mathbb{C}^n$ [10].

To define the operators associated with a fusion frame, we assume the Hilbert space

$$\oplus_{i=1}^N W_i = \left\{ \{\mathbf{x}_i\}_{i=1}^N : \mathbf{x}_i \in W_i \right\},$$

with the inner product

$$\langle \{\mathbf{x}_i\}_{i=1}^N, \{\mathbf{y}_i\}_{i=1}^N \rangle = \sum_{i=1}^N \langle \mathbf{x}_i, \mathbf{y}_i \rangle.$$

The analysis operator $T_W : \mathbb{C}^n \to \oplus_{i=1}^N W_i$ for any $\mathbf{x} \in \mathbb{C}^n$ is defined as

$$T_W(\mathbf{x}) = \{P_{W_i}\mathbf{x}\}_{i=1}^N.$$

The adjoint of the analysis operator, $T_W^* : \oplus_{i=1}^N W_i \to \mathbb{C}^n$ is called the synthesis operator, and it is given by

$$T_W^*\left(\{\mathbf{x}_i\}_{i=1}^N\right) = \sum_{i=1}^N \mathbf{x}_i.$$

Let $\{W_i\}_{i=1}^N$ be a fusion frame. The fusion frame operator $S_W : \mathbb{C}^n \to \mathbb{C}^n$ is defined by

$$S_W\mathbf{x} = \sum_{i=1}^N P_{W_i}\mathbf{x},$$

which is an invertible and positive operator. Consequently, we have

$$\mathbf{x} = \sum_{i=1}^N S_W^{-1} P_{W_i}\mathbf{x}$$

for every $\mathbf{x} \in \mathbb{C}^n$. The family $\{S_W^{-1}W_i\}_{i=1}^N$ is called the canonical dual fusion frame associated with $\{W_i\}_{i=1}^N$. We also call a Bessel fusion sequence $\{V_i\}_{i=1}^N$ as a dual fusion frame of $\{W_i\}_{i=1}^N$ if and only if there exists an operator $\Phi_{VW} : \oplus_{i=1}^N W_i \to \oplus_{i=1}^N V_i$ such that $T_V \Phi_{VW} T_W^* = I_{\mathbb{C}^n}$.

Moreover, a Hilbert C^*-module over a C^*-algebra $\mathcal{A}$ is a left $\mathcal{A}$-module E with an $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle : E \times E \to \mathcal{A}$ such that E is a Banach space with respect to the norm $\|\mathbf{x}\| = \|\langle \mathbf{x}, \mathbf{x} \rangle\|^{\frac{1}{2}}$. Recall that the inner product on E has the following properties:

- $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$,
- $\langle \mathbf{x}, \mathbf{x} \rangle = 0 \Leftrightarrow \mathbf{x} = 0$,
- $\langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle$,

- $\langle a\mathbf{x}, \mathbf{y}\rangle = a\langle \mathbf{x}, \mathbf{y}\rangle$,
- $\langle \mathbf{x}, \mathbf{y}\rangle^* = \langle \mathbf{y}, \mathbf{x}\rangle$,

where $\mathbf{x}, \mathbf{y}, \mathbf{z} \in E$ and $a \in \mathcal{A}$. We say that E is countably generated if there exists a sequence $\{x_i\}_{i\in I}$ in E such that the closed linear span of the set $\{\mathbf{x}_i a : i \in I, a \in \mathcal{A}\}$ is equal to E. It is clear that $\mathcal{B}(\mathbb{C}^n)$ is a finitely generated Hilbert C^*-module over itself.

Let E be a Hilbert C^*-module. A finite sequence $\{\mathbf{x}_i\}_{i=1}^N$ in E is called a frame for E if there exist two positive constants A and B such that for every $\mathbf{x} \in E$

$$A\|\langle \mathbf{x}, \mathbf{x}\rangle\| \leq \|\sum_{i=1}^{N} \langle \mathbf{x}, \mathbf{x}_i\rangle\langle \mathbf{x}_i, \mathbf{x}\rangle\| \leq B\|\langle \mathbf{x}, \mathbf{x}\rangle\|. \tag{2.3}$$

If only the second inequality of (2.3) is satisfied, then we say that $\{\mathbf{x}_i\}_{i=1}^N$ is a Bessel sequence. The constants A and B are called frame bounds. If $A = B = 1$, that is, if for every $x \in E$

$$\sum_{i=1}^{N} \langle \mathbf{x}, \mathbf{x}_i\rangle\langle \mathbf{x}_i, \mathbf{x}\rangle = \langle \mathbf{x}, \mathbf{x}\rangle,$$

then the sequence $\{\mathbf{x}_i\}_{i=1}^N$ is called a Parseval frame for E.

Finally, we provide a brief summary of Gabor frames, which is useful to construct our tight fusion frames. We index the components of a vector $\mathbf{x} \in \mathbb{C}^n$ by $\{0, 1, \ldots, n-1\}$, that is, the cyclic group $\mathbb{Z}_N$. We write $\mathbf{x}(m)$ instead of $\mathbf{x}_m$ to avoid algebraic operations on indices.

