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ISBELL CONVEXITY IN FUZZY QUASI-METRIC SPACES

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Abstract. We introduce the concept of Isbell convexity in fuzzy quasi-metric spaces, which we call fuzzy Isbell convexity. This idea extends Isbell convexity (or q-hyperconvexity) in quasi-metric spaces to fuzzy quasi-metric spaces. We show that fuzzy Isbell convexity is preserved by certain F-bounded subsets and the space of nonnegative function pairs of the fuzzy quasi-metric space.

1. Introduction

Fuzzy metric spaces were introduced by Kramosil and Michalek [6]. This concept is related to the class of probabilistic metric spaces (or generalized Menger spaces). George and Veeramani [2, 3] studied a stronger form of fuzziness. After that, some authors examined this concept for fuzzy metric spaces by using the definition of fuzzy metric introduced by George and Veeremani. Due to many applications of quasi-metric spaces in theoretical computer science, approximation theory, and topological algebra, different researchers have studied the concept of a fuzzy quasi-metric space. For example, Gregori and Romaguera [4] introduced two definitions of fuzzy quasi-metric spaces that generalize the corresponding notions of fuzzy metric spaces by Kramosil and Michalek and by George and Veeraamani to the framework of quasi-metric spaces, and several properties were obtained.

The notion of a hyperconvex metric space is due to Aronszajn and Panichpakdi [1]. In 1979, Sine [11] and Soardi [12] proved, independently, that the fixed point property for nonexpansive mappings holds in bounded hyperconvex spaces.

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Since then, the theory of hyperconvex metric spaces has been widely studied and many results have been obtained. Furthermore, a number of results in hyperconvex metric spaces have found applications in many fields of mathematics like geometry, topology, and operator theory. These applications are not only limited to mathematics, but also to other areas such as the study of phylogenetic trees in biology and medicine. They have been motivations of extending results on hyperconvexity from the metric setting to the quasi-metric setting (see [5, 7–10]) and, is still, a motivation of extending the results from the quasi-metric setting to the fuzzy quasi-metric setting. Also, since hyperconvexity plays an important role in the study of fixed point and best approximation theorems in metric and quasi-metric spaces, it is our hope that this article forms a basis of such studies in fuzzy quasi-metric spaces. Therefore, it only seems natural first to establish the theory of hyperconvexity in fuzzy quasi-metric spaces.

In this article, we start by investigating the concept of Isbell convexity in fuzzy quasi-metric spaces, which we call fuzzy Isbell convexity. We are aware that this idea was studied in fuzzy metric spaces by Yiǵit and Efe [13], and they called it fuzzy hyperconvexity. We show that the set of real numbers equipped with the standard fuzzy quasi-metric is fuzzy Isbell convex (see Theorem 2.12). However, the set of real numbers equipped with the standard fuzzy metric is not fuzzy Isbell convex (see Example 2.13), but it is fuzzy hyperconvex in the sense of [13]. It should be pointed out that the concept of Isbell convexity in quasi-pseudometric spaces has been extensively studied in [5, 7, 9]. Thus we will adapt some of the results from these articles in our study.

We start by introducing the concept of fuzzy metric convexity in fuzzy quasi-pseudo-metric spaces; thereafter, we introduce the concept of fuzzy Isbell hypercompleteness. We show that a fuzzy quasi-pseudometric space is fuzzy Isbell convex if and only if it is fuzzy metrically convex and fuzzy Isbell hypercomplete. Thereafter, we introduce the concept of fuzzy admissible subsets and show that every fuzzy admissible subset of a fuzzy Isbell convex quasi-metric space is fuzzy Isbell convex. Finally, we introduce the concept of a space of nonnegative function pairs of a fuzzy quasi-metric space and show that this space is fuzzy Isbell convex.

2. Isbell-convexity in fuzzy quasi-metric spaces

In this section, we introduce the concept of Isbell convexity in fuzzy quasi-metric spaces, which we call fuzzy Isbell convexity. For more details on fuzzy quasi-metric spaces, we refer the reader to [4]. We begin this section by introducing two important concepts; fuzzy metric convexity and fuzzy Isbell hypercompleteness and show how they are related to our concept of Isbell convexity in fuzzy quasi-metric spaces.

In order to understand the definition of fuzzy metric convexity in fuzzy quasi-pseudometric spaces, we need the following lemma.

Lemma 2.1 (Compare [13, Lemma 1]). *Let $(X, M, *)$ be a fuzzy quasi-pseudometric space, let $x, y \in X$, let $r_1, s_2 \in (0, 1)$, and let $t_1, t_2 \in (0, \infty)$. If $C_M(x, r_1, t_1) \cap$*

$C_{M^{-1}}(y, s_2, t_2) \neq \emptyset$, then $M(x, y, t_1 + t_2) \geq (1 - r_1) * (1 - s_2)$ for any $x, y \in X$ and each pair $r_1, t_1 > 0$ and $s_2, t_2 > 0$.

