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SOME PROPERTIES OF GEODESIC (α, E)-PREINVEX FUNCTIONS ON RIEMANNIAN MANIFOLDS

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Abstract. We introduce the concept of *geodesic* (α, E)*-invex set*, and by using this concept the notion of *geodesic* (α, E)*-preinvex functions* and *geodesic* (α, E)*-invex functions* is developed on Riemannian manifolds. Moreover, several properties and results are deduced within aforesaid functions. An example is also constructed to illustrate the definition of geodesic (α, E)-invex set. We also establish an important relation between a geodesic (α, E)-preinvex function and geodesic (α, E)-invex function in a complete Riemannian manifold.

1. Introduction

Convex sets and convex functions play an indispensable and significant role in the theory of nonlinear programming and optimization. Since, the notion of convexity has great influence in real world problems, various authors have developed the new concept of convexity in order to burgeon the result to the larger class of optimization. Hanson in 1981, made a significant step by introducing the concept of invexity. Hanson's work is responsible for further development of the significance of invexity in mathematical programming. Later, the role of preinvex functions and invex sets in optimization theory, variational inequalities, and equilibrium problems was investigated by Jeyakumar [5], Weir and Mond [15], and various authors. In 1999, Younees [18] generalized the concept of convexity by introducing an operator E and named it as E-convex set and E-convex function. However, Yang [16] proved that some of the results given by Youness

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are not correct. Moreover, the concept of strong E-convexity and semi-strong E-convexity was defined in [19].

Over the past few years, many results in the theory of nonlinear analysis and optimization on Riemannian manifolds have been extended from the Euclidean spaces. Convex functions in Riemannian manifolds, were studied by many authors; see [3, 9, 12, 13, 17]. Udrişte [14] and Rapcsak [11] developed the concept of geodesic convexity on Riemannian manifolds. Pini [10] introduced the concept of Riemannian invexity, while Mititelu [8] investigated its generalization. In 2012, Iqbal, Ali, and Ahmad [4] established the notion of geodesic E-convex set and functions and also they have discussed their properties and results. Barani and Pouryayevali [2] introduced the concepts of geodesic invex set and geodesic preinvex functions on Riemannian manifolds with respect to particular maps. In 2012, Agarwal et al. [1] generalized the notion of invexity and developed the concept of α-invex sets and α-preinvex functions in Riemannian manifold. Moreover, Kumari and Jayswal [7] introduced the notion of geodesic E-preinvex function and geodesic semi E-preinvex function on Riemannian manifold and investigated some of its properties. Motivating the work of Agarwal et al. and Kumari and Jayswal, we have developed the concept of (α, E)-invex sets and (α, E)-preinvex functions in a complete Riemannian manifold and investigated some of its properties.

The paper is structured as follows: Section 2 deals with some well-known facts of Riemannian manifolds, geodesic convexity, and geodesic invexity. In the next section, we define *geodesic* (α, E)*-invex set* in a complete Riemannian manifold. By using this definition we develop the concept of *geodesic* (α, E)*-preinvex* and *geodesic* (α, E)*-invex functions*. In the last section, we deduce a relation between geodesic (α, E)-preinvex and geodesic (α, E)-invex function, (see Theorem 4.2), which is the main result of this paper. Also we deduce some properties of (α, E)-preinvex and geodesic (α, E)-invex functions.

2. Notations and preliminaries

In this section, we recall from [6] some basic concept of a Riemannian manifold (M, g), which is necessary throughout this paper. In this paper, the notation M will always mean the complete Riemannina manifold unless specified separately. The length $L(\gamma)$ of the curve $\gamma : [a, b] \to M$ is given by

$$\begin{aligned} L(\gamma) &= \int_a^b \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))}\, dt \\ &= \int_a^b |\dot{\gamma}(t)| dt. \end{aligned}$$

