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ALMOST KENMOTSU MANIFOLDS ADMITTING CERTAIN VECTOR FIELDS

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ABSTRACT. The object of the present article is to characterize almost Kenmotsu manifolds admitting holomorphically planar conformal vector (in short, HPCV) fields. It is shown that an almost Kenmotsu manifold M^{2n+1} admitting a nonzero HPCV field V such that V is pointwise collinear with the Reeb vector field ξ, is locally a warped product of an almost Kaehler manifold and an open interval. Furthermore, if an almost Kenmotsu manifold with constant ξ-sectional curvature admits a nonzero HPCV field V, then M^{2n+1} is locally a warped product of an almost Kaehler manifold and an open interval. Moreover, a $(k,\mu)'$-almost Kenmotsu manifold admitting an HPCV field V such that $\phi V \neq 0$ is either locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$ or V is an eigenvector of h'.

1. INTRODUCTION

The study of existence of Killing vector fields in Riemannian manifolds is a very interesting topic as their flow preserves a given metric. Conformal vector fields whose flow preserves a conformal class of metrics are very important in the study of several kind of almost contact metric manifolds.

A smooth vector field V on a Riemannian manifold (M, g) is said to be a conformal vector field if there exist a smooth function f on M such that

$$\pounds_V g = 2fg,$$

where $\pounds_V g$ is the Lie derivative of g with respect to V. The vector field V is called homothetic or Killing according as f is constant or zero, respectively. Moreover,

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V is said to be a closed conformal vector field if the metrically equivalent 1-form of V is closed. If the conformal vector field V is gradient of some smooth function λ, then V is called a gradient conformal vector field. The geometry of conformal vector fields was investigated in [4,5].

A vector field V on a contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is said to be a holomorphically planar conformal vector field if it satisfies

$$\nabla_X V = aX + b\phi X \tag{1.1}$$

for any vector field X, where a and b are smooth functions on M^{2n+1}. As a generalization of closed conformal vector fields, Sharma [15] introduced the notion of holomorphically planar conformal vector (in short, HPCV) fields on almost Hermitian manifolds. An almost Hermitian manifold admitting an HPCV field was characterized by Ghosh and Sharma [11]. They have shown that if V is a strictly nongeodesic nonvanishing HPCV field on an almost Hermitian manifold, then V is homothetic and almost analytic. Furthermore, Sharma [16] proved that among all complete and simply connected K-contact manifolds, only the unit sphere admits a non-Killing HPCV field and a (k, μ)-contact manifold admitting a nonzero HPCV field is either Sasakian or locally isometric to E^3 or $E^{n+1} \times S^n(4)$. Ghosh [10] studied HPCV fields in the framework of contact metric manifolds under certain conditions and proved that a contact metric manifold with pointwise constant ξ-sectional curvature admitting a nonclosed HPCV field V is either K-contact or V is homothetic.

Motivated by the above studies, we consider HPCV fields in the framework of a special type of almost contact metric manifolds, named almost Kenmotsu manifolds. The article is organized as follows: In section 2, we present some preliminary notion on almost Kenmotsu manifolds existing in the literature. Section 3 deals with HPCV fields on almost Kenmotsu manifolds, and section 4 is devoted to the study of HPCV fields on $(k, \mu)'$-almost Kenmotsu manifolds.

2. Preliminaries

In this section, we first recall some basic definitions and formulas, which will be used in what follows. We begin with the definition of contact manifold.

Definition 2.1. [1] A contact manifold is a $(2n+1)$-dimensional smooth manifold M together with a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$.

The 1-form η is usually known as the contact form on M^{2n+1}. It is well known that a contact metric manifold admits a contact metric structure (ϕ, ξ, η, g), where ϕ is a $(1,1)$-tensor, ξ is a global vector field, η is a 1-form, and g is the Riemannian metric satisfying (see [1,2]):

$$\begin{cases} \phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \\ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \end{cases} \tag{2.1}$$

for any smooth vector fields X and Y on M^{2n+1}. Here $\phi\xi = 0$ and $\eta \circ \phi = 0$ hold; both can be derived from (2.1) easily.