The discrete Fourier transform is the base of Gabor analysis, which is defined as

$$\mathcal{F}\mathbf{x}(l) = \hat{\mathbf{x}}(l) = \sum_{m=0}^{n-1} \mathbf{x}(m)\, e^{-2\pi i l \frac{m}{n}}.$$

Gabor analysis is concerned with the interplay of the Fourier transform, translation operators, and modulation operators. The translation operator $T_k : \mathbb{C}^N \to \mathbb{C}^N$ is given by

$$T_k\mathbf{x}(m) = \mathbf{x}(m-k).$$

The operator T_k alters the position of the entries of $\mathbf{x}$. Note that $m-k$ is achieved modulo n. The modulation operator $M_\ell : \mathbf{C}^n \to \mathbb{C}^n$ is given by

$$M_l\mathbf{x}(m) = e^{2\pi i l \frac{m}{n}}\mathbf{x}(m).$$

Modulation operators are implemented as the pointwise product of the vector with harmonics $e^{-2\pi i \ell \frac{\cdot}{n}}$.

Translation and modulation operators are referred to as time-shift and frequency-shift operators, respectively. The time-frequency shift operator $\pi(k, \ell)$ is the combination of a translation operator and a modulation operator

$$\pi(k, \ell) : \mathbb{C}^n \to \mathbb{C}^n \quad \pi(k, \ell)\,\mathbf{x} = M_l T_k \mathbf{x}.$$

Hence, the short time-Fourier transform $V_{\mathbf{y}} : \mathbb{C}^n \to \mathbb{C}^{n\times n}$ with respect to the window $\mathbf{y} \in \mathbb{C}^n$ can be written as

$$V_{\mathbf{y}}\mathbf{x}\left(k,l\right) = \langle \mathbf{x}, \pi\left(k,l\right)\mathbf{y}\rangle = \sum_{m=0}^{n-1} \mathbf{x}\left(m\right)\overline{\mathbf{y}\left(m-k\right)}e^{-2\pi i l\frac{m}{n}}, \quad \mathbf{x} \in \mathbb{C}^n.$$

The short time-Fourier transform generally uses a window function $\mathbf{y}$, supported at a neighborhood of zero that is translated by k. Therefore, the pointwise product with $\mathbf{x}$ selects a portion of $\mathbf{x}$ centered at k, and this portion is analyzed by using the Fourier transform. The inversion formula for the short time-Fourier transform is (see [2, 3, 18])

$$\begin{aligned}\mathbf{x}\left(m\right) &= \frac{1}{n\|\mathbf{y}\|_2^2}\sum_{k=0}^{n-1}\sum_{l=0}^{n-1} V_{\phi}\mathbf{x}\left(k,\ell\right)\mathbf{y}\left(m-k\right)e^{2\pi i\ell\frac{m}{n}}\\ &= \frac{1}{n\|\mathbf{y}\|_2^2}\sum_{k=0}^{n-1}\sum_{l=0}^{n-1}\langle \mathbf{x}, \pi\left(k,l\right)\mathbf{y}\rangle\pi\left(k,l\right)\mathbf{y}\left(m\right), \quad \mathbf{x} \in \mathbb{C}^n.\end{aligned}$$

Hence, it is obvious that for all $\mathbf{y} \neq 0$, the system is an $n\|\mathbf{y}\|_2^2$-tight frame.

3. Correspondence between Fusion frames in $\mathbb{C}^n$ and frames in Hilbert C^*-module $\mathcal{B}\left(\mathbb{C}^n\right)$

We start by briefly reviewing fusion frames in finite-dimensional Hilbert spaces and frames in Hilbert C^*-modules. In this section, we present a relationship between fusion frames in $\mathbb{C}^n$ and frames in Hilbert C^*-module $\mathcal{B}\left(\mathbb{C}^n\right)$. Moreover, $\mathcal{B}\left(\mathbb{C}^n\right)$ is a C^*-algebra with multiplication as the matrix multiplication. Therefore, it is a Hilbert C^*-module on itself with the inner product defined as

$$\langle A, B\rangle = AB^*, \quad A, B \in \mathcal{B}\left(\mathbb{C}^n\right),$$

where $*$ is the usual conjugate transpose of a matrix. Then, we explore some properties of these two notions and find out how they are related.

First, we investigate how the fusion frame elements can be represented in $\mathcal{B}\left(\mathbb{C}^n\right)$. Let W be a subspace of $\mathbb{C}^n$ and let $\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ be an orthonormal basis for W. We present the subspace W by the projection matrix P_W, which is equal to $U_W U_W^*$, where the matrix U_W is of the form

$$U_W = \left[\mathbf{e}_1|\cdots|\mathbf{e}_m\right],$$

and $\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ are the columns of U_W. We define a map $\tilde{\ }^{i} : \mathbb{C}^n \to \mathbb{C}^{n\times n}$ by adding some zero vectors to rows of a matrix, so that a vector $\mathbf{x} \in \mathbb{C}^n$ is transferred to an $n \times n$ matrix $\tilde{X}^i := [\mathbf{0}|\cdots|\mathbf{x}|\mathbf{0}|\cdots|\mathbf{0}]^T$, where the ith row of $\tilde{X}$ is equal to $\mathbf{x}$ and other rows are equal to $\mathbf{0}$ and T is the transpose of a matrix. By this process, we have P_W and $\tilde{X}^1$ as two elements of $\mathcal{B}\left(\mathbb{C}^n\right)$, which lead the product of these two elements are well-defined.

Our next theorem shows that fusion frames are frames in Hilbert C^*-module, and as a result, they share similar properties.

Theorem 3.1. *Assume that $\{W_i\}_{i=1}^N$ is a family of subspaces of $\mathbb{C}^n$ and that P_{W_i} is the matrix associated to W_i. Then, the following statements are equivalent:*

(i) $\{W_i\}_{i=1}^N$ *is a fusion frame for* $\mathbb{C}^n$.
(ii) $\{P_{W_i}\}_{i=1}^N$ *is a frame for* $\mathcal{B}(\mathbb{C}^n)$.