Proof. Let $C_M(x, r_1, t_1) \cap C_{M^{-1}}(y, s_2, t_2) \neq \emptyset$. Then there exists $z \in X$ such that $z \in C_M(x, r_1, t_1) \cap C_{M^{-1}}(y, s_2, t_2)$. This implies $z \in C_M(x, r_1, t_1)$ and $z \in C_{M^{-1}}(y, s_2, t_2)$. Therefore, $M(x, z, t_1) \geq 1 - r_1$ and $M^{-1}(y, z, t_2) = M(z, y, t_2) \geq (1 - s_2)$. Thus $M(x, z, t_1) * M(z, y, t_2) \geq (1 - r_1) * (1 - s_2)$. This implies that

$$M(x, y, t_1 + t_2) \geq M(x, z, t_1) * M(z, y, t_2) \geq (1 - r_1) * (1 - s_2).$$

□

Definition 2.2 (Compare [13, Definition 11]). Let $(X, M, *)$ be a fuzzy quasi-pseudometric space. We say that $(X, M, *)$ is fuzzy-metrically convex if for any points $x, y \in X$ and for each pair $r_1, t_1 > 0$ and $s_2, t_2 > 0$ ($r_1, s_2 \in (0, 1)$ and $t_1, t_2 \in (0, \infty)$), such that

$$M(x, y, t_1 + t_2) \geq (1 - r_1) * (1 - s_2),$$

there exists $z \in X$ such that $M(x, z, t_1) \geq (1 - r_1)$ and $M^{-1}(y, z, t_2) = M(z, y, t_2) \geq (1 - s_2)$ or equivalently

$$z \in C_M(x, r_1, t_1) \cap C_{M^{-1}}(y, s_2, t_2).$$

Example 2.3 (Compare [13, Example 5]). Let (X, d) be a metrically convex T_0-quasi-metric space and let $a * b = a \cdot b$ for all $a, b \in [0, 1]$ be a continuous t-norm. Let M be the fuzzy set on $X \times X \times (0, \infty)$ defined as follows:

$$M(x, y, t) = e^{\frac{-d(x,y)}{t}}.$$

Then one can easily check that $(X, M, *)$ is a fuzzy quasi-metric space. We show that $(X, M, *)$ is fuzzy metrically convex.

Since (X, d) is metrically convex, then for any $x, y \in X$ and $\alpha, \beta \in (0, \infty)$ such that $d(x, y) \leq \alpha + \beta$, there exists $z \in X$ such that $d(x, z) \leq \alpha$ and $d(z, y) \leq \beta$. Take $\alpha = -t_1 \ln(1 - r_1)$ and $\beta = -t_2 \ln(1 - s_2)$. Then

$$\begin{aligned} M(x, y, t_1 + t_2) &\geq M(x, z, t_1) * M(z, y, t_2) \\ &\geq e^{\frac{-d(x,z)}{t_1}} * e^{\frac{-d(z,y)}{t_2}} \\ &= e^{\ln(1-r_1)} * e^{\ln(1-s_2)} \\ &= (1 - r_1) * (1 - s_2) \end{aligned}$$

is satisfied and also $r_1, s_2 \in (0, 1)$. By the metric convexity of (X, d), we have

$$\begin{aligned} & d(x, z) \leq -t_1 \ln(1 - r_1) \quad \text{and} \quad d(z, y) \leq -t_2 \ln(1 - s_2) \\ \Longrightarrow\ & \frac{-d(x, z)}{t_1} \geq \ln(1 - r_1) \quad \text{and} \quad \frac{-d(z, y)}{t_2} \geq \ln(1 - s_2) \\ \Longrightarrow\ & e^{\frac{-d(x,z)}{t_1}} \geq e^{\ln(1-r_1)} \quad \text{and} \quad e^{\frac{-d(z,y)}{t_2}} \geq e^{\ln(1-s_2)} \\ \Longrightarrow\ & M(x, z, t_1) \geq (1 - r_1) \quad \text{and} \quad M(z, y, t_2) \geq (1 - s_2). \end{aligned}$$

This implies that

$$z \in C_M(x, r_1, t_1) \cap C_{M^{-1}}(y, s_2, t_2).$$

Therefore, $(X, M, *)$ is fuzzy metrically convex.

Proposition 2.4. *Let $(X, M, *)$ be a fuzzy quasi-metric space. If $(X, M, *)$ is fuzzy metrically convex, then $(X, M^{-1}, *)$ is fuzzy metrically convex.*

Proof. Assume that $(X, M, *)$ is fuzzy metrically convex. Let $x, y \in X$, $r_1, s_2 \in (0, 1)$, and $t_1, t_2 \in (0, \infty)$ be such that $M^{-1}(x, y, t_1 + t_2) \geq (1 - r_1) * (1 - s_2)$. By the fuzzy metric convexity of $(X, M, *)$ and $M(y, x, t_1+t_2) = M^{-1}(x, y, t_1+t_2) \geq (1 - r_1) * (1 - s_2)$, we have

$$C_M(y, s_2, t_2) \cap C_{M^{-1}}(x, r_1, t_1) \neq \emptyset.$$

Hence $(X, M^{-1}, *)$ is fuzzy metrically convex. □

Definition 2.5. Let $(X, M, *)$ be a fuzzy quasi-pseudometric space. A family of double balls $(C_M(x_i, r_i, t_i), C_{M^{-1}}(x_i, s_i, t_i))_{i \in I}$, where $r_i, s_i \in (0, 1)$ and $t_i \in (0, \infty)$ whenever $i \in I$, is said to have a mixed binary intersection property if for all indices $i, j \in I$,