The fundamental 2-form Φ on an almost contact metric manifold is defined by

$$\Phi(X,Y)=g(X,\phi Y)$$

for any smooth vector fields X and Y on M^{2n+1}. The condition for an almost contact metric manifold being normal is equivalent to vanishing of the $(1,2)$-type torsion tensor N_ϕ, defined by

$$N_\phi=[\phi,\phi]+2d\eta\otimes\xi,$$

where $[\phi,\phi]$ is the Nijenhuis tensor of ϕ; see [1]. Recently in [7–9], almost contact metric manifolds with closed η and $d\Phi=2\eta\wedge\Phi$ have been studied and they are called almost Kenmotsu manifolds. For more details on almost Kenmotsu manifolds we refer the reader to go through the references [6, 7, 9]. A normal almost Kenmotsu manifold is a Kenmotsu manifold. Also, Kenmotsu manifolds can be characterized by

$$(\nabla_X\phi)Y=g(\phi X,Y)\xi-\eta(Y)\phi X$$

for any smooth vector fields X and Y. Let the distribution orthogonal to ξ be denoted by $\mathcal{D}$; then $\mathcal{D}=Im(\phi)=Ker(\eta)$. Since η is closed, $\mathcal{D}$ is an integrable distribution.

The study of nullity distributions is a very interesting topic on almost contact metric manifolds. The notion of k-nullity distribution was introduced by Gray [12] and Tanno [17] in the study of Riemannian manifolds. Blair, Koufogiorgos, and Papantoniou [3] introduced the generalized notion of the k-nullity distribution, named the (k,μ)-nullity distribution on a contact metric manifold. In [8], Dileo and Pastore introduced the notion of $(k,\mu)'$-nullity distribution, another generalized notion of the k-nullity distribution, on an almost Kenmotsu manifold $(M^{2n+1},\phi,\xi,\eta,g)$, which is defined for any $p\in M^{2n+1}$ and $k,\mu\in\mathbb{R}$ as follows:

$$\begin{aligned}N_p(k,\mu)'=\{Z\in T_pM:R(X,Y)Z &= k[g(Y,Z)X-g(X,Z)Y]\\ &\quad+\mu[g(Y,Z)h'X-g(X,Z)h'Y]\},\end{aligned}\tag{2.2}$$

where $h'=h\circ\phi$.

Let M^{2n+1} be an almost Kenmotsu manifold with structure (ϕ,ξ,η,g). The Levi-Civita connection ∇ satisfies $\nabla_\xi\xi=0$ and $\nabla_\xi\phi=0$. We denote by $h=\frac{1}{2}\pounds_\xi\phi$ and $l=R(\cdot,\xi)\xi$ on M^{2n+1}. The tensor fields l and h are symmetric operators and satisfy the following relations [14]:

$$h\xi=0,\ l\xi=0,\ tr(h)=0,\ tr(h\phi)=0,\ h\phi+\phi h=0.$$

We also have the following formulas (see [7–9]):

$$\nabla_X\xi=X-\eta(X)\xi-\phi hX,\tag{2.3}$$

$$R(X,Y)\xi=\eta(X)(Y-\phi hY)-\eta(Y)(X-\phi hX)+(\nabla_Y\phi h)X-(\nabla_X\phi h)Y,\tag{2.4}$$

$$(\nabla_X\phi)Y+(\nabla_{\phi X}\phi)\phi Y=-\eta(Y)\phi X-2g(X,\phi Y)\xi-\eta(Y)hX.\tag{2.5}$$

Using (2.3), we can easily obtain

$$(\nabla_X\eta)Y=g(X,Y)-\eta(X)\eta(Y)-g(\phi hX,Y).\tag{2.6}$$

3. HPCV fields on almost Kenmotsu manifolds

In this section, we characterize almost Kenmotsu manifolds admitting a holomorphically planar conformal vector field V. Before proving our main theorem, we first state and prove the following lemma.