The curve γ is said to be a geodesic if its velocity vector $\gamma'(t)$ is parallel along $\gamma(t)$, that is, $\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) = 0$ for all $t \in [a, b]$, where ∇ is the Riemannian connection of g. For any point $p \in M$ and the unique geodesic γ_u with $\gamma(0) = p$ and $\dot{\gamma}_u(0) = u$, the exponential map $\exp_p : V_p \to M$ is defined by

$$\exp_p(u) = \gamma_u(1),$$

where and V_p is a collection of vectors of T_pM such that for each element $u \in V_p$, the geodesic with initial tangent vector u is defined on $[0,1]$. It can be easily seen that the norm of a tangent vector is constant for a geodesic γ. A smooth vector field is a smooth function $X : M \to TM$ such that $\pi \circ X = id_M$, where $\pi : TM \to M$ is the projection map, that is, in each point of the manifold M, we smoothly choose a tangent vector. The gradient of a function $f \in C^\infty(M)$ at the point $p \in M$ is defined by $\nabla f(p) = g^{ij} \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_i} \ |_p$, where $\{\frac{\partial}{\partial x_i}\}_{i=1}^n|_p$ is an orthonormal coordinate system for T_pM. We will denote the parallel translation from $T_{\gamma(t_0)}M$ to $T_{\gamma(t)}M$ along the curve γ by the notation $P^t_{t_0,\gamma}$.

Definition 2.1 ([14])**.** A real valued function f on M is called convex if

$$f \circ \gamma(t) \le (1-t) f \circ \gamma(0) + t f \circ \gamma(1) \quad \text{ for all } t \in [0,1],$$

for every geodesic $\gamma : [0,1] \to M$.

Definition 2.2 ([2])**.** Let $\eta : M \times M \to TM$ be a function such that $\eta(x,y) \in T_yM$ for every $x, y \in M$. A nonempty subset S of M is said to be geodesic invex with respect to η if, for every $x, y \in S$, there exists exactly one geodesic $\gamma_{x,y} : [0,1] \to M$ such that

$$\gamma_{x,y}(0) = y, \ \gamma'_{x,y}(0) = \eta(x,y), \ \gamma_{x,y}(t) \in S \quad \text{ for all } t \in [0,1].$$

Definition 2.3 ([2])**.** Let S be an open set of M that is a geodesic invex set with respect to $\eta : M \times M \to TM$. A differentiable function $f : S \to \mathbb{R}$ is said to be η-invex on S if the following condition holds:

$$f(x) - f(y) \ge df_y(\eta(x,y)) \qquad \text{ for all } x, y \in S.$$

Moreover, f is called geodesic η-preinvex if for every $x, y \in S$,

$$f(\gamma_{x,y}(t)) \le t f(x) + (1-t) f(y), \quad \text{for all } t \in [0,1].$$

3. (α, E)-invex set and (α, E)-preinvex function

Suppose that $E : M \to M$ is a function and that $\alpha : M \times M \to \mathbb{R} - \{0\}$ is a bifunction.

Definition 3.1. Let $\eta : M \times M \to TM$ be a function such that $\alpha(x,y)\eta(x,y) \in T_yM$ for every $x, y \in M$. A subset $S \neq \phi$ of M is said to be a geodesic (α, E)-invex set with respect to η and α if for every $x, y \in S$, there exists exactly one geodesic $\gamma_{E(x),E(y)} : [0,1] \to M$ such that $\gamma_{E(x),E(y)}(0) = E(y)$, $\gamma'_{E(x),E(y)}(0) = \alpha(E(x),E(y))\eta(E(x),E(y))$, and $\gamma_{E(x),E(y)}(t) \in S$, for all $t \in [0,1]$.

If $\alpha(p,q) = 1$ for all $p, q \in M$ and E is the identity map, then the geodesic (α, E)-invex set becomes the geodesic invex set; see [2]. Every geodesic (α, E)-invex set reduces to a geodesic α-invex set [1] if $E(p) = p$ for all $p \in M$. If only $\alpha \equiv 1$, then the geodesic (α, E)-invex set becomes a geodesic E-invex set [7].

Definition 3.2. Let S be an open subset of M that is a geodesic (α, E)-invex set with respect to $\eta : M \times M \to TM$ and $\alpha : M \times M \to \mathbb{R} - \{0\}$. A function $f : S \to \mathbb{R}$ is said to be geodesic (α, E)-preinvex if

$$f(\gamma_{E(x),E(y)}(t)) \le t f(E(x)) + (1-t) f(E(y)), \quad \text{ for all } x, y \in S \text{ and } t \in [0,1],$$

where $\gamma_{E(x),E(y)}$ is the unique geodesic.

