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A NOTE ON CRITICAL POINT EQUATIONS ON THREE-DIMENSIONAL COSYMPLECTIC MANIFOLDS

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Abstract. The goal of this article is to investigate some properties of the critical point equations on three-dimensional f-cosymplectic manifolds. We obtain some geometric equations on the three-dimensional f-cosymplectic manifolds, which admit critical point equations. We give a relation between f and $\tilde{f}$ for a CPE metric on the three-dimensional f-cosymplectic manifold to be Einstein. Also we obtain an eigenvalue of the Laplace operator on the three-dimensional f-cosymplectic manifolds with CPE metrics.

1. Introduction

Let (M, g) be an n-dimensional compact (without boundary) oriented Riemannian manifold with dimension at least three. As we know, the total scalar curvature functional $\mathcal{R} : \mathcal{M} \to \mathbb{R}$ is

$$\mathcal{R}(g) = \int_M R_g dvol_g,$$

where R is scalar curvature and $\mathcal{M}$ is the space of Riemannian metrics on the manifold M. The Euler–Lagrangian equation of the total scalar curvature functional restricted to the space of metrics with constant scalar curvature of unit volume is given by

$$Ric - \frac{R}{n}g = Hess(f) - (Ric - \frac{R}{n-1}g)f, \tag{1.1}$$

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where Ric, R, and $Hess$ stand, respectively, for the Ricci tensor, the scalar curvature, and the Hessian form on M^n [4, 6]. We recall the definition of critical point equations (CPE metrics).

Definition 1.1 ([2])**.** A CPE metric is a three-tuple (M^n, g, f), where (M^n, g), is a compact oriented Riemannian manifold of dimension at least three with constant scalar curvature and $f : M^n \to \mathbb{R}$ is a nonconstant smooth function satisfying (1.1). Such a function f is called a potential.

It was conjectured in 1980s that a critical metric of the total scalar curvature functional, restricted to the space of metrics with constant scalar curvature, must be Einstein [6]. The conjecture proposed in [6] may be restated in terms of CPE. The geometric structure of an Einstein solution of (1.1) is known to be simple. Indeed, Obata [11] showed that such a solution is isometric to a standard n-sphere. The conjecture was proposed in [6] and here we will present this problem in the following way.

Conjecture 1.2. ([2]). A CPE metric is always Einstein.

There are many researches who are working on critical point equations with an attempt to prove Conjecture 1.2; for example, you can see [3, 5, 10, 12] and references therein. In [10], it was provided a necessary and sufficient condition on the norm of the gradient of the potential function for a CPE metric to be Einstein as follows.

Theorem 1.3 ([10])**.** *Let (M^n, g, f) be an n-dimensional CPE metric. Then M^n is Einstein if and only if*

$$|\nabla f|^2 + \frac{Rf^2}{n(n-1)} = \Lambda, \tag{1.2}$$

where Λ a constant.

1.1. **Cosymplectic manifolds.** The geometry of several kinds of almost contact metric manifolds has been the subject of many researches in the last decade. In this subsection, we summarize some basic definitions on almost contact manifolds, which will be used in the next section (for more details you can see [7]). An almost contact structure on a $(2n+1)$-dimensional smooth manifold M is a triple (ϕ, ξ, η), where ϕ is a $(1, 1)$-type tensor field, ξ is a global vector field, and η a 1-form, such that

$$\phi^2 = -\mathrm{id} + \eta \otimes \xi, \quad \eta(\xi) = 1,$$

where, id denotes the identity mapping, which implies that $\phi(\xi) = 0$, $\eta \circ \phi = 0$, and $\mathrm{rank}(\phi) = 2n$. Generally, ξ is called the characteristic vector field or the Reeb vector field.

A Riemannian metric g on M^{2n+1} is said to be compatible with the almost contact structure (ϕ, ξ, η) if for every $X, Y \in \mathcal{X}(M)$, we have

$$g(\phi(X), \phi(Y)) = g(X, Y)) - \eta(X)\eta(Y). \tag{1.3}$$

An almost contact structure endowed with a compatible Riemannian metric is said to be an almost contact metric structure. The fundamental 2-form Φ of an almost contact metric manifold M^{2n+1} is defined by

$$\Phi(X,Y) = g(X,\phi(Y))$$

for any vector fields X and Y on M^{2n+1}.