Lemma 3.1. *Let M^{2n+1} be an almost Kenmotsu manifold admitting an HPCV field V. Then the following relation*

$$\phi Va = 2nb\eta(V) + (\xi b)\eta(V) - Vb$$

holds on M^{2n+1}.

Proof. Using the formula $R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$ in (1.1), we obtain

$$\begin{aligned} R(X,Y)V &= (Xa)Y - (Ya)X + (Xb)\phi Y - (Yb)\phi X \\ &\quad + b[(\nabla_X\phi)Y - (\nabla_Y\phi)X]. \end{aligned} \tag{3.1}$$

Putting ϕX for X and ϕY for Y in the preceding equation yields

$$\begin{aligned} R(\phi X,\phi Y)V &= (\phi Xa)\phi Y - (\phi Ya)\phi X + (\phi Xb)[-Y + \eta(Y)\xi] \\ &\quad -(\phi Yb)[-X + \eta(X)\xi] + b[(\nabla_{\phi X}\phi)\phi Y - (\nabla_{\phi Y}\phi)\phi X]. \end{aligned} \tag{3.2}$$

Now, adding equations (3.1) and (3.2) and using (2.5), we have

$$\begin{aligned} R(X,Y)V + R(\phi X,\phi Y)V &= (Xa)Y - (Ya)X + (Xb)\phi Y - (Yb)\phi X \\ &\quad +(\phi Xa)\phi Y - (\phi Ya)\phi X - (\phi Xb)Y \\ &\quad +(\phi Xb)\eta(Y)\xi + (\phi Yb)X - (\phi Yb)\eta(X)\xi \\ &\quad +b[-\eta(Y)\phi X - 2g(X,\phi Y)\xi - \eta(Y)hX \\ &\quad +\eta(X)\phi Y + 2g(\phi X,Y)\xi + \eta(X)hY]. \end{aligned} \tag{3.3}$$

Taking the inner product of (3.3) with V and then substituting ϕX for X and ϕY for Y yield

$$\begin{aligned} &(\phi Xa)g(\phi Y,V) - (\phi Ya)g(\phi X,V) + (\phi Xb)[-g(Y,V) + \eta(Y)\eta(V)] \\ &-(\phi Yb)[-g(X,V) + \eta(X)\eta(V)] + [-X + \eta(X)\xi](a)[-g(Y,V) + \eta(Y)\eta(V)] \\ &-[-Y + \eta(Y)\xi](a)[-g(X,V) + \eta(X)\eta(V)] - [-X + \eta(X)\xi](b)g(\phi Y,V) \\ &+[-Y + \eta(Y)\xi](b)g(\phi X,V) - 4bg(X,\phi Y)\eta(V) = 0. \end{aligned} \tag{3.4}$$

Now, replacing Y by ϕY in (3.4), we obtain

$$\begin{aligned} &g(Da,\phi X)[-g(Y,V) + \eta(Y)\eta(V)] - [-g(Da,Y) + \eta(Y)(\xi a)]g(\phi X,V) \\ &-g(Db,\phi X)g(\phi Y,V) - [-g(Db,Y) + \eta(Y)(\xi b)][-g(X,V) + \eta(X)\eta(V)] \\ &+[g(Da,X) - \eta(X)(\xi a)]g(\phi Y,V) + g(Da,\phi Y)[-g(X,V) + \eta(X)\eta(V)] \\ &+[g(Db,X) - \eta(X)(\xi b)][-g(Y,V) + \eta(Y)\eta(V)] - g(Db,\phi Y)g(\phi X,V) \\ &+4b[g(X,Y) - \eta(X)\eta(Y)] = 0. \end{aligned} \tag{3.5}$$