Proof. (i) $\to$ (ii) Let $\{W_i\}_{i=1}^N$ be a fusion frame with the minimum and the maximum bounds A and B, respectively. Therefore, for every $\mathbf{x} \in \mathbb{C}^n$

$$A\|\mathbf{x}\|^2 \leq \sum_{i=1}^{N} \|P_{W_i}\mathbf{x}\|^2 \leq B\|\mathbf{x}\|^2.$$

Let $\tilde{X}^1$ be the associated matrix with x. Then,

$$\begin{aligned}\langle \tilde{X}^1, P_{W_i}\rangle &= [\mathbf{x}, 0, \ldots, 0]^T \left[e_1^i, \ldots, e_m^i\right] \left[\overline{e_1^i}, \ldots, \overline{e_m^i}\right]^T \\ &= \left[\sum_{j=1}^{m} \langle x, \overline{e_j^i}\rangle \overline{e_j^i}, 0, \ldots, 0\right]^T = \widetilde{P_{W_i}\mathbf{x}}^1.\end{aligned}$$

Assume that $Z = [\mathbf{x}_1, \ldots, \mathbf{x}_n]^T \in \mathcal{B}(\mathbb{C}^n)$ is given. We have

$$\begin{aligned}\|\sum_{k=1}^{N}\langle Z, P_{W_k}\rangle\langle P_{W_k}, Z\rangle\|_{1,1} &= \|\sum_{k=1}^{N}\langle\sum_{j=1}^{n}\tilde{X^j}_j, P_{W_k}\rangle\langle P_{W_k}, \sum_{l=1}^{n}\tilde{X^l}_l\rangle\|_{1,1} \\ &= \|\sum_{k=1}^{N}\sum_{j=1}^{n}\sum_{l=1}^{n}\tilde{X^j}_j P_{W_k}^* P_{W_k}\tilde{X^{l*}}_l\|_{1,1} \\ &= \|\sum_{k=1}^{N}\sum_{j=1}^{n}\sum_{l=1}^{n}\tilde{X^j}_j P_{W_k}\tilde{X^{l*}}_l\|_{1,1} \\ &= \sum_{j=1}^{n}\sum_{l=1}^{n}|\sum_{k=1}^{N}\sum_{i=1}^{m}\langle\mathbf{x}_j, \overline{e_i}^k\rangle\langle\overline{e_i}^k, \mathbf{x}_l\rangle| \\ &= \sum_{j=1}^{n}\sum_{k=1}^{N}\sum_{i=1}^{m}|\langle\mathbf{x}_j, \overline{e_i}^k\rangle|^2 \\ &\quad + \sum_{j,l=1, j\neq l}^{n}|\sum_{k=1}^{N}\sum_{i=1}^{m}\langle\mathbf{x}_j, \overline{e_i}^k\rangle\langle\overline{e_i}^k, x_l\rangle| \\ &= \sum_{j=1}^{n}\sum_{k=1}^{N}\|P_{W_k}\mathbf{x}_j\|^2 + \sum_{j,l=1, j\neq l}^{n}|\sum_{k=1}^{N}\sum_{i=1}^{m}\langle\mathbf{x}_j, \overline{e_i}^k\rangle\langle\overline{e_i}^k, \mathbf{x}_l\rangle| \\ &\geq A\sum_{j=1}^{n}\|\mathbf{x}_j\|^2.\end{aligned} \tag{3.1}$$

On the other hand,

$$
\begin{aligned}
\|\langle Z, Z\rangle\|_{1,1} = \sum_{j,l=1}^{n} |\langle \mathbf{x}_j, \mathbf{x}_l\rangle| &\le \sum_{j=1}^{n} \|\mathbf{x}_j\|^2 + \sum_{j,l=1^n, j\neq l} \|\mathbf{x}_j\|\|x_l\| \\
&\le \sum_{j=1}^{n} \|\mathbf{x}_j\|^2 + \sum_{j,l=1^n, j\neq l} \frac{1}{2}\left(\|\mathbf{x}_j\|^2 + \|\mathbf{x}_l\|^2\right) \\
&= \sum_{j=1}^{n} \|\mathbf{x}_j\|^2 \\
&\quad + (n-1)\sum_{j=1}^{n} \|\mathbf{x}_j\|^2 = n\sum_{j=1}^{n} \|\mathbf{x}_j\|^2.
\end{aligned} \tag{3.2}
$$

By equations (3.1) and (3.2), we have

$$
\|\sum_{k=1}^{N} \langle Z, P_{W_k}\rangle\langle P_{W_k}, Z\rangle\|_{1,1} \ge A\sum_{j=1}^{n} \|\mathbf{x}_j\|^2 \ge \frac{A}{n}\|\langle Z, Z\rangle\|_{1,1}. \tag{3.3}
$$

All norms are equivalent on the finite-dimensional Hilbert C^*-module $\mathcal{B}(\mathbb{C}^n)$, so inequality (3.3) satisfies in the operator norm.