$$C_M(x_i, r_i, t_i) \cap C_{M^{-1}}(x_j, s_j, t_j) \neq \emptyset.$$

Definition 2.6. Let $(X, M, *)$ be a fuzzy quasi-pseudometric space. Then $(X, M, *)$ is said to be fuzzy-Isbell hypercomplete if for every family of double balls $(C_M(x_i, r_i, t_i), C_{M^{-1}}(x_i, s_i, t_i))_{i \in I}$, where $r_i, s_i \in (0, 1)$ and $t_i \in (0, \infty)$ whenever $i \in I$, with mixed binary intersection property has

$$\bigcap_{i \in I} C_M(x_i, r_i, t_i) \cap C_{M^{-1}}(x_i, s_i, t_i) \neq \emptyset.$$

Proposition 2.7. *Let $(X, M, *)$ be a fuzzy quasi-pseudometric space. If $(X, M, *)$ is fuzzy Isbell hypercomplete, then $(X, M^{-1}, *)$ and $(X, M^i, *)$ are fuzzy Isbell hypercomplete and fuzzy hypercomplete, respectively.*

Proof. Assume that $(X, M, *)$ is fuzzy-Isbell hypercomplete. Let a family of balls $(C_M(x_i, r_i, t_i), C_{M^{-1}}(x_i, s_i, t_i))_{i \in I}$ be given with mixed binary intersection property. Since $(X, M, *)$ is fuzzy-Isbell hypercomplete,
$(C_M(x_i, s_i, t_i), C_{M^{-1}}(x_i, r_i, t_i))_{i \in I}$ has

$$\bigcap_{i \in I} C_M(x_i, s_i, t_i) \cap C_{M^{-1}}(x_i, r_i, t_i) \neq \emptyset.$$

Then $(X, M^{-1}, *)$ is fuzzy Isbell hypercomplete.

Also, suppose that $(X, M, *)$ is fuzzy Isbell hypecomplete. Let the family $(C_{M^i}(x_i, r_i, t_i))_{i \in I}$ have the binary intersection property. Then
$(C_M(x_i, r_i, t_i), C_{M^{-1}}(x_i, r_i, t_i))_{i \in I}$ has the mixed binary intersection property. Thus,

$$\begin{aligned} \emptyset &\neq \bigcap_{i \in I} C_M(x_i, r_i, t_i) \cap C_{M^{-1}}(x_i, r_i, t_i) \\ &= \bigcap_{i \in I} C_{M^i}(x_i, r_i, t_i). \end{aligned}$$

Therefore, $(X, M^i, *)$ is fuzzy hypercomplete.

□

Definition 2.8 (Compare [13, Definition 13]). A fuzzy quasi-pseudometric space $(X, M, *)$ is said to be fuzzy Isbell convex if for any family of points $(x_i)_{i\in I}$ in X, families $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ of points in $(0,1)$, and the family of points $(t_i)_{i\in I}$ in $(0,\infty)$ satisfying

$$M(x_i, x_j, t_i + t_j) \geq (1 - r_i) * (1 - s_j),$$

whenever $i, j \in I$, we have

$$\bigcap_{i\in I} C_M(x_i, r_i, t_i) \cap C_{M^{-1}}(x_i, s_i, t_i) \neq \emptyset.$$

Proposition 2.9. *Let $(X, M, *)$ be a fuzzy quasi-pseudometric space. If $(X, M, *)$ is fuzzy-Isbell convex, then $(X, M^{-1}, *)$ is fuzzy Isbell convex and $(X, M^i, *)$ is fuzzy hyperconvex.*

Proof. Assume that $(X, M, *)$ is fuzzy-Isbell convex. Let $(x_i)_{i\in I}$ be a family of points in X, $(r_i)_{i\in I}$, let $(s_i)_{i\in I}$ be families of points in $(0,1)$, and let $(t_i)_{i\in I}$ be families of points in $(0,\infty)$ such that

$$M^{-1}(x_i, x_j, t_i + t_j) \geq (1 - r_i) * (1 - s_j),$$

whenever $i, j \in I$. Find $x \in C_M(x_i, r_i, t_i) \cap C_{M^{-1}}(x_i, s_i, t_i)$ for any $i \in I$. By the fuzzy-Isbell convexity of $(X, M, *)$, $M^{-1}(x_i, x_j, t_i + t_j) = M(x_j, x_i, t_i + t_j) \geq (1 - r_i) * (1 - s_j)$ whenever $i, j \in I$ implies that there exists $x_0 \in C_M(x_i, s_i, t_i) \cap C_{M^{-1}}(x_i, r_i, t_i)$. Take $x = x_0$, and then $(X, M^{-1}, *)$ is Isbell convex.