Contracting X and Y in (3.5), we have

$$\phi Va = 2nb\eta(V) + (\xi b)\eta(V) - Vb.$$

□

Theorem 3.2 ([7]). *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold and assume that $h = 0$. Then, M^{2n+1} is locally a warped product of an almost Kaehler manifold and an open interval.*

Proposition 3.3. *If an almost Kenmotsu manifold M^{2n+1} admits a nonzero HPCV field V such that V is pointwise collinear with ξ, then M^{2n+1} is locally a warped product of an almost Kaehler manifold and an open interval.*

Proof. Let M^{2n+1} be an almost Kenmotsu manifold admitting a nonzero HPCV field V such that V is pointwise collinear with ξ. Then there exists a nonzero smooth function α on M^{2n+1} such that

$$V = \alpha\xi. \tag{3.6}$$

Using (3.6) in Lemma 3.1, we have $2nb\alpha = 0$, which implies $b = 0$ as α is a nonzero smooth function, and therefore, (1.1) reduces to $\nabla_X V = aX$. From (3.6), we get $\alpha = \eta(V)$. Now,

$$X\alpha = \nabla_X\alpha = \nabla_X\eta(V) = (\nabla_X\eta)V + \eta(\nabla_X V).$$

Applying (2.6), (3.6), and $\nabla_X V = aX$ in the preceding equation yields $X\alpha = a\eta(X)$. Differentiating (3.6) covariantly along any vector field X and using $b = 0$, we obtain

$$aX = a\eta(X)\xi + \alpha[X - \eta(X)\xi - \phi hX]. \tag{3.7}$$

Taking the inner product of (3.7) with Y and then contracting X and Y yield $a = \alpha$. Substituting the value of a in (3.7), we get

$$a\phi hX = 0. \tag{3.8}$$

Since $a = \alpha$ and α is a nonzero smooth function, then $a \neq 0$. Hence, from (3.8), we get $h' = 0$, which implies $h = 0$. Now, the conclusion follows from Theorem 3.2. □

Remark 3.4. [7, Proposition 1] says that "In an almost Kenmotsu manifold M^{2n+1}, the integral manifolds of $\mathcal{D}$ are totally umbilical submanifolds of M^{2n+1} if and only if h vanishes". Hence, we can state the following: Let M^{2n+1} be an almost Kenmotsu manifold admitting a nonzero HPCV field V such that V is pointwise collinear with ξ. Then the integral manifolds of $\mathcal{D}$ are totally umbilical submanifolds of M^{2n+1}.

From [7, Theorem 3], we can say that if a locally symmetric almost Kenmotsu manifold M^{2n+1} admits a nonzero HPCV field V such that V is pointwise collinear with ξ, then M^{2n+1} is a Kenmotsu manifold. [18, Proposition 2.1] states that "Any 3-dimensional almost Kenmotsu manifold is Kenmotsu if and only if h vanishes ". Thus we arrive to the following: A 3-dimensional almost Kenmotsu manifold M^{2n+1} admitting a nonzero HPCV field V such that V is pointwise collinear with ξ is a Kenmotsu manifold.

Remark 3.5. If an almost Kenmotsu manifold admits a nonzero HPCV field V such that V is pointwise collinear with ξ, then $h = 0$. The converse of this statement is not true in general. So, Proposition 3.3 may be considered as a generalization of [7, Theorem 2].

Theorem 3.6. *Let M^{2n+1} be a complete almost Kenmotsu manifold admitting a nonzero HPCV field V. If M^{2n+1} has a constant ξ-sectional curvature, then M^{2n+1} is locally a warped product of an almost Kaehler manifold and an open interval.*