On the other hand, by (3.1), we have

$$
\begin{aligned}
\|\sum_{k=1}^{N} \langle Z, P_{W_k}\rangle\langle P_{W_k}, Z\rangle\|_{1,1} &= \sum_{j=1}^{n}\sum_{k=1}^{N} \|P_{W_k}\mathbf{x}_j\|^2 + \sum_{j,l=1, j\neq l}^{n} |\sum_{k=1}^{N}\sum_{i=1}^{m} \langle \mathbf{x}_j, \overline{e_i}^k\rangle\langle \overline{e_i}^k, \mathbf{x}_l\rangle| \\
&= \sum_{j=1}^{n}\sum_{k=1}^{N} \|P_{W_k}\mathbf{x}_j\|^2 + \sum_{j,l=1, j\neq l}^{n} |\sum_{k=1}^{N} |\langle P_{W_k}\mathbf{x}_j, P_{W_k}\mathbf{x}_l\rangle|.
\end{aligned} \tag{3.4}
$$

By the same process as we have done in (3.2), the following inequalities are resulted from (3.4):

$$
\|\sum_{k=1}^{N} \langle Z, P_{W_k}\rangle\langle P_{W_k}, Z\rangle\|_{1,1} \le n\sum_{k=1}^{N}\sum_{j=1}^{n} \|P_{W_k}\mathbf{x}_j\|^2 \le nB\sum_{j=1}^{n} \|\mathbf{x}_j\|^2. \tag{3.5}
$$

Moreover,

$$
\|\langle Z, Z\rangle\|_{1,1} = \sum_{j,l=1}^{n} |\langle \mathbf{x}_j, \mathbf{x}_l\rangle| \ge \sum_{j=1}^{n} \|\mathbf{x}_j\|^2. \tag{3.6}
$$

By equations (3.5) and (3.6), we have

$$
\|\sum_{k=1}^{N} \langle Z, P_{W_k}\rangle\langle P_{W_k}, Z\rangle\|_{1,1} \le nB\|\langle Z, Z\rangle\|_{1,1}. \tag{3.7}
$$

The inequality (3.7) satisfies in the operator norm. Because, all norms are equivalent in $\mathcal{B}(\mathbb{C}^n)$.

(ii) $\rightarrow$ (i) Let $\{P_{W_i}\}_{i=1}^N$ be a frame in the Hilbert C^*-module $\mathcal{B}(\mathbb{C}^n)$. Therefore, there exist two positive constants $A \leq B$ such that for every $Z \in \mathcal{B}(\mathbb{C}^n)$, we have

$$A\|\langle Z, Z\rangle\|_{op} \leq \|\sum_{k=1}^N \langle Z, P_{W_k}\rangle\langle P_{W_k}, Z\rangle\|_{op} \leq B\|\langle Z, Z\rangle\|_{op}.$$

Since, all norms are equivalent in $\mathcal{B}(\mathbb{C}^n)$, we have

$$A\|\langle Z, Z\rangle\|_{1,1} \leq \|\sum_{k=1}^N \langle Z, P_{W_k}\rangle\langle P_{W_k}, Z\rangle\|_{1,1} \leq B\|\langle Z, Z\rangle\|_{1,1}. \tag{3.8}$$

For every $\mathbf{x} \in \mathbb{C}^n$, we assume the associated matrix $\tilde{X}^1$. Therefore,

$$\|\langle \tilde{X}^1, \tilde{X}^1\rangle\|_{1,1} = |\langle \mathbf{x}, \mathbf{x}\rangle| = \|\mathbf{x}\|^2 \tag{3.9}$$

and

$$\|\sum_{k=1}^N \langle \tilde{X}^1, P_{W_k}\rangle\langle P_{W_k}, \tilde{X}^1\rangle\|_{1,1} = |\sum_{k=1}^N \sum_{i=1}^m |\langle \mathbf{x}, \overline{e_i}^k\rangle|^2| = \sum_{k=1}^N \|P_{W_k}\mathbf{x}\|^2. \tag{3.10}$$

By (3.8)–(3.10), the following inequality satisfies for every $x \in \mathbb{C}^n$:

$$A\|\mathbf{x}\|^2 \leq \sum_{i=1}^N \|P_{W_i}\mathbf{x}\|^2 \leq B\|\mathbf{x}\|^2.$$

$\square$

Now we check the correspondence between the analysis, synthesis, and frame operators of a fusion frame $\{W_i\}_{i=1}^N$ in $\mathbb{C}^n$ and its associated frame $\{P_{W_i}\}_{i=1}^N$ in the C^*-algebra $\mathcal{B}(\mathbb{C}^n)$. The analysis operator for $\{P_{W_i}\}_{i=1}^N$ in the C^*-algebra $\mathcal{B}(\mathbb{C}^n)$ is

$$T : \mathcal{B}(\mathbb{C}^n) \rightarrow (\mathcal{B}(\mathbb{C}^n))^N$$
$$X \mapsto \{\langle X, P_{W_i}\rangle\}_{i=1}^N,$$

and the analysis operator for $\tilde{X}^1 = [\mathbf{x}|\mathbf{0}|\cdots|\mathbf{0}]^T$ is

$$T\left(\tilde{X}^1\right) = \{\langle \tilde{X}^1, P_{W_i}\rangle\}_{i=1}^N = \{\widetilde{P_{W_i}\mathbf{x}}^1\}_{i=1}^N,$$

which corresponds to the analysis operator of the fusion frame $\{W_i\}_{i=1}^N$ in $\mathbb{C}^n$. As a result, the synthesis and frame operators of $\{P_{W_i}\}_{i=1}^N$ in the C^*-algebra $\mathcal{B}(\mathbb{C}^n)$ must be equal to those of the fusion frame $\{W_i\}_{i=1}^N$ in $\mathbb{C}^n$;

$$T^* = (\mathcal{B}(\mathbb{C}^n))^N \rightarrow \mathcal{B}(\mathbb{C}^n),$$
$$\{X_i\}_{i=1}^N \mapsto \sum_{i=1}^N X_i P_{W_i}.$$