Also, let us assume that $(X, M, *)$ is fuzzy Isbell convex. Let $(x_i)_{i\in I}$ be a family of points in X, $(r_i)_{i\in I}$, let $(s_i)_{i\in I}$ be a family of points in $(0,1)$, and let $(t_i)_{i\in I}$ be a family of points in $(0,\infty)$ such that

$$M^i(x_i, x_j, t_i + t_j) \geq (1 - r_i) * (1 - s_j),$$

whenever $i, j \in I$. By the fuzzy Isbell convexity of $(X, M, *)$, we have

$$\begin{aligned} \emptyset &\neq \bigcap_{i\in I} C_M(x_i, r_i, t_i) \cap C_{M^{-1}}(x_i, r_i, t_i) \\ &= \bigcap_{i\in I} C_{M^i}(x_i, r_i, t_i). \end{aligned}$$

It follows that $(X, M^i, *)$ is fuzzy hyperconvex in the sense of [13]. □

Corollary 2.10. *Each fuzzy Isbell-convex quasi-metric space $(X, M, *)$ is bicomplete.*

Proof. By Proposition 2.9, $(X, M^i, *)$ is fuzzy hyperconvex. Since fuzzy hyperconvex spaces are complete by [13, Theorem 4], $(X, M^i, *)$ is bicomplete. □

Lemma 2.11. *Let $(X, M, *)$ be a fuzzy quasi-pseudometric space. Then $(X, M, *)$ is fuzzy Isbell convex if and only if $(X, M, *)$ is fuzzy metrically convex and fuzzy Isbell hypercomplete.*

Proof. Suppose that $(X, M, *)$ is fuzzy Isbell convex. Let $x_1, x_2 \in X$, $r_1, s_2 \in (0,\infty)$, and $t_1, t_2 \in (0,\infty)$ be such that

$$M(x_1, x_2, t_1 + t_2) \geq (1 - r_1) * (1 - s_2).$$

By the fuzzy Isbell convexity of $(X, M, *)$, we have

$$C_M(x_1, r_1, t_1) \cap C_{M^{-1}}(x_2, s_2, t_2) \neq \emptyset.$$

This implies that

$$M(x_1, z, t_1) \geq 1 - r_1 \text{ and } M^{-1}(x_2, z, t_2) = M(z, x_2, t_2) \geq 1 - s_2.$$

So $(X, M, *)$ is fuzzy metrically convex. Let $(C_M(x_i, r_i, t_i), C_{M^{-1}}(x_i, s_i, t_i))_{i\in I}$ have a mixed binary intersection property. Thus

$$M(x_i, x_j, t_i + t_j) \geq (1 - r_i) * (1 - s_j),$$

whenever $i, j \in I$. Then

$$\bigcap_{i\in I} C_M(x_i, r_i, t_i) \cap C_{M^{-1}}(x_i, s_i, t_i) \neq \emptyset,$$

by the fuzzy Isbell convexity of $(X, M, *)$.

Conversely, suppose that $(X, M, *)$ is fuzzy metrically convex and fuzzy Isbell hypercomplete. Suppose that $(x_i)_{i\in I}$ is a family of points in X, that $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ are families of points in $(0, 1)$, and that $(t_i)_{i\in I}$ is a family of points in $(0, \infty)$ such that

$$M(x_i, x_j, t_i + t_j) \geq (1 - r_i) * (1 - s_j),$$

whenever $i, j \in I$. Then $(C_M(x_i, r_i, t_i), C_{M^{-1}}(x_i, s_i, t_i))_{i\in I}$ has a mixed binary intersection property by the fuzzy metric convexity of $(X, M, *)$. Therefore,

$$\bigcap_{i\in I} C_M(x_i, r_i, t_i) \cap C_{M^{-1}}(x_i, s_i, t_i) \neq \emptyset.$$

Hence $(X, M, *)$ is fuzzy Isbell convex. □

Theorem 2.12 (Compare [13, Theorem 3])**.** *Let X be the set of real numbers equipped with the T_0 quasi-metric $d(x, y) = \max\{x - y, 0\}$ for any $x, y \in X$. Let $(X, M, *)$ be a fuzzy quasi-metric space, where $*$ is a continuous t-norm defined by $a * b = a \cdot b$ for all $a, b \in [0, 1]$ and M is a fuzzy set in $X \times X \times (0, \infty)$ defined by*

$$M(x, y, t) = \frac{t}{t + d(x, y)},$$

whenever $x, y \in X$ and $t \in (0, \infty)$. Then $(X, M, *)$ is fuzzy Isbell convex.

Proof. It is known that X equipped with $d(x, y) = \max\{x - y, 0\}$ is Isbell convex (see [9, Example 3.1.1]). Thus for any family $(x_i)_{i\in I}$ of points in X and families of nonnegative real numbers $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ satisfying $d(x_i, x_j) \leq r_i + s_j$ whenever $i, j \in I$, we have

$$\bigcap_{i\in I} C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i) \neq \emptyset.$$