Proof. If the sectional curvature $K(\xi, X) = c$ of an almost Kenmotsu manifold is a positive constant, then we can easily obtain the following:

$$R(\xi, X)\xi = -c[X - \eta(X)\xi]. \tag{3.9}$$

Putting $X = \xi$ in (3.1), we have

$$R(\xi, Y)V = (\xi a)Y - (Ya)\xi + (\xi b)\phi Y + b\phi Y + bhY. \tag{3.10}$$

Taking the inner product of (3.10) with ξ, we get

$$g(R(\xi, Y)V, \xi) = (\xi a)\eta(Y) - (Ya). \tag{3.11}$$

Again, using (3.9), we have

$$g(R(\xi, Y)V, \xi) = -g(R(\xi, Y)\xi, V) = c[g(Y, V) - \eta(Y)\eta(V)]. \tag{3.12}$$

Hence, from (3.11) and (3.12), we obtain

$$Da - (\xi a)\xi + cV - c\eta(V)\xi = 0. \tag{3.13}$$

Taking the inner product of (3.10) with V, we get

$$(\xi a)V - \eta(V)(Da) - (\xi b)\phi V - b\phi V + bhV = 0. \tag{3.14}$$

Eliminating Da from (3.13) and (3.14), we have

$$-(\xi a)\phi^2 V - c\eta(V)\phi^2 V - (\xi b)\phi V - b\phi V + bhV = 0. \tag{3.15}$$

Now, differentiating (3.13) covariantly along any vector field X and then taking the inner product of the resulting equation with Y, we infer

$$\begin{aligned}&g(\nabla_X Da, Y) - (\xi a)[g(X, Y) - \eta(X)\eta(Y) - g(\phi hX, Y)] - (X(\xi a))\eta(Y)\\&+c[ag(X, Y) + bg(\phi X, Y)] - c\eta(Y)[g(X, V) - \eta(X)\eta(V) + a\eta(X)]\\&-c\eta(V)[g(X, Y) - \eta(X)\eta(Y) - g(\phi hX, Y)] = 0.\end{aligned} \tag{3.16}$$

Antisymmetrizing (3.16) and using the symmetry of the Hessian operator, that is, $\text{Hess}_a(X, Y) = g(\nabla_X Da, Y) = g(\nabla_Y Da, X)$, we obtain

$$\begin{aligned}&(Y(\xi a))\eta(X) - (X(\xi a))\eta(Y) + 2bcg(\phi X, Y)\\&-c\eta(Y)g(X, V) + c\eta(X)g(Y, V) = 0.\end{aligned} \tag{3.17}$$

Replacing X by ϕX and Y by ϕY in (3.17), we get $2bcg(\phi X, Y) = 0$, which implies $b = 0$ as c is a nonzero constant by hypothesis. Then from (3.14), we have

$$(\xi a)V = (Da)\eta(V). \tag{3.18}$$

Also from (3.15), we obtain

$$[(\xi a) + c\eta(V)]\phi^2 V = 0,$$

which implies either $\phi^2 V = 0$ or $(\xi a) = -c\eta(V)$.

Case 1: If $\phi^2 V = 0$, then we have $V = \eta(V)\xi$ and this implies V is pointwise collinear with ξ. Thus, from Proposition 3.3, we infer that M^{2n+1} is locally a

warped product of an almost Kaehler manifold and an open interval.
Case 2: If $(\xi a) = -c\eta(V)$, then from (3.18), we have

$$(Da + cV)\eta(V) = 0.$$

Now, if $\eta(V) = 0$, then from (3.18) we have $\xi a = 0$ as V is nonzero. Hence, from (3.13), we get $Da = -cV$. Thus, in either cases, we obtain $Da = -cV$. Differentiating this covariantly along any vector field X and using (1.1), we have $\nabla_X Da = -caX$. We are now in a position to apply Obata's theorem [13]: "In order for a complete Riemannian manifold of dimension $n \geq 2$ to admit a non-constant function λ with $\nabla_X D\lambda = -c^2\lambda X$ for any vector X, it is necessary and sufficient that the manifold is isometric with a sphere $S^n(c)$ of radius $\frac{1}{c}$ " to conclude that the manifold is isometric to the sphere $S^{2n+1}(\sqrt{c})$ of radius $\frac{1}{\sqrt{c}}$. This is a contradiction to the well-known fact that an almost Kenmotsu manifold cannot be compact. □