An almost contact metric manifold is defined as an almost cosymplectic manifold such that $d\eta = 0$ and $d\Phi = 0$. In particular, an almost cosymplectic manifold is said to be a cosymplectic manifold if $\nabla\phi = 0$. An almost contact structure is said to be almost α-Kenmotsu if $d\eta = 0$ and $d\omega = 2\alpha\eta\wedge\omega$ for a nonzero constant α. More generally, if the constant α is any real number, then an almost contact structure is said to be almost α-cosymplectic [9]. Moreover, Aktan, Yildirim, and Murathan [1] generalized the real number α to a smooth function f on M and defined an almost f-cosymplectic manifold, which is an almost contact metric manifold (M,ϕ,ξ,η,g) such that $d\omega = 2f\eta\wedge\omega$ and $d\eta = 0$ for a smooth function f satisfying $df\wedge\eta = 0$. In addition, if the almost f-cosymplectic structure on M is normal, then we say that M is an f-cosymplectic manifold. Obviously, if f is constant, then an f-cosymplectic manifold is either cosymplectic under condition $f = 0$, or α-Kenmotsu ($\alpha = f \neq 0$). Chen proved the following results.

Proposition 1.4 ([8])**.** *For any f-cosymplectic manifold, if $\xi(\tilde{f}) = 0$, then $\tilde{f} =$ const, where $\tilde{f} =^{\delta} \xi(f) + f^2$.*

Proposition 1.5 ([8])**.** *A compact f-cosymplectic manifold M^{2n+1} with $\xi(\tilde{f}) = 0$ is α-cosymplectic. In particular, if $\tilde{f} = 0$, then M is cosymplecitc.*

Moreover, for the three-dimensional case, it was proved the following result.

Lemma 1.6 ([8])**.** *For a three-dimensional f-cosymplectic manifold M^3, the Ricci operator of M for any vector filed Y will be obtained as follows:*

$$QY = (-3\tilde{f} - \frac{R}{2})\eta(Y)\xi + (\tilde{f} + \frac{R}{2})Y. \tag{1.4}$$

In this article, we investigate some properties of the critical point equations on three-dimensional f-cosymplectic manifolds. We obtain some geometric equations on the three-dimensional f-cosymplectic manifolds that admit critical point equations. We give a relation only between f and $\tilde{f}$ for a CPE metric on the three-dimensional f-cosymplectic manifold to be Einstein. Also we obtain an eigenvalue of the Laplace operator on the three-dimensional f-cosymplectic manifolds with a CPE metric.

2. Main results

In this section, we present our main results and provide their proofs. At first, we show the following lemma.

Lemma 2.1. *Let M be a three-dimensional f-cosymplectic manifold with a critical point equation (M^n, g, f). Then the following equation holds:*

$$\nabla^2 f(\xi,\xi) = \tilde{f}(2f-2) + R(\frac{f}{2} - \frac{1}{3}). \tag{2.1}$$

Proof. Let M be a CPE metric on the three-dimensional f-cosymplectic manifold. Then from (1.1), we have

$$\begin{aligned} Ric(X,Y) - \frac{R}{n}g(X,Y) &= Hess(f)(X,Y) - (Ric(X,Y) - \frac{R}{n-1}g(X,Y))f, \\ g(X,QY) - \frac{R}{n}g(X,Y) &= \nabla^2(f)(X,Y) - (g(X,QY) - \frac{R}{n-1}g(X,Y))f. \end{aligned}$$

Now using (1.4), we get

$$\begin{aligned} &g(X,(-3\tilde{f} - \frac{R}{2})\eta(Y)\xi + (\tilde{f} + \frac{R}{2})Y) - \frac{R}{n}g(X,Y) \\ &\quad = \nabla^2(f)(X,Y) - g(X,(-3\tilde{f} - \frac{R}{2})\eta(Y)\xi + (\tilde{f} + \frac{R}{2})Y) - \frac{R}{n-1}g(X,Y))f. \end{aligned}$$

Then

$$\begin{aligned} \nabla^2 f(X,Y) =& g(X,(-3\tilde{f} - \frac{R}{2})\eta(Y)\xi) + g(X,(\tilde{f} + \frac{R}{2})Y) - \frac{R}{n}g(X,Y) \\ &+ fg(X,(3\tilde{f} + \frac{R}{2})\eta(Y)\xi) - fg(X,(\tilde{f} + \frac{R}{2})Y) + \frac{fR}{n-1}g(X,Y)) \\ =& (-3\tilde{f} - \frac{R}{2})\eta(Y)g(X,\xi) + (\tilde{f} + \frac{R}{2})g(X,Y) - \frac{R}{n}g(X,Y) \\ &+ f(3\tilde{f} + \frac{R}{2})\eta(Y)g(X,\xi) - f(\tilde{f} + \frac{R}{2})g(X,Y) + \frac{fR}{n-1}g(X,Y) \\ =& (f-1)(3\tilde{f} + \frac{R}{2})\eta(Y)g(X,\xi) + [(1-f)(\tilde{f} + \frac{R}{2}) + \frac{fR}{2} - \frac{R}{3}]g(X,Y). \end{aligned}$$