Let

$$R_i = \frac{r_i}{t_i + r_i} \text{ and } S_i = \frac{s_i}{t_i + s_i},$$

where $(t_i)_{i\in I}$ is a family of points in $(0,\infty)$. Then $(R_i)_{i\in I}$ and $(S_i)_{i\in I}$ are families of points in $(0,1)$. Also, using the metric convexity of (X,d), a calculation shows that

$$\begin{aligned} M(x_i,x_j,t_i+t_j) &\geq M(x_i,z,t_i)*M(z,x_j,t_j) \\ &= \left(\frac{t_i}{t_i+d(x_i,z)}\right)*\left(\frac{t_j}{t_j+d(z,x_j)}\right) \\ &\geq \left(\frac{t_i}{t_i+r_i}\right)*\left(\frac{t_j}{t_j+s_j}\right) \\ &= (1-R_i)*(1-S_j), \end{aligned}$$

whenever $i,j\in I$ is satisfied. Also, by the Isbell convexity of (X,d), we have

$$\bigcap_{i\in I} C_d(x_i,r_i)\cap C_{d^{-1}}(x_i,s_i)\neq\emptyset.$$

Then there exists $z\in\bigcap_{i\in I} C_d(x_i,r_i)\cap C_{d^{-1}}(x_i,s_i)$ such that $d(x_i,z)\leq r_i$ and $d(z,x_i)\leq s_i$ for all $i\in I$. Therefore, we have

$$\begin{aligned} & t_i+d(x_i,z)\leq t_i+r_i \quad\text{and}\quad t_i+d(z,x_i)\leq t_i+s_i \\ \Longrightarrow\ & \frac{t_i}{t_i+d(x_i,z)}\geq\frac{t_i}{t_i+r_i} \quad\text{and}\quad \frac{t_i}{t_i+d(z,x_i)}\geq\frac{t_i}{t_i+s_i} \\ \Longrightarrow\ & M(x_i,z,t_i)\geq 1-R_i \quad\text{and}\quad M(z,x_i,t_i)\geq 1-S_i. \end{aligned}$$

This implies that $z\in C_M(x_i,R_i,t_i)\cap C_{M^{-1}}(x_i,S_i,t_i)$ for all $i\in I$ and so

$$\bigcap_{i\in I} C_M(x_i,R_i,t_i)\cap C_{M^{-1}}(x_i,S_i,t_i)\neq\emptyset.$$

Therefore, $(X,M,*)$ is fuzzy Isbell convex. □

Example 2.13. Let $\mathbb{R}$ be equipped with its standard metric $d^s(x,y)=|x-y|$ whenever $x,y\in\mathbb{R}$. Then $(\mathbb{R},d^s)$ is not Isbell convex by [5, Example 2]. Let $(\mathbb{R},M,*)$ be a fuzzy quasi-metric space, where $*$ is a continuous t-norm defined by $a*b=a\cdot b$ for all $a,b\in[0,1]$ and M is a fuzzy set in $\mathbb{R}\times\mathbb{R}\times(0,\infty)$ defined by

$$M(x,y,t)=\frac{t}{t+d(x,y)},$$

whenever $x,y\in\mathbb{R}$, $t\in(0,\infty)$, and $d(x,y)=\max\{x-y,0\}$. Then $(\mathbb{R},M^i,*)$, where

$$M^i(x,y,t)=\frac{t}{t+d^s(x,y)}$$

in which $d^s(x,y)=|x-y|$ whenever $x,y\in\mathbb{R}$, is a fuzzy set in $\mathbb{R}\times\mathbb{R}\times(0,\infty)$; therefore, $(\mathbb{R},M^i,*)$ is a fuzzy metric space. We show that $(\mathbb{R},M^i,*)$ is not fuzzy Isbell convex.

For any $i\in[0,1]$, set $r_i=\frac{1}{4}$ and $s_i=\frac{3}{4}$. If $t=1$, then we define R_i and S_i by

$$R_i=\frac{\frac{1}{4}}{1+\frac{1}{4}}=\frac{1}{5} \text{ and } S_i=\frac{\frac{3}{4}}{1+\frac{3}{4}}=\frac{3}{7}.$$

Thus for any $i, j \in [0,1]$, we have

$$M^i(i,j,1+1) = \frac{2}{2+d^s(i,j)} \geq \frac{2}{3} > \left(1-\frac{1}{5}\right)\cdot\left(1-\frac{3}{7}\right) = \frac{16}{35}.$$

Moreover,

$$\begin{aligned}\bigcap_{i\in[0,1]} C_{M^i}(i,R_i,t_i)\cap C_{M^i}(i,S_i,t_i) &\subseteq C_{M^i}\left(0,\frac{1}{5},1\right)\cap C_{M^i}\left(1,\frac{1}{5},1\right)\\ &= \left[-\frac{1}{4},\frac{1}{4}\right]\cap\left[\frac{3}{4},\frac{5}{4}\right]=\emptyset.\end{aligned}$$

Hence, $(\mathbb{R}, M^i, *)$ is not fuzzy Isbell convex.

Remark 2.14. Note that $(\mathbb{R}, M^i, *)$, where $*$ is a continuous t-norm defined by $a * b = a \cdot b$ for all $a, b \in [0,1]$ and M is a fuzzy set in $\mathbb{R}\times\mathbb{R}\times(0,\infty)$ defined by

$$M^i(x,y,t) = \frac{t}{t+|x-y|},$$

whenever $x, y \in \mathbb{R}$ and $t \in (0,\infty)$, is fuzzy hyperconvex by [13, Theorem 3]. Therefore, $(\mathbb{R}, M, *)$ is an example of a space for which $(\mathbb{R}, M^i, *)$ is fuzzy hyperconvex but not fuzzy Isbell convex.