Definition 3.3. Let S be an open subset of M that is the geodesic (α, E)-invex set with respect to $\eta : M \times M \to TM$ and $\alpha : M \times M \to \mathbb{R} - \{0\}$. A differentiable function $f : S \to \mathbb{R}$ is said to be geodesic (α, E)-invex if

$$f(E(x)) - f(E(y)) \geq df_{E(y)}(\alpha(E(x), E(y))\eta(E(x), E(y))), \quad \text{for all } x, y \in S.$$

Example 3.4. Let M be a Cartan–Hadamard manifold and let $x_0, y_0 \in M$ such that $x_0 \neq y_0$. Consider two open balls $B(x_0, r_1)$ and $B(y_0, r_2)$ of radius r_1 and r_2, respectively, such that $B(x_0, r_1) \cap B(y_0, r_2) = \phi$ for some $0 < r_1, r_2 < \frac{1}{2}d(x_0, y_0)$. Now take

$$A = B(x_0, r_1) \cup B(y_0, r_2).$$

Then it is obvious that A is not geodesic convex. Now define the functions $E : M \to M$ and $\eta : M \times M \to M$ such that

$$\eta(x, y) = \begin{cases} \exp_y^{-1} x, & x, y \in B(x_0, r_1) \text{ or } x, y \in B(y_0, r_2), \\ 0_y & \text{otherwise.} \end{cases}$$

We have

$$E(x) = \{y \in \gamma_{x,y} : d(x_0, y) = r_1/2\} \text{ for all } x \in A,$$

where $\gamma_{x,y}$ denotes the geodesic joining x and y whose existence is ensured in Cartan–Hadamard manifolds.

For every $x, y \in M$, consider a bifunction $\alpha : M \times M \to \mathbb{R} - \{0\}$ and $\gamma_{E(x),E(y)} : [0, 1] \to M$ defined by

$$\gamma_{E(x),E(y)}(t) = \exp_{E(y)}(t\alpha(E(x), E(y))\eta(E(x), E(y))), \text{ for all } t \in [0, 1].$$

Then,

$$\gamma_{E(x),E(y)}(0) = E(y), \ \gamma'_{E(x),E(y)}(0) = \alpha(E(x), E(y))\eta(E(x), E(y)).$$

Now we will show that A is a geodesic (α, E)-invex set. Let $x, y \in B(x_0, r_1)$. Therefore, using the convexity of $B(x_0, r_1)$, we get

$$\gamma_{E(x),E(y)}(t) = \exp_{E(y)}(t\alpha(E(x), E(y)) \exp_{E(y)}^{-1} E(x)) \in B(x_0, r_1) \subset A,$$

for all $t \in [0, 1]$. Similarly, for $x, y \in B(y_0, r_2)$, we obtain

$$\gamma_{E(x),E(y)}(t) \in A, \text{ for all } t \in [0, 1].$$

If $x \in B(x_0, r_1)$ and $y \in B(y_0, r_2)$ or $x \in B(y_0, r_2)$ and $y \in B(x_0, r_1)$, then, we get

$$\gamma_{E(x),E(y)}(t) = \exp_{E(y)}(t\alpha(E(x), E(y))0_{E(y)}) = E(y) \in A, \text{ for all } t \in [0, 1].$$

Hence, A is a geodesic (α, E)-invex set.

4. Main results

Proposition 4.1. *Suppose that S is an open subset of M that is a geodesic (α, E)-invex set with respect to $\eta : M \times M \to TM$ and $\alpha : M \times M \to \mathbb{R} - \{0\}$. Suppose that $f : S \to \mathbb{R}$ is a geodesic (α, E)-preinvex function. Then every lower section of f defined by*

$$S_\lambda := \{x \in S : f(x) \le \lambda\}, \quad \lambda \in \mathbb{R},$$

is a geodesic (α, E)-invex set with respect to η and α.