Let $\mathbf{x}_i \in W_i$ for each $i = 1, \ldots, N$ and let $\tilde{X}_i^1$ be the associated matrix of $\mathbf{x}_i$. Then, the synthesis operator of $\{P_{W_i}\}_{i=1}^N$ in the C^*-algebra $\mathcal{B}(\mathbb{C}^n)$ is equal to

$$T^*\left(\{\tilde{X}_i^1\}_{i=1}^N\right) = \sum_{i=1}^N \tilde{X}_i^1 P_{W_i}^* = \sum_{i=1}^N \widetilde{P_{W_i}\mathbf{x}_i}^1 = \sum_{i=1}^N \tilde{X}_i^1.$$

Note that the equality $XP_{W_i} = \widetilde{P_{W_i}\mathbf{x}}^1$ is true, since $\{\bar{e}_{ij}\}_{j=1}^{n_i}$ is also an orthonormal basis for W_i, where $\{\mathbf{e}_{ij}\}_{j=1}^{n_i}$ is an orthonormal basis for W_i. The frame operator is concluded as the combination of the synthesis and analysis operators, so it is the same as the frame operator of the fusion frame $\{W_i\}_{i=1}^N$. More precisely,

$$\begin{aligned} S\left(\tilde{X}^1\right) &= \sum_{i=1}^{N} \langle \tilde{X}^1, P_{W_i}\rangle P_{W_i} = \sum_{i=1}^{N} \tilde{X}^1 P_{W_i}^* P_{W_i} = \sum_{i=1}^{N} \widetilde{P_{W_i}\mathbf{x}}^1 P_{W_i} \\ &= \sum_{i=1}^{N} \widetilde{P_{W_i} P_{W_i}\mathbf{x}}^1 = \sum_{i=1}^{N} \widetilde{P_{W_i}\mathbf{x}}^1. \end{aligned}$$

One of the most favorite types of frames is tight frames, which attract the attention of many researchers. In the next theorem, we show that the fusion frame $\{W_i\}_{i=1}^N$ is tight if and only if $\{P_{W_i}\}_{i=1}^N$ is a tight frame in the Hilbert C^*-module $\mathcal{B}(\mathbb{C}^n)$.

Theorem 3.2. *Assume that $\{W_i\}_{i=1}^N$ is a sequence of subspaces in $\mathbb{C}^n$. The following statements are equivalent:*

(i) *$\{W_i\}_{i=1}^N$ is an A-tight fusion frame in $\mathbb{C}^n$.*
(ii) *$\{P_{W_i}\}_{i=1}^N$ is an A-tight frame in the Hilbert C^*-module $\mathcal{B}(\mathbb{C}^n)$.*

Proof. (i) → (ii) By the assumption, $\{W_i\}_{i=1}^N$ is an A-tight fusion frame, so for every $\mathbf{x} \in \mathbb{C}^n$, we have

$$\mathbf{x} = \frac{1}{A} \sum_{i=1}^{N} P_{W_i}\mathbf{x} \tag{3.11}$$

or

$$\tilde{X}^1 = \frac{1}{A} \sum_{i=1}^{N} \langle \tilde{X}^1, P_{W_i}\rangle P_{W_i}. \tag{3.12}$$

We assume $X = [\mathbf{x}_1|\cdots|\mathbf{x}_n] \in \mathcal{B}(\mathbb{C}^n)$ as $\sum_{i=1}^N \tilde{X}_i^i$, where each $\tilde{X}_i^i$ contains the ith row of X and other rows are equal to zero. Since (3.11) holds for every $\mathbf{x}_i$, (3.12) is valid for every $X \in \mathcal{B}(\mathbb{C}^n)$. Thus, $\{P_{W_i}\}_{i=1}^N$ is an A-tight frame in the Hilbert C^*-module $\mathcal{B}(\mathbb{C}^n)$.

(ii) → (i) Assume that $\{P_{W_i}\}_{i=1}^N$ is an A-tight frame, so for every $X \in \mathcal{B}(\mathbb{C}^n)$, we have

$$X = \frac{1}{A} \sum_{i=1}^{N} \langle X, P_{W_i}\rangle P_{W_i}.$$

Thus, for every $\mathbf{x} \in \mathbb{C}^n$

$$\tilde{X}^1 = \frac{1}{A} \sum_{i=1}^{N} \langle \tilde{X}^1, P_{W_i}\rangle P_{W_i}. \tag{3.13}$$

Now equality (3.13) can be written as

$$\tilde{X}^1 = \frac{1}{A} \sum_{i=1}^{N} \tilde{X}^1 P_{W_i}^* P_{W_i} = \frac{1}{A} \sum_{i=1}^{N} \tilde{X}^1 P_{W_i} = \frac{1}{A} \sum_{i=1}^{N} \widetilde{P_{W_i}\mathbf{x}}^1.$$

Therefore, $\{W_i\}_{i=1}^N$ is an A-tight frame. □

Now we introduce the notion of Riesz fusion bases using Riesz bases in Hilbert C^*-module and show that it coincides with Riesz decomposition of fusion frames. We recall that a frame $\{X_j\}_{j=1}^N$ is a Riesz basis if and only if for each $j = 1, \ldots, N$, $X_j \neq 0$ and that if $\sum_{j=1}^N A_j X_j = 0$ for some sequence $\{A_j\}_{j=1}^N \in \ell^2(\mathcal{B}(\mathbb{C}^n))$, then $A_j X_j = 0$, for each $j = 1, \ldots, N$; see [17]. The next theorem shows that every fusion Riesz basis in $\mathbb{C}^n$ is a Riesz basis in the Hilbert C^*-module $\mathcal{B}(\mathbb{C}^n)$ and vice versa.