4. $(k,\mu)'$-almost Kenmotsu manifolds

In this section, we study HPCV fields on almost Kenmotsu manifolds with ξ belonging to the $(k,\mu)'$-nullity distribution. Let $X \in \mathcal{D}$ be the eigenvector of h' corresponding to the eigenvalue λ. Then from (4.2), it is clear that $\lambda^2 = -(k+1)$, a constant. Therefore, $k \leq -1$ and $\lambda = \pm\sqrt{-k-1}$. We denote by $[\lambda]'$ and $[-\lambda]'$, the corresponding eigenspaces related to the nonzero eigenvalues λ and $-\lambda$ of h', respectively. Before proving our main theorem in this section, we recall some existing results.

Lemma 4.1 ([8, Proposition 4.1]). *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that ξ belongs to the $(k,\mu)'$-nullity distribution and let $h' \neq 0$. Then $k < -1$, $\mu = -2$, and Spec $(h') = \{0, \lambda, -\lambda\}$, with 0 as a simple eigenvalue and $\lambda = \sqrt{-k-1}$.*

Lemma 4.2 ([8, Lemma 4.1]). *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with $h' \neq 0$ and ξ belongs to the $(k,-2)'$-nullity distribution. Then, for any $X,\ Y \in \chi(M^{2n+1})$,*

$$(\nabla_X h')Y = -g(h'X + h'^2X, Y)\xi - \eta(Y)(h'X + h'^2X).$$

Lemma 4.3 ([8, Proposition 4.2]). *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that $h' \neq 0$ and ξ belongs to the $(k,-2)'$-nullity distribution. Then for any $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$, the Riemann curvature tensor satisfies*

$$R(X_\lambda, Y_\lambda)Z_{-\lambda} = 0,$$

$$R(X_{-\lambda}, Y_{-\lambda})Z_\lambda = 0,$$

$$R(X_\lambda, Y_{-\lambda})Z_\lambda = (k+2)g(X_\lambda, Z_\lambda)Y_{-\lambda},$$

$$R(X_\lambda, Y_{-\lambda})Z_{-\lambda} = -(k+2)g(Y_{-\lambda}, Z_{-\lambda})X_\lambda,$$

$$R(X_\lambda, Y_\lambda)Z_\lambda = (k-2\lambda)[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda],$$

$$R(X_{-\lambda},Y_{-\lambda})Z_{-\lambda} = (k+2\lambda)[g(Y_{-\lambda},Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda},Z_{-\lambda})Y_{-\lambda}].$$

From (2.2), we have

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)h'X - \eta(X)h'Y], \tag{4.1}$$

where $k,\ \mu \in \mathbb{R}$. Also, we get from (4.1) that

$$R(\xi,X)Y = k[g(X,Y)\xi - \eta(Y)X] + \mu[g(h'X,Y)\xi - \eta(Y)h'X].$$

for any $X,\ Y$ on M^{2n+1}. The $(1,1)$-type symmetric tensor field $h' = h\circ\phi$ is anticommuting with ϕ and $h'\xi = 0$. Also, it is clear that (see [8, 19])

$$h = 0 \Leftrightarrow h' = 0,\ \ h'^2 = (k+1)\phi^2 (\Leftrightarrow h^2 = (k+1)\phi^2). \tag{4.2}$$

Theorem 4.4. *A $(k,\mu)'$-almost Kenmotsu manifold with $h' \neq 0$ admitting an HPCV field V such that $\phi V \neq 0$ is either locally isometric to $\mathbb{H}^{n+1}(-4)\times\mathbb{R}^n$ or V is an eigenvector of h'.*