Proof. Let $x, y \in S_\lambda$. Since S is a geodesic (α, E)-invex set with respect to η and α, there exists exactly one geodesic $\gamma_{E(x),E(y)} : [0,1] \to M$ such that

$$\gamma_{E(x),E(y)}(0) = E(y),\ \gamma'_{E(x),E(y)}(0) = \alpha(E(x), E(y))\eta(E(x), E(y)),\ \gamma_{E(x),E(y)}(t) \in S,$$

for all $t \in [0,1]$. Now by the geodesic (α, E)-preinvexity of f we have

$$\begin{aligned} f(\gamma_{E(x),E(y)}(t)) &\le tf(E(x)) + (1-t)f(E(y)) \\ &\le t\lambda + (1-t)\lambda = \lambda. \end{aligned}$$

Hence $\gamma_{E(x),E(y)}(t) \in S_\lambda$, for all $t \in [0,1]$. Therefore, S_λ is a geodesic (α, E)-invex set with respect to η and α. □

Theorem 4.2. *Let $S \subset M$ be a geodesic (α, E)-invex set with respect to η and α. If a function $f : S \to \mathbb{R}$ is differentiable and geodesic (α, E)-preinvex on S, then f is a geodesic (α, E)-invex function on S.*

Proof. Since S is a geodesic (α, E)-invex set with respect to η and α, there exists exactly one geodesic $\gamma_{E(x),E(y)} : [0,1] \to M$ such that

$$\gamma_{E(x),E(y)}(0) = E(y),\ \gamma'_{E(x),E(y)}(0) = \alpha(E(x), E(y))\eta(E(x), E(y)),\ \gamma_{E(x),E(y)}(t) \in S,$$

for all $t \in [0,1]$. Moreover, f is a geodesic (α, E)-preinvex function. Therefore, we have

$$f(\gamma_{E(x),E(y)}(t)) \le tf(E(x)) + (1-t)f(E(y)),$$

that is,

$$f(\gamma_{E(x),E(y)}(t)) - f(E(y)) \le t(f(E(x)) - f(E(y))).$$

On dividing by t, we get

$$\frac{1}{t}[f(\gamma_{E(x),E(y)}(t)) - f(E(y))] \le (f(E(x)) - f(E(y))).$$

Now taking limit as $t \to 0$ implies

$$df_{\gamma_{E(x),E(y)}(0)}(\gamma'_{E(x),E(y)}(0)) \le f(E(x)) - f(E(y)).$$

Therefore,

$$df_{E(y)}(\alpha(E(x), E(y))\eta(E(x), E(y))) \le f(E(x)) - f(E(y)),$$

which shows that f is a geodesic (α, E)-invex function. □

Definition 4.3 (Property (P))**.** Let $\gamma : [0,1] \to M$ be a curve on M such that $\gamma_{E(x),E(y)}(0) = E(y)$ and $\gamma_{E(x),E(y)}(1) = E(x)$. Then $\gamma_{E(x),E(y)}$ is said to possess the Property (P) with respect to $x, y \in M$ if

$$\gamma'_{E(x),E(y)}(s)(t-s) = \alpha(\gamma_{E(x),E(y)}(t), \gamma_{E(x),E(y)}(s))\eta(\gamma_{E(x),E(y)}(t), \gamma_{E(x),E(y)}(s)),$$

for all $s, t \in [0,1]$.

Remark 4.4. If $\alpha = 1$, then the above property reduces to the property defined by Kumari and Jayswal [7]. Agarwal et. al. [1] defined the above property when E is the identity map. If $\alpha = 1$ and E is the identity map, then the above property was defined by Pini [10].

Let $\gamma_{E(x),E(y)}$ possess the Property (P) with respect to $x, y \in M$. Then

$$\begin{aligned}
&\alpha(E(x), E(y))\eta(E(x), E(y))\\
&= \alpha(\gamma_{E(x),E(y)}(1), \gamma_{E(x),E(y)}(0))\eta(\gamma_{E(x),E(y)}(1), \gamma_{E(x),E(y)}(0))\\
&= \gamma'_{E(x),E(y)}(0).
\end{aligned}$$