Theorem 3.3. *Let $\{W_j\}_{j=1}^N$ be a fusion frame in $\mathbb{C}^n$. The following statements are equivalent:*

(i) *$\{P_{W_j}\}_{j=1}^N$ is a Riesz basis in $\mathcal{B}(\mathbb{C}^n)$.*
(ii) *$\{W_j\}_{j=1}^N$ is a Riesz fusion basis in $\mathbb{C}^n$.*

Proof. (i) $\to$ (ii) Assume $\sum_{i=1}^N \mathbf{x}_i = \mathbf{0}$. So, $\sum_{i=1}^N P_{W_i}\mathbf{x}_i = \mathbf{0}$ and $\sum_{i=1}^N \tilde{X}_i^1 P_{W_i} = \mathbf{0}$. Since $\{P_{W_i}\}_{i=1}^N$ is a Riesz basis in $\mathcal{B}(\mathbb{C}^n)$, we have $\tilde{X}_i^1 P_{W_i} = \mathbf{0}$ for each $i = 1, \ldots, N$ or $P_{W_i}\mathbf{x}_i = \mathbf{x}_i = \mathbf{0}$, which means that $\{W_j\}_{j=1}^N$ is a Riesz fusion basis in $\mathbb{C}^n$.

(ii) $\to$ (i) Assume that $\{W_i\}_{i=1}^N$ is a Riesz fusion basis. Then every $\mathbf{x} \in \mathbb{C}^n$ has a unique representation based on $\{W_i\}_{i=1}^N$, which means that if $\sum_{i=1}^N P_{W_i}\mathbf{x}_i = 0$, then $P_{W_i}\mathbf{x}_i = 0$ for all $i = 1, \ldots, N$ for all $\{\mathbf{x}_i\}_{i=1}^N \in \oplus_{i=1}^N W_i$. Now assume $\sum_{i=1}^N X_i P_{W_i} = 0$. We can rewrite it as $\sum_{i=1}^N \sum_{j=1}^n \tilde{X}_{i_j}^j P_{W_i} = 0$, where $\tilde{X}_{i_j}^j$ is the matrix with jth row is equal to the jth row of X_i and other rows are equal to zero. Therefore, for each $i = 1, \ldots, n$, we have $\sum_{j=1}^N \tilde{X}_{i_j}^j P_{W_i} = 0$. As a result $\sum_{ji=1}^N P_{W_i} x_{i_j} = 0$. Hence, $P_{W_i}\mathbf{x}_{i_j} = 0$ since $\{W_i\}_{i=1}^N$ is a Riesz fusion basis. Therefore, $\tilde{X}_{i_j}^j P_{W_i} = \mathbf{0}$ and then $X_i P_{W_i} = \mathbf{0}$. □

We recall that a sequence $\{X\}_{i=1}^N$ is a dual frame of $\{Y_i\}_{i=1}^N$ in $\mathcal{B}(\mathbb{C}^n)$ if

$$X = \sum_{i=1}^N \langle X, Y_i \rangle X_i$$

for all $X \in \mathcal{B}(\mathbb{C}^n)$. Therefore, we call the sequence of subspaces $\{V_i\}_{i=1}^N$ of $\mathbb{C}^n$ a dual of the fusion frame $\{W_i\}_{i=1}^N$ if for all $\mathbf{x} \in \mathbb{C}^n$,

$$\tilde{X}^1 = \sum_{i=1}^N \langle \tilde{X}^1, P_{W_i} \rangle P_{V_i} = \sum_{i=1}^N \widetilde{P_{V_i} P_{W_i}\mathbf{x}}^1$$

or

$$\mathbf{x} = \sum_{i=1}^N P_{V_i} P_{W_i}\mathbf{x}.$$

Now we study the dual of Riesz fusion bases in $\mathbb{C}^n$. It is worth mentioning that contrary to the Hilbert space situation, Riesz bases of Hilbert C^*-modules may possess infinitely many dual frames due to the existence of zero divisors in C^*-algebra of coefficients. Consequently, this property is satisfied for Riesz fusion bases; see [17].

In the next corollary, we characterize the dual sequence of Riesz fusion bases, which is a direct conclusion of Theorem 4.12 in [17].

Corollary 3.4. *Suppose that $\{W_i\}_{i=1}^N$ is a fusion Riesz basis in $\mathbb{C}^n$. Let $\{V_i\}_{i=1}^N$ be a sequence of subspaces in $\mathbb{C}^n$. The following statements are equivalent.*

(i) *$\{V_i\}_{i=1}^N$ is a dual fusion frame of $\{W_i\}_{i=1}^N$.*
(ii) *$\{V_i\}_{i=1}^N$ is a dual fusion Bessel sequence of $\{W_i\}_{i=1}^N$.*
(iii) *For each $i = 1, \ldots, N$, $P_{V_i} = S^{-1}P_{W_i} + Z_i$, where S is the frame operator of $\{W_i\}_{i=1}^N$ and $\{Z_i\}_{i=1}^N$ is a Bessel sequence of $\mathcal{B}(\mathbb{C}^n)$ satisfying $\langle X, Z_i\rangle P_{W_i} = 0$ for all $X \in \mathcal{B}(\mathbb{C}^n)$ and $i = 1, \ldots, N$.*

Proof. Using Theorem 3.3, one may conclude that $\{P_{W_i}\}_{i=1}^N$ is a Riesz basis in Hilbert C^*-module $\mathcal{B}(\mathbb{C}^n)$. So by [17, Theorem 4.12], the proof is completed. □

The next Corollary concerns the dual fusion Riesz basis of a fusion Riesz basis in $\mathbb{C}^n$.