Proof. Substituting $X = \xi$ in (3.1), we have

$$R(\xi,Y)V = (\xi a)Y - (Ya)\xi + (\xi b)\phi Y + b\phi Y + bhY. \tag{4.3}$$

Taking the inner product of (4.3) with ξ, we obtain

$$g(R(\xi,Y)V,\xi) = (\xi a)\eta(Y) - (Ya). \tag{4.4}$$

Making use of (4.1) in the foregoing equation, we get

$$\begin{aligned} g(R(\xi,Y)V,\xi) &= -g(R(\xi,Y)\xi,V) \\ &= -k\eta(Y)\eta(V) + kg(Y,V) - 2g(h'Y,V). \end{aligned} \tag{4.5}$$

Equations (4.4) and (4.5) together imply

$$-k\eta(Y)\eta(V) + kg(Y,V) - 2g(h'Y,V) = (\xi a)\eta(Y) - (Ya), \tag{4.6}$$

which implies

$$-k\eta(V)\xi + kV - 2h'V = (\xi a)\xi - Da. \tag{4.7}$$

Now, the inner product of (4.3) with V gives

$$(\xi a)g(Y,V) - (Ya)\eta(V) + (\xi b)g(\phi Y,V) + bg(\phi Y,V) + bg(hY,V) = 0,$$

which implies

$$(\xi a)V - (Da)\eta(V) - (\xi b)\phi V - b\phi V + bhV = 0. \tag{4.8}$$

Eliminating Da from (4.7) and (4.8), we have

$$-(\xi a)\phi^2 V - k\eta(V)\phi^2 V - 2\eta(V)h'V - (\xi b)\phi V - b\phi V + bhV = 0. \tag{4.9}$$

Differentiating (4.7) covariantly along any vector field X and using (1.1), (2.3), Lemma 4.2, and the value of μ from Lemma 4.1, we infer

$$\begin{aligned} &-k[g(X-\eta(X)\xi-\phi hX,V) + g(\xi,aX+b\phi X)]\xi - k\eta(V)[X-\eta(X)\xi-\phi hX] \\ &+k[aX+b\phi X] - 2[-g(h'X+h'^2X,V)\xi - \eta(V)(h'X+h'^2X) + h'(aX+b\phi X)] \\ &\quad = (\xi a)[X-\eta(X)\xi-\phi hX] + (X(\xi a))\xi - \nabla_X Da. \end{aligned}$$

Taking the inner product of the foregoing equation with Y, we obtain

$$\begin{aligned}&-k[g(X,V)-\eta(X)\eta(V)-g(\phi hX,V)+a\eta(X)]\eta(Y)-k\eta(V)[g(X,Y)\\&-\eta(X)\eta(Y)-g(\phi hX,Y)]+k[ag(X,Y)+bg(\phi X,Y)]-2[-g(h'X\\&+h'^2X,V)\eta(Y)-\eta(V)g(h'X+h'^2X,Y)+g(ah'X-bhX,Y)]\\&\quad=(\xi a)[g(X,Y)-\eta(X)\eta(Y)-g(\phi hX,Y)]\\&\qquad+(X(\xi a))\eta(Y)-g(\nabla_X Da,Y).\end{aligned} \tag{4.10}$$

Antisymmetrizing the above equation (4.10) and using the symmetry of the Hessian operator, that is, $\mathrm{Hess}_a(X,Y)=g(\nabla_X Da,Y)=g(\nabla_Y Da,X)$, we obtain

$$\begin{aligned}&-k[g(X,V)\eta(Y)-g(Y,V)\eta(X)-g(\phi hX,V)\eta(Y)+g(\phi hY,V)\eta(X)]\\&+2kbg(\phi X,Y)-2[-g(h'X+h'^2X,V)\eta(Y)+g(h'Y+h'^2Y,V)\eta(X)]\\&\quad=(X(\xi a))\eta(Y)-(Y(\xi a))\eta(X).\end{aligned} \tag{4.11}$$