In this case where $\gamma_{E(x),E(y)}$ is a geodesic, then

$$\begin{aligned}
&\alpha(\gamma_{E(x),E(y)}(0), \gamma_{E(x),E(y)}(s))\eta(\gamma_{E(x),E(y)}(0), \gamma_{E(x),E(y)}(s))\\
&= -s\gamma'_{E(x),E(y)}(s)\\
&= -sP^{s}_{0,\gamma_{E(x),E(y)}}[\gamma'_{E(x),E(y)}(0)]\\
&= -sP^{s}_{0,\gamma_{E(x),E(y)}}[\alpha(E(x), E(y))\eta(E(x), E(y))].
\end{aligned}$$

or,

$$\begin{aligned}
&P^{0}_{s,\gamma_{E(x),E(y)}}[\alpha(E(y), \gamma_{E(x),E(y)}(s))\eta(E(y), \gamma_{E(x),E(y)}(s))]\\
&= -s\alpha(E(x), E(y))\eta(E(x), E(y)).
\end{aligned}$$

Also

$$\begin{aligned}
&\alpha(\gamma_{E(x),E(y)}(1), \gamma_{E(x),E(y)}(s))\eta(\gamma_{E(x),E(y)}(1), \gamma_{E(x),E(y)}(s))\\
&= (1-s)\gamma'_{E(x),E(y)}(s)\\
&= (1-s)P^{s}_{0,\gamma_{E(x),E(y)}}[\gamma'_{E(x),E(y)}(0)]\\
&= (1-s)P^{s}_{0,\gamma_{E(x),E(y)}}[\alpha(E(x), E(y))\eta(E(x), E(y))].
\end{aligned}$$

or,

$$\begin{aligned}
&P^{0}_{s,\gamma_{E(x),E(y)}}[\alpha(E(x), \gamma_{E(x),E(y)}(s))\eta(E(x), \gamma_{E(x),E(y)}(s))]\\
&= (1-s)\alpha(E(x), E(y))\eta(E(x), E(y)).
\end{aligned}$$

Therefore,

$$\text{(C)}\begin{cases}
P^{0}_{s,\gamma_{E(x),E(y)}}[\alpha(E(y), \gamma_{E(x),E(y)}(s))\eta(E(y), \gamma_{E(x),E(y)}(s))]\\
= -s\alpha(E(x), E(y))\eta(E(x), E(y))\\
P^{0}_{s,\gamma_{E(x),E(y)}}[\alpha(E(x), \gamma_{E(x),E(y)}(s))\eta(E(x), \gamma_{E(x),E(y)}(s))]\\
= (1-s)\alpha(E(x), E(y))\eta(E(x), E(y)),
\end{cases}$$

for all $s \in [0,1]$. We call the above two conditions as Condition (C).

Theorem 4.5. *Let S be an open subset of M that is geodesic (α, E)-invex set with respect to $\eta : M \times M \to TM$ and $\alpha : M \times M \to \mathbb{R} - \{0\}$. Let $f : S \to \mathbb{R}$ be a differentiable function and satisfy Condition* (C). *Then f is a geodesic (α, E)-preinvex on S if f is (α, E)-invex on S.*

Proof. Since S is a geodesic (α, E)-invex set with respect to η and α, there exists a unique geodesic $\gamma_{E(x),E(y)} : [0, 1] \to M$ such that

$$\gamma_{E(x),E(y)}(0) = E(y),\ \gamma'_{E(x),E(y)}(0) = \alpha(E(x), E(y))\eta(E(x), E(y)),\ \gamma_{E(x),E(y)}(t) \in S,$$

for all $t \in [0, 1]$. Now, fix $t \in [0, 1]$ and set $q = \gamma_{E(x),E(y)}(t)$. Then using (α, E)-invexity of f on S, we have

$$f(E(x)) - f(E(q)) \geq df_{E(q)}(\alpha(E(x), E(q))\eta(E(x), E(q))) \tag{4.1}$$

and

$$f(E(y)) - f(E(q)) \geq df_{E(q)}(\alpha(E(y), E(q))\eta(E(y), E(q))). \tag{4.2}$$

On multiplying (4.1) by t and (4.2) by $(1 - t)$, respectively, and then adding we get

$$\begin{aligned} tf(E(x)) + (1 - t)f(E(y)) - f(E(q)) \geq & df_{E(q)}[t\alpha(E(x), E(q))\eta(E(x), E(q)) \\ & + (1 - t)\alpha(E(y), E(q))\eta(E(y), E(q))]. \end{aligned} \tag{4.3}$$