3. Fuzzy admissible subsets

In this section, we introduce the concept of fuzzy admissible subsets and show that every fuzzy admissible subset of a fuzzy Isbell convex quasi-metric space is fuzzy Isbell convex.

Definition 3.1. Let $(X, M, *)$ be a fuzzy quasi-metric space and let A be a subset of X. Then A is said to be F-bounded if there exist $t > 0$ and $r \in (0,1)$ such that

$$M(x,y,t) > 1-r$$

for all $x, y \in A$.

Let $(X, M, *)$ be a fuzzy quasi-metric space and let $A \subset X$ be F-bounded. Then

$$\begin{aligned}cov(A)_M &= \bigcap\{C_M(x,r,t) : A\subseteq C_M(x,r,t), x\in X, r\in(0,1) \text{ and } t\in(0,\infty)\}, \text{and}\\ cov(A)_{M^{-1}} &= \bigcap\{C_{M^{-1}}(x,s,t) : A\subseteq C_{M^{-1}}(x,s,t), x\in X, s\in(0,1) \text{ and } t\in(0,\infty)\}.\end{aligned}$$

Also, $bicov(A) = cov(A)_M \cap cov(A)_{M^{-1}}$.

Definition 3.2. Let $(X, M, *)$ be a fuzzy metric space. An F-bounded subset D of X is said to be fuzzy admissible if $D = bicov(D)$.

The collection of all fuzzy admissible subsets of a fuzzy metric space $(X, M, *)$ will be denoted by $\mathcal{A}_M(X)$.

Remark 3.3. A subset of X is fuzzy admissible if and only if it can be written as the intersection of a family of sets of the form

$$C_M(x,r,t)\cap C_{M^{-1}}(x,s,t),$$

where $r,s\in(0,1)$, $t\in(0,\infty)$ and $x\in X$. For this reason, the family $\mathcal{A}_M(X)$ is closed under nonempty intersections.

Proposition 3.4. *(Compare [9, Proposition 4.1.1]) Suppose that $(X,M,*)$ is a fuzzy Isbell convex quasi-metric space. Then $D\in\mathcal{A}_M(X)$ is fuzzy Isbell convex.*

Proof. Since $D\in\mathcal{A}_M(x)$, then

$$D=\bigcap_{i\in I}C_M(x_i,r_i,t_i)\cap C_{M^{-1}}(x_i,s_i,t_i)\neq\emptyset,$$

where $x_i\in X$, $r_i,s_i\in(0,1)$, and $t_i\in(0,\infty)$ whenever $i\in I$. Let $(C_M(x_\alpha,r_\alpha,t_\alpha),C_{M^{-1}}(x_\alpha,s_\alpha,t_\alpha))_{\alpha\in A}$, where $x_\alpha\in D$, $r_\alpha,s_\alpha\in(0,1)$, and $t_\alpha\in(0,\infty)$ whenever $\alpha\in A$ and $M(x_\alpha,x_\beta,t_\alpha+t_\beta)\geq(1-r_\alpha)*(1-s_\beta)$ whenever $\alpha,\beta\in A$. Then by the fuzzy Isbell convexity of X, we have

$$\bigcap_{\alpha\in A}C_M(x_\alpha,r_\alpha,t_\alpha)\cap C_{M^{-1}}(x_\alpha,s_\alpha,t_\alpha)\neq\emptyset.$$

Now consider the family of balls $[(C_M(x_\alpha,r_\alpha,t_\alpha))_{\alpha\in A},(C_{M^{-1}}(x_\alpha,s_\alpha,t_\alpha))_{\alpha\in A},(C_M(x_i,r_i,t_i))_{i\in I},(C_{M^{-1}}(x_i,s_i,t_i))_{i\in I}]$. We have for each $\alpha\in A$ and $i\in I$,

$$\begin{aligned}M(x_\alpha,x_i,t_\alpha+t_i)&\geq M(x_\alpha,z,t_\alpha)*M(z,x_i,t_i)\\&\geq(1-r_\alpha)*(1-s_i)\end{aligned}$$

and

$$\begin{aligned}M(x_i,x_\alpha,t_i+t_\alpha)&\geq M(x_i,z,t_i)*M(z,x_\alpha,t_\alpha)\\&\geq(1-r_i)*(1-s_\alpha)\end{aligned}$$

for some $z\in D$. Furthermore, for all $i,j\in I$, $t_i,t_j\in(0,\infty)$, we have

$$\begin{aligned}M(x_i,x_j,t_i+t_j)&\geq M(x_i,x_\alpha,t_i)*M(x_\alpha,x_j,t_j)\\&\geq(1-r_i)*(1-s_j).\end{aligned}$$