Thus

$$\nabla^2 f(X,Y) = (f-1)(3\tilde{f} + \frac{R}{2})\eta(Y)g(X,\xi) + [(1-f)(\tilde{f} + \frac{R}{2}) + \frac{fR}{2} - \frac{R}{3}]g(X,Y). \quad (2.2)$$

Putting $X = Y = \xi$, we obtain the following equation:

$$\nabla^2 f(\xi,\xi) = (f-1)(3\tilde{f} + \frac{R}{2})\eta(\xi)g(\xi,\xi) + [(1-f)(\tilde{f} + \frac{R}{2}) + \frac{fR}{2} - \frac{R}{3}]g(\xi,\xi).$$

Then

$$\begin{aligned} \nabla^2 f(\xi,\xi) &= (f-1)(3\tilde{f} + \frac{R}{2}) + (1-f)(\tilde{f} + \frac{R}{2}) + \frac{fR}{2} - \frac{R}{3} \\ &= \tilde{f}(2f-2) + R(\frac{f}{2} - \frac{1}{3}). \end{aligned}$$

Consequently

$$\nabla^2 f(\xi,\xi) = \tilde{f}(2f-2) + R(\frac{f}{2} - \frac{1}{3}).$$

□

Applying (2.1), we obtain the scalar curvature as follows:

$$R = \frac{6[\nabla^2 f(\xi,\xi) - 2\tilde{f}(f-1)]}{3f-2}.$$

Using it in (1.2), we get

$$\begin{aligned} |\nabla f|^2 + \frac{f^2}{3\times 2}\left(\frac{6[\nabla^2 f(\xi,\xi) - 2\tilde{f}(f-1)]}{3f-2}\right) &= \Lambda, \\ |\nabla f|^2 + f^2\left(\frac{[\nabla^2 f(\xi,\xi) - 2\tilde{f}(f-1)]}{3f-2}\right) &= \Lambda. \end{aligned}$$

Therefore we get the following result.

Theorem 2.2. *Let M be a three-dimensional f-cosymplectic manifold with a critical point equation (M^n, g, f). The f-cosymplectic manifold is Einstein if and only if*

$$|\nabla f|^2 + f^2\left(\frac{[\nabla^2 f(\xi,\xi) - 2\tilde{f}(f-1)]}{3f-2}\right) = \Lambda, \tag{2.3}$$

where Λ is a constant.

Note that the above theorem gives a relation only between f and $\tilde{f}$ for a CPE metric on the three-dimensional f-cosymplectic manifold to be Einstein and consequently, by the Obata theorem (see [11]), the f-cosymplectic manifold is isometric to a standard three-sphere.

Proposition 2.3. *Let (M^3, g, f) be a critical point equation on the three-dimensional f-cosymplectic manifold. If ξ is a Killing vector field, then $\frac{3}{2}R$ is an eigenvalue of the Laplace operator with eigenfunction f.*

Proof. Considering (2.2), we have

$$\nabla^2 f(X,Y) = (f-1)(3\tilde{f} + \frac{R}{2})\eta(X)\eta(Y) + [(1-f)(\tilde{f} + \frac{R}{2}) + \frac{fR}{2} - \frac{R}{3}]g(X,Y).$$

Let $\tilde{f} = 0$. Applying (1.3), we get

$$\begin{aligned} \nabla^2 f(X,Y) &= \frac{R}{2}(f-1)(g(X,Y) - g(\phi X, \phi Y)) + \frac{R}{6}g(X,Y) \\ &= R[\frac{1}{2}(f-1) + \frac{1}{6}]g(X,Y) - \frac{R}{2}(f-1)g(\phi X, \phi Y) \\ &= R[\frac{f}{2} - \frac{1}{3}]g(X,Y) - \frac{R}{2}(f-1)(\mathcal{L}_\xi g)(X,Y). \end{aligned}$$

If ξ is a Killing vector field, then

$$\nabla^2 f(X,Y) = R[\frac{f}{2} - \frac{1}{3}]g(X,Y).$$

Taking trace, we get

$$\Delta f = \frac{3}{2}Rf - R,$$

which shows that $\frac{3}{2}R$ is an eigenvalue of the Laplace operator with eigenfunction f. □

Remark 2.4. Studying this structure for general odd-dimensional f-cosymplectic manifolds is the topic of a forthcoming paper.

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