Corollary 3.5. *Let $\{W_i\}_{i=1}^N$ be a fusion Riesz basis and let $\{V_i\}_{i=1}^N$ be a collection of subspaces in $\mathbb{C}^n$. Then $\{V_i\}_{i=1}^N$ is a dual fusion Riesz basis of $\{W_i\}_{i=1}^N$ if and only if for each $i = 1, \ldots, N$, $P_{V_i} = S^{-1}P_{W_i} + Z_i$, where S is the frame operator of $\{P_{W_i}\}_{i=1}^N$ and $\{Z_i\}_{i=1}^N$ is a Bessel sequence of $\mathcal{B}(\mathbb{C}^n)$ with the property that for each $i = 1, \ldots, N$, there exists $A_i \in \mathcal{B}(\mathbb{C}^n)$ such that $Z_i = A_iS^{-1}P_{W_i}$ and $\langle \tilde{X}, P_{W_i}\rangle A_i\tilde{X}_i = 0$ for all $\mathbf{x} \in \mathbb{C}^n$.*

Proof. Suppose that $\{W_i\}_{i=1}^N$ is a Riesz fusion basis. Then by the Theorem 3.3, $\{P_{W_i}\}_{i=1}^N$ is a Riesz basis for $\mathcal{B}(\mathbb{C}^n)$. Now by the Theorem 4.13 in [17], the proof is completed. □

Note that the only dual of a Riesz fusion basis of a Riesz fusion basis is its canonical dual; see [1,10]. Therefore, $\{Z_i\}_{i=1}^N$ in Corollary 3.5 is the zero sequence.

4. Finite Gabor Fusion frames

The aim of this section is to introduce the Gabor fusion frame through modular frames in Hilbert C^*-module $\mathcal{B}(\mathbb{C}^n)$. Gabor frames on a finite-dimensional Hilbert space were discussed in [13, 18]. We recall that a unitary system $\mathcal{U}$ on $\mathcal{B}(\mathbb{C}^n)$ is a set of unitary operators on $\mathcal{B}(\mathbb{C}^n)$, which contains the identity operator. The translation operator $\mathcal{T}_k$ on $\mathcal{B}(\mathbb{C}^n)$ is defined as

$$\begin{aligned}\mathcal{T}_k :&\mathcal{B}(\mathbb{C}^n) \to \mathcal{B}(\mathbb{C}^n),\\ &\mathcal{T}_k(X) = \mathcal{T}_k\left((\mathbf{x}_1, \ldots, \mathbf{x}_n)^T\right) = (T_k(\mathbf{x}_1), \ldots, T_k(\mathbf{x}_n))^T,\end{aligned}$$

where T_k is the usual translation operator on $\mathbb{C}^n$. It is clear that the matrix $\mathcal{T}_k$ is a unitary operator for $\mathcal{B}(\mathbb{C}^n)$ and that $\mathcal{T}_k^* = \mathcal{T}_k^{-1} = \mathcal{T}_{n-k}$.

Now the modulation operator on $\mathcal{B}(\mathbb{C}^n)$ is

$$\begin{aligned}\mathcal{M}_l :&\mathcal{B}(\mathbb{C}^n) \to \mathcal{B}(\mathbb{C}^n),\\ &\mathcal{M}_l(X) = \mathcal{M}_l\left((\mathbf{x}_1, \ldots, \mathbf{x}_n)^T\right) = (M_l(\mathbf{x}_1), \ldots, M_l(\mathbf{x}_n))^T,\end{aligned}$$

where M_l is the usual modulation operator on $\mathbb{C}^n$. Like the translation operator, the modulation operator is also a unitary operator and $\mathcal{M}_l^* = \mathcal{M}_l^{-1} = \mathcal{M}_{n-l}$.

It is obvious that the combination of two unitary operators is a unitary operator. Therefore, the set $\{\mathcal{M}_l\mathcal{T}_k\}_{k,l=0}^{N-1}$ constitutes a unitary system, which includes the identity operator when $k = l = 0$. The Gabor transform is then defined on $\mathcal{B}(\mathbb{C}^n)$ for the window matrix $\Phi \in \mathcal{B}(\mathbb{C}^n)$ as

$$\mathcal{V}_\Phi X(k,l) = \langle X, \mathcal{M}_l \mathcal{T}_k \Phi \rangle.$$

Now we assume W as a subspace of $\mathbb{C}^n$ and $P_W \in \mathcal{B}(\mathbb{C}^n)$ as the projection associated to W. Moreover, $\tilde{X} = (\mathbf{x}, \mathbf{0}, \ldots, \mathbf{0})^T \in \mathcal{B}(\mathbb{C}^n)$ for any $\mathbf{x} \in \mathbb{C}^n$. Then, the Gabor transform on $\mathcal{B}(\mathbb{C}^n)$ can easily be transfered to the Gabor fusion transform on $\mathbb{C}^n$, which is defined by

$$\mathcal{V}_\Phi X(k,l) = \langle \tilde{X}^1, \mathcal{M}_l \mathcal{T}_k P_W \rangle.$$

Indeed, the Gabor fusion transform is obtained through the projection of a signal on the translation and modulation of a window subspace. The set $\{\mathcal{M}_l \mathcal{T}_k P_W\}_{k,l=0}^{N-1}$ is called a Gabor fusion system. If it constitutes a fusion frame in $\mathbb{C}^n$, we call it a Gabor fusion frame. In the next theorem, we show that for any subspace W of $\mathbb{C}^n$ that satisfies some specified conditions, the Gabor fusion system $\{\mathcal{M}_l \mathcal{T}_k P_W\}_{k,l=0}^{N-1}$ is a Gabor fusion frame.