It follows from the fuzzy Isbell convexity of X that

$$\begin{aligned}&\left(\bigcap_{\alpha\in A}C_M(x_\alpha,r_\alpha,t_\alpha)\cap C_{M^{-1}}(x_\alpha,s_\alpha,t_\alpha)\right)\cap\left(\bigcap_{i\in I}C_M(x_i,r_i,t_i)\cap C_{M^{-1}}(x_i,s_i,t_i)\right)\\&=\left(\bigcap_{\alpha\in A}C_M(x_\alpha,r_\alpha,t_\alpha)\cap C_{M^{-1}}(x_\alpha,s_\alpha,t_\alpha)\right)\cap D\neq\emptyset.\end{aligned}$$

Hence the subspace D of X is fuzzy Isbell convex. □

Definition 3.5. Let $(X,M,*)$ be a fuzzy quasi-metric space. For a fuzzy quasi-metric subspace A of X, we define for $\epsilon_1,\epsilon_2\in(0,1)$ and $t\in(0,\infty)$ the ϵ_1,ϵ_2-parallel set of A as

$$N_{\epsilon_1,\epsilon_2}(A)=\bigcup_{a\in A}C_M(a,\epsilon_1,t)\cap C_{M^{-1}}(a,\epsilon_2,t).$$

Lemma 3.6 (Compare [9, Lemma 4.2.1])**.** *Let $(X, M, *)$ be a fuzzy Isbell-convex fuzzy convex fuzzy quasi-metric space. Let A be a fuzzy admissible subset of X, that is, $\emptyset \neq \bigcap_{i\in I} C_M(x_i, r_i, t_i) \cap C_{M^{-1}}(x_i, s_i, t_i)$ with $x_i \in X$, $r_i, s_i \in (0,1)$ and $t_i \in (0,\infty)$ whenever $i \in I$. Then for each $\epsilon_1, \epsilon_2 \in (0,1)$ and $t \in (0,\infty)$, we have*

$$N_{\epsilon_1,\epsilon_2}(A) = \bigcap_{i\in I} C_M(x_i, r_i + \epsilon_2, t + t_i) \cap C_{M^{-1}}(x_i, s_i + \epsilon_1, t + t_i).$$

Proof. Suppose $y \in N_{\epsilon_1,\epsilon_2}(A)$. Then

$$M(a, y, t) \geq (1 - \epsilon_2) \text{ and } M(y, a, t) \geq (1 - \epsilon_2)$$

for some $a \in A$ and for every $t \in (0,\infty)$. Therefore

$$\begin{aligned} M(x_i, y, t + t_i) &\geq M(x_i, a, t_i) * M(a, y, t) \\ &\geq (1 - r_i) * (1 - \epsilon_2) \\ &\geq (1 - (r_i + \epsilon_2)) \end{aligned}$$

and

$$\begin{aligned} M(y, x_i, t_i + t) &\geq M(y, a, t) * M(a, x_i, t_i) \\ &\geq (1 - \epsilon_1) * (1 - s_i) \\ &\geq (1 - (s_i + \epsilon_1)). \end{aligned}$$

Then for each $i \in I$, we have $y \in C_M(x_i, r_i + \epsilon_2, t + t_i)$ and $y \in C_{M^{-1}}(x_i, s_i + \epsilon_1, t + t_i)$. So

$$N_{\epsilon_1,\epsilon_2}(A) \subseteq \bigcap_{i\in I} C_M(x_i, r_i + \epsilon_2, t + t_i) \cap C_{M^{-1}}(x_i, s_i + \epsilon_1, t + t_i).$$

Conversely, suppose

$$y \in \bigcap_{i\in I} C_M(x_i, r_i + \epsilon_2, t + t_i) \cap C_{M^{-1}}(x_i, s_i + \epsilon_1, t + t_i).$$

Then

$$M(x_i, y, t + t_i) \geq 1 - (r_i + \epsilon_2) \text{ and } M(y, x_i, t + t_i) \geq 1 - (s_i + \epsilon_1).$$

Since A is nonempty and by the definition of A, we must have for any $i, j \in I$,

$$\begin{aligned} M(x_i, x_j, t_i + t_j) &\geq M(x_i, a, t_i) * M(a, x_j, t_j) \\ &\geq (1 - r_i) * (1 - s_j) \\ &\geq (1 - (r_i + s_j)). \end{aligned}$$

So by the fuzzy Isbell convexity of X, we have

$$\begin{aligned} \emptyset &\neq \left(\bigcap_{i\in I} C_M(x_i, r_i, t_i) \cap C_M(y, \epsilon_1, t)\right) \cap \left(C_{M^{-1}}(x_i, s_i, t_i) \cap C_{M^{-1}}(y, \epsilon_2, t)\right) \\ &= \left(\bigcap_{i\in I} C_M(x_i, r_i, t_i) \cap C_{M^{-1}}(x_i, s_i, t_i)\right) \cap (C_M(y, \epsilon_1, t) \cap C_{M^{-1}}(y, \epsilon_2, t)) \\ &= A \cap C_M(y, \epsilon_1, t) \cap C_{M^{-1}}(y, \epsilon_2, t). \end{aligned}$$

Therefore, $a \in A$ such that

$$M(y,a,t) \geq 1-\epsilon_1 \text{ and } M(a,y,t) \geq 1-\epsilon_2.$$

Hence $y \in N_{\epsilon_1,\epsilon_2}(A)$. □

4. A space of nonnegative function pairs of a fuzzy quasi-metric space $(X, M, *)$

In this section, we introduce the concept of a space of nonnegative function pairs of a fuzzy quasi-metric space and show that this space is fuzzy Isbell convex.