By Condition (C), we have

$$\begin{aligned} & t\alpha(E(x), E(q))\eta(E(x), E(q)) + (1 - t)\alpha(E(y), E(q))\eta(E(y), E(q)) \\ & = t(1 - t)P^t_{0,\gamma}[\alpha(E(x), E(y))\eta(E(x), E(y))] \\ & \quad -(1 - t)tP^t_{0,\gamma}[\alpha(E(x), E(y))\eta(E(x), E(y))] \\ & = 0. \end{aligned}$$

Now combining the above relation with inequality (4.3), we get

$$tf(E(x)) + (1 - t)f(E(y)) - f(E(q)) \geq 0,$$

which implies that

$$f(\gamma_{E(x),E(y)}(t)) \leq tf(E(x)) + (1 - t)f(E(y)).$$

Therefore, the function f is geodesic (α, E)-preinvex on S. □

Theorem 4.6. *Let $S \subset M$ be a geodesic (α, E)-invex set with respect to $\eta : M \times M \to TM$ and $\alpha : M \times M \to R$. Suppose that a function $f : S \to R$ is a geodesic (α, E)-preinvex and $\phi : I \to R$ is an increasing geodesic preinvex such that the range f is a subset I, where I is any interval. Then the composite function $\phi \circ f$ is a geodesic (α, E)-preinvex on S.*

Proof. Since S is a geodesic (α, E)-invex set with respect to η and α, there exists a unique geodesic $\gamma_{E(x),E(y)} : [0, 1] \to M$ such that

$$\gamma_{E(x),E(y)}(0) = E(y),\ \gamma'_{E(x),E(y)}(0) = \alpha(E(x), E(y))\eta(E(x), E(y)),\ \gamma_{E(x),E(y)}(t) \in S,$$

for all $t \in [0, 1]$. By the definition of geodesic (α, E)-preinvex function, we have

$$f(\gamma_{E(x),E(y)}(t)) \leq tf(E(x)) + (1 - t)f(E(y)).$$

As ϕ is an increasing geodesic preinvex function, we get

$$\phi \circ f(\gamma_{E(x),E(y)}(t)) \leq \phi(tf(E(x)) + (1-t)f(E(y)))$$

$$\begin{aligned}(\phi \circ f)(\gamma_{E(x),E(y)}(t)) &\leq t\phi(f(E(x))) + (1-t)\phi(f(E(y))) \\ &= t(\phi \circ f)(E(x)) + (1-t)(\phi \circ f)(E(y)).\end{aligned}$$

Therefore, $\phi \circ f$ is a geodesic (α, E)-preinvex function on S. □

Theorem 4.7. *Let $S \subset M$ be a geodesic (α, E)-invex set with respect to $\eta : M \times M \to TM$ and $\alpha : M \times M \to R$. If $f_j : S \to R$, $j \in J$ are geodesic (α, E)-preinvex functions on S such that $\sup_{j \in J} f_j$ exists in $\mathbb{R}$, then the function $f : S \to R$ defined by*

$$f(x) = \sup_{j \in J} f_j(x), \text{ for } x \in S,$$

is a geodesic (α, E)-preinvex function on S.

Proof. Since S is a geodesic (α, E)-invex set with respect to η and α, there exists a unique geodesic $\gamma_{E(x),E(y)} : [0,1] \to M$ such that

$$\gamma_{E(x),E(y)}(0) = E(y),\ \gamma'_{E(x),E(y)}(0) = \alpha(E(x), E(y))\eta(E(x), E(y)),\ \gamma_{E(x),E(y)}(t) \in S,$$

for all $t \in [0,1]$. From the geodesic (α, E)-preinvexity of f_j, $j \in J$, we have

$$f_j(\gamma_{E(x),E(y)}(t)) \leq tf_j(E(x)) + (1-t)f_j(E(y)).$$

Then,

$$\begin{aligned}\sup_{j \in J} f_j(\gamma_{E(x),E(y)}(t)) &\leq \sup_{j \in J} \Big(tf_j(E(x)) + (1-t)f_j(E(y))\Big) \\ &\leq t\Big(\sup_{j \in J} f_j(E(x))\Big) + (1-t)\Big(\sup_{j \in J} f_j(E(y))\Big) \\ &= tf(E(x)) + (1-t)f(E(y)).\end{aligned}$$