Theorem 4.1. *Let W be a subspace of $\mathbb{C}^n$ and let $\{\mathbf{e}_1, \ldots, \mathbf{e}_t\}$ be an orthonormal basis for W with $t \leq n$. Then, $\{\mathcal{T}_k \mathcal{M}_l P_W\}_{k,l=0}^{N-1}$ is a tight fusion frame for $\mathbb{C}^n$.*

Proof. For every $x \in \mathbb{C}^n$, we have

$$\begin{aligned}
\sum_{k,l=0}^{n-1} \|\mathcal{M}_l \mathcal{T}_k P_W \mathbf{x}\|^2 &= \sum_{k,l=0}^{n-1} \sum_{i=1}^{t} |\langle \mathbf{x}, M_l T_k \mathbf{e}_i \rangle|^2 \\
&= \sum_{i=1}^{t} \sum_{k,l=0}^{n-1} |\langle \mathbf{x}, M_l T_k \mathbf{e}_i \rangle|^2 \\
&= \sum_{i=1}^{t} \|V_{\mathbf{e}_i} \mathbf{x}\|_2^2 \\
&= n \sum_{i=1}^{t} \|\mathbf{e}_i\|_2^2 \|\mathbf{x}\|_2^2 \\
&= tn \|\mathbf{x}\|_2^2.
\end{aligned}$$

Thus, $\{\mathcal{M}_l \mathcal{T}_k P_W\}_{k,l=0}^{N-1}$ is a tight fusion frame in $\mathbb{C}^n$. □

The next theorem contains a relation between the notion of the canonical dual of a Gabor fusion frame and of a Gabor fusion frame.

Theorem 4.2. *If $\{\mathcal{T}_k \mathcal{M}_l P_W\}_{k,l=0}^{n-1}$ is a Gabor fusion frame for $\mathbb{C}^n$ in which W is a subspace of $\mathbb{C}^n$, then there exists a dual window subspace V in $\mathbb{C}^n$ such that $\{\mathcal{T}_k \mathcal{M}_l P_V\}_{k,l=0}^{n-1}$ is a Gabor fusion frame.*

Proof. Assume that S_W is the Gabor fusion frame operator for $\{\mathcal{M}_l \mathcal{T}_k P_W\}_{k,l=0}^{n-1}$. We first show that S_W commutes with time-frequency shifts $\mathcal{M}_s \mathcal{T}_r$ for $r, s =$

$0, \ldots, n-1$. Indeed

$$\begin{aligned}
(\mathcal{M}_s\mathcal{T}_r)^{-1} S_W \mathcal{M}_s\mathcal{T}_r X &= (\mathcal{M}_s\mathcal{T}_r)^{-1} \sum_{k,l=0}^{n-1} \langle \mathcal{M}_s\mathcal{T}_r X, \mathcal{M}_l\mathcal{T}_k P_W\rangle \mathcal{M}_l\mathcal{T}_k P_W \\
&= \sum_{k,l=0}^{n-1} \langle X, (\mathcal{M}_s\mathcal{T}_r)^{-1} \mathcal{M}_l\mathcal{T}_k P_W\rangle (\mathcal{M}_s\mathcal{T}_r)^{-1} \mathcal{M}_l\mathcal{T}_k P_W \\
&= \sum_{k,=0}^{n-1} \langle X, \mathcal{M}_{l-s}\mathcal{T}_{k-r} P_W\rangle \mathcal{M}_{l-s}\mathcal{T}_{k-r} P_W \qquad (4.1) \\
&= \sum_{k,l=0}^{n-1} \langle X, \mathcal{M}_l\mathcal{T}_k P_W\rangle \mathcal{M}_l\mathcal{T}_k P_W \\
&= S_W X,
\end{aligned}$$

for $\mathbf{x} \in \mathbb{C}^n$. Equality (4.1) is achieved, since

$$\begin{aligned}
&(\mathcal{M}_s\mathcal{T}_r)^{-1} \mathcal{M}_l\mathcal{T}_k P_W \\
&= \left((M_sT_r)^{-1} M_lT_k\mathbf{y}_1, \ldots, (M_sT_r)^{-1} M_lT_k\mathbf{y}_n\right)^T \\
&= \left(e^{+2\pi il(k-r)} M_{l-s}T_{k-r}\mathbf{y}_1, \ldots, e^{+2\pi il(k-r)} M_{l-s}T_{k-r}\mathbf{y}_n\right)^T \\
&= e^{-2\pi il(k-r)} \mathcal{M}_{l-s}\mathcal{T}_{k-r} P_W.
\end{aligned}$$

So it is shown that S_W commutes with $\mathcal{M}_s\mathcal{T}_r$ for all $r, s = 0, \ldots, n-1$. Now replacing X by $S_W^{-1}X$, we get

$$S_W \mathcal{M}_s\mathcal{T}_r S_W^{-1} X = \mathcal{M}_s\mathcal{T}_r S_W S_W^{-1} X = \mathcal{M}_s\mathcal{T}_r X.$$

□

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