Definition 4.1. Let (X,d) be a quasi-pseudo-metric space and let $\mathcal{FP}(X,d)$ be the set of functions $f=(f_1,f_2)$, where $f_i : X \to [0,\infty), i=1,2$. We define a T_0-quasi-metric D on $\mathcal{FP}(X,d)$ as follows:

$$D(f,g) = \sup_{x\in X}(f_1(x) \dot{-} g_1(x)) \vee \sup_{x\in X}(g_2(x) \dot{-} f_2(x)).$$

Definition 4.2. Let $(X,M,*)$ be a fuzzy quasi-metric space and let $\mathcal{FP}(X,M,*)$ be the set of functions $f=(f_1,f_2)$, where $f_i : X \to [0,\infty)$, $i=1,2$, and $*$ is the binary operation $a*b=a\cdot b$ for any $a,b\in[0,1]$. We define a fuzzy set in $\mathcal{FP}(X,M,*)\times\mathcal{FP}(X,M,*)\times(0,\infty)$ as follows:

$$M(f,g,t)=\frac{t}{t+D(f,g)}.$$

Theorem 4.3. *Let (X,d) be a T_0-quasi-metric space and $\mathcal{FP}(X,d)$. If $\mathcal{FP}(X,d)$ is Isbell convex, then $\mathcal{FP}(X,M,*)$ is a fuzzy Isbell convex quasi-metric space, where $*$ is the binary operation $a*b=a\cdot b$ for any $a,b\in[0,1]$ and M is a fuzzy set on $\mathcal{FP}(X,M,*)\times\mathcal{FP}(X,M,*)\times(0,\infty)$ defined by*

$$M(f,g,t)=\frac{t}{t+D(f,g)}.$$

Proof. Since $\mathcal{FP}(X,d)$ is Isbell convex, then for any family $((f_i)_1,(f_i)_2)_{i\in I}$ of points in $\mathcal{FP}(X,d)$ and families $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ of nonnegative real numbers such that
$D(((f_i)_1,(f_i)_2),((f_j)_1,(f_j)_2)) \leq r_i+s_j$ whenever $i,j\in I$, we have

$$\bigcap_{i\in I} C_M(((f_i)_1,(f_i)_2),r_i)\cap C_{M^{-1}}(((f_i)_1,(f_i)_2),s_i)\neq\emptyset.$$

Let $R_i=\frac{r_i}{t_i+r_i}$ and $S_i=\frac{s_i}{t_i+s_i}$, where $(t_i)_{i\in I}$ is a family of points in $(0,\infty)$. Then $(R_i)_{i\in I}$ and $(S_i)_{i\in I}$ are families of points in $(0,1)$. Using the metric convexity of $\mathcal{FP}(X,d)$, we find that $M(((f_i)_1,(f_i)_2),((f_j)_1,(f_j)_2),t_i+t_j)\geq(1-R_i)*(1-S_j)$ whenever $i,j\in I$, is satisfied. Also, by the Isbell convexity of $\mathcal{FP}(X,d)$, we have

$$(g_1,g_2)\in\bigcap_{i\in I} C_M(((f_i)_1,(f_i)_2),r_i)\cap C_{M^{-1}}(((f_i)_1,(f_i)_2),s_i).$$

Thus $D(((f_i)_1,(f_i)_2),(g_1,g_2)) \leq r_i$ and $D(((g_1,g_2),(f_i)_1,(f_i)_2)) \leq s_i$, whenever $i \in I$. Therefore, we have

$$\begin{aligned} t_i + D(((f_i)_1,(f_i)_2),(g_1,g_2)) &\leq t_i + r_i \\ \Longrightarrow \frac{t_i}{t_i + D(((f_i)_1,(f_i)_2),(g_1,g_2))} &\geq \frac{t_i}{t_i + r_i} \\ \Longrightarrow M(((f_i)_1,(f_i)_2),(g_1,g_2),t_i) &\geq 1 - R_i \end{aligned}$$

and

$$\begin{aligned} t_i + D((g_1,g_2),((f_i)_1,(f_i)_2)) &\leq t_i + s_i \\ \Longrightarrow \frac{t_i}{t_i + D((g_1,g_2),((f_i)_1,(f_i)_2)),(g_1,g_2))} &\geq \frac{t_i}{t_i + s_i} \\ \Longrightarrow M((g_1,g_2),((f_i)_1,(f_i)_2),t_i) &\geq 1 - S_i. \end{aligned}$$

This implies that $(g_1,g_2) \in C_M(((f_i)_1,(f_i)_2),R_i,t_i) \cap C_{M^{-1}}(((f_i)_1,(f_i)_2),S_i,t_i)$ for all $i \in I$ and so $\bigcap_{i\in I} C_M(((f_i)_1,(f_i)_2),R_i,t_i) \cap C_{M^{-1}}(((f_i)_1,(f_i)_2),S_i,t_i) \neq \emptyset$. Therefore, $\mathcal{FP}(X,M,*)$ is fuzzy Isbell convex. □

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