Therefore, we get

$$f(\gamma_{E(x),E(y)}(t)) \leq tf(E(x)) + (1-t)f(E(y)).$$

This proves the (α, E)-preinvexity of f. □

Theorem 4.8. *Let S be a geodesic (α, E)-invex set with respect to $\eta : M \times M \to TM$ and $\alpha : M \times M \to \mathbb{R}$. Suppose that $F : S \times S \to \mathbb{R}$ is continuous geodesic (α, E)-preinvex, that is, F is geodesic (α, E)-preinvex with respect to each variable. Then the function $f : S \to \mathbb{R}$ defined by*

$$f(p) = \inf_{q \in S} F(p, q),$$

is a geodesic (α, E)-preinvex function on K.

Proof. Suppose that $\epsilon > 0$ is an arbitrary small number and that $p_0, p_1 \in S$. Since S is a geodesic (α, E)-invex set with respect to η and α, there exists a unique geodesic $\gamma_{E(p_0),E(p_1)} : [0,1] \to M$ such that, for all $t \in [0,1]$,

$$\gamma_{E(p_0),E(p_1)}(0) = E(p_1),\ \gamma'_{E(p_0),E(p_1)}(0) = \alpha(E(p_0), E(p_1))\eta(E(p_0), E(p_1)),$$

and $\gamma_{E(p_0),E(p_1)}(t) \in S$. Now from the definition of f, there exist $q_0, q_1 \in S$ such that

$$F(p_1, q_1) < f(p_1) + \epsilon, \ \ F(p_0, q_0) < f(p_0) + \epsilon.$$

By the geodesic of (α, E)-set with respect to η and α, there exists exactly one geodesic $\lambda_{E(q_0),E(q_1)} : [0, 1] \to M$ such that, for all $s \in [0, 1]$,

$$\lambda_{E(q_0),E(q_1)}(0) = E(q_1), \ \ \lambda'_{E(q_0),E(q_1)}(0) = \alpha(E(q_0), E(q_1))\eta(E(q_0), E(q_1)),$$

and $\lambda_{E(q_0),E(q_1)}(s) \in S$. Hence, the curve $\Gamma = (\gamma_{E(p_0),E(p_1)}, \lambda_{E(q_0),E(q_1)}) : [0, 1] \to M \times M$ is a geodesic in $S \times S$, with

$$\Gamma(0) = (E(p_1), E(q_1) \text{ and}$$
$$\Gamma'(0) = (\alpha(E(p_0), E(p_1))\eta(E(p_0), E(p_1)), \alpha(E(q_0), E(q_1))\eta(E(q_0), E(q_1))).$$

Since Γ is a geodesic in $S \times S$,

$$\Gamma(s) = (\gamma_{E(p_0),E(p_1)}(s), \lambda_{E(q_0),E(q_1)}(s)) \in S \times S \ \text{ for all } s \in [0, 1].$$

Now from the definition of f and the geodesic (α, E)-preinvexity of F, we get

$$\begin{aligned} f(\gamma_{E(p_0),E(p_1)}(s)) &= \inf_{q\in S} F(\gamma_{E(p_0),E(p_1)}(s), E(q)) \\ &\leq F(\gamma_{E(p_0),E(p_1)}(s), \lambda_{E(q_0),E(q_1)}(s)) \\ &\leq sF(E(p_0), E(q_0)) + (1 - s)F(E(p_1), E(q_1)) \\ &\leq s(f(E(p_0)) + \epsilon) + (1 - s)(f(E(p_1)) + \epsilon) \\ &= sf(E(p_0)) + (1 - s)f(E(p_1)) + \epsilon. \end{aligned}$$

Since ϵ is an arbitrary number, therefore

$$f(\gamma_{E(p_0),E(p_1)}(s)) \leq sf(E(p_0)) + (1 - s)f(E(p_1)).$$

□

5. Conclusion

In the present paper, we introduced the concept of geodesic (α, E)-invex set in a complete Riemannian manifold. We also developed the notion of geodesic (α, E)-preinvex functions and geodesic (α, E)-invex functions and discussed some properties of the functions. As a future work, the concept of geodesic (α, E)-preinvex functions can be generalized such as strongly geodeisc (α, E)-preinvex functions on Riemnnian manifolds.

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