Putting ϕX for X and ϕY for Y in (4.11), we infer that $2kbg(\phi X,Y)=0$, which implies $b=0$ as $k<-1$. Hence, from (4.8), we have

$$(\xi a)V=(Da)\eta(V).$$

Now, letting $Y\in[\lambda]'$ in (4.6) yields

$$(k-2\lambda)g(Y,V)=-(Ya),$$

which implies

$$Da=(2\lambda-k)V \quad \text{and} \quad (\xi a)=(2\lambda-k)\eta(V). \tag{4.12}$$

Using $b=0$ and the value of (ξa) from (4.12) in (4.9), we have

$$2(\lambda+1)\eta(V)(h'V+\phi^2V)=0,$$

which implies either $\lambda=-1$ or $\eta(V)=0$ or $h'V=-\phi^2V$.
Case 1: If $\lambda=-1$, then from $\lambda^2=-k-1$, we obtain $k=-2$. Now letting $X,\ Y,\ Z\in[\lambda]'$ and noting that $k=-2,\ \lambda=-1$, from Lemma 4.3, we have

$$R(X_\lambda,Y_\lambda)Z_\lambda=0$$

and

$$R(X_{-\lambda},Y_{-\lambda})Z_{-\lambda}=-4[g(Y_{-\lambda},Z_{-\lambda})X_{-\lambda}-g(X_{-\lambda},Z_{-\lambda})Y_{-\lambda}]$$

for any $X_\lambda,Y_\lambda,Z_\lambda\in[\lambda]'$ and $X_{-\lambda},Y_{-\lambda},Z_{-\lambda}\in[-\lambda]'$. Also, noting $\mu=-2$, it follows from Lemma 4.1 that $K(X,\xi)=-4$ for any $X\in[-\lambda]'$ and $K(X,\xi)=0$ for any $X\in[\lambda]'$. Again, from Lemma 4.1, we see that $K(X,Y)=-4$ for any $X,Y\in[-\lambda]'$ and $K(X,Y)=0$ for any $X,Y\in[\lambda]'$. It was shown in [8] that the distribution $[\xi]\oplus[\lambda]'$ is integrable with totally geodesic leaves and the distribution $[-\lambda]'$ is integrable with totally umbilical leaves by $H=-(1-\lambda)\xi$, where H is the mean curvature tensor field for the leaves of $[-\lambda]'$ immersed in M^{2n+1}. Here $\lambda=-1$; then the two orthogonal distributions $[\xi]\oplus[\lambda]'$ and $[-\lambda]'$ are both integrable with totally geodesic leaves immersed in M^{2n+1}. Then we can say that M^{2n+1} is locally isometric to $\mathbb{H}^{n+1}(-4)\times\mathbb{R}^n$.
Case 2: If $\eta(V)=0$, then from (4.12), we have $(\xi a)=0$. Then from (4.7), we have $Da=2h'V-kV$. Now, equating the value of Da from this and (4.12), we

get $h'V = \lambda V$. This shows that V is an eigenvector of h'.
Case 3: If $h'V = -\phi^2 V = V - \eta(V)\xi$, then applying h' on both side of it, we have $h'^2 V = h'V$. Hence, using (4.2), we obtain $-(k+2)(V - \eta(V)\xi) = 0$. Now, $V - \eta(V)\xi \neq 0$ as $\phi V \neq 0$ by hypothesis. Therefore, we have $k = -2$. Now, from $\lambda^2 = -k - 1$, we obtain $\lambda^2 = 1$. Without loss of generality, we assume that $\lambda = -1$. Then by the same argument as in Case 1, we get M^{2n+1} is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$. □

Remark 4.5. The result of Theorem 4.4 says that a $(k,\mu)'$-almost Kenmotsu manifold is locally isometric to a product space. As far as known, in almost every article on $(k,\mu)'$-almost Kenmotsu manifold, several authors proved the same result under different conditions. So, the problem is to find different conditions that produce the same result. We prove the same result as in [8] under a different condition.

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