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MINIMAL USCO AND MINIMAL CUSCO MAPS

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ABSTRACT. The main aim of this paper is to present a survey of known results concerning minimal usco and minimal cusco maps. We give characterizations of minimal usco and minimal cusco maps in the class of all set-valued maps using quasicontinuous selections. If X is a topological space and Y is a Banach space, there is a bijection between the space of minimal usco maps from X to Y and the space of minimal cusco maps from X to Y . We study this bijection with respect to various topologies on underlying spaces. Some new results are also given.

1. INTRODUCTION

The acronym usco (cusco) stands for a (convex) upper semicontinuous non-empty compact-valued set-valued map. Such set-valued maps are interesting because they describe common features of maximal monotone operators, of the convex subdifferential and of Clarke generalized gradient. Examination of cuscos and uscous leads to serious insights into the underlying topological properties of the convex subdifferential and the Clarke generalized gradient. (It is known that Clarke subdifferential of a locally Lipschitz function and, in particular, the subdifferential of a convex continuous functions are weak* cuscous.) (see [8])

Minimal usco and minimal cusco maps are used in many papers (see [8, 9, 16, 20, 25, 27, 32, 60]). Historically, minimal usco maps seem to have appeared first in complex analysis (in the second half of the 19th century), in the form of a bounded holomorphic function and its “cluster sets”, see e.g. [10]. Minimal usco maps are a very convenient tool in the theory of games (see [14]) or in functional

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analysis (see [5]), where a differentiability property of single-valued functions is characterized by their Clarke subdifferentials being minimal cuscós. Minimal uscós/cuscós also appear in:

- (i) optimization, [11, 12, 40];
- (ii) the study of weak Asplund spaces, [18, 37, 52, 58, 59];
- (iii) selection theorems, [13, 51];
- (iv) the study of differentiability of Lipschitz functions, [6, 50, 61].

2. PRELIMINARIES

In what follows, let X and Y be Hausdorff topological spaces, \mathbb{R} be the space of real numbers with the usual metric and Z^+ be the set of positive integers. Also, for $x \in X$ and $y \in Y$, $\mathcal{U}(x)$ and $\mathcal{V}(y)$ are always used to denote a base of open neighborhoods of x in X and $y \in Y$, respectively. The symbol \bar{A} and $\text{Int}A$ will stand for the closure and interior of the set A in a topological space.

A set-valued map, or multifunction, from X to Y is a function that assigns to each element of X a subset of Y . If F is a set-valued map from X to Y , we denote it by $F : X \rightsquigarrow Y$.

If $F : X \rightsquigarrow Y$, then its graph is the set

$$\{(x, y) \in X \times Y : y \in F(x)\}.$$

Conversely, if F is a subset of $X \times Y$ and $x \in X$, define $F(x) = \{y \in Y : (x, y) \in F\}$. Then we can assign to each subset F of $X \times Y$ a set-valued map which takes the value $F(x)$ at each point $x \in X$ and which graph is F . In this way, we identify set-valued maps with their graphs. Following [16] the term map is reserved for a set-valued map.

Notice that if $f : X \rightarrow Y$ is a single-valued function, we will use the symbol f also for the graph of f .

Given two maps F and G from X to Y , we write $G \subset F$ and say that G is contained in F if $G(x) \subset F(x)$ for every $x \in X$.

A map $F : X \rightsquigarrow Y$ is upper semi-continuous at a point $x \in X$ if, for every open set V containing $F(x)$, there exists $U \in \mathcal{U}(x)$ such that

$$F(U) = \cup\{F(u) : u \in U\} \subset V.$$

F is upper semi-continuous if it is upper semi-continuous at each point of X . Following Christensen [14] we say that a map F is usco if it is upper semi-continuous and takes nonempty compact values.

In this paper we are primarily interested in selections of set-valued maps. If $F : X \rightsquigarrow Y$ is a set-valued map, then a function $f : X \rightarrow Y$ is called a selection of F if $f(x) \in F(x)$ for every $x \in X$.

A function $f : X \rightarrow Y$ is called quasicontinuous [54] at $x \in X$ if, for every $V \in \mathcal{V}(f(x))$ and every $U \in \mathcal{U}(x)$, there is a nonempty open set $W \subset U$ such that $f(W) \subset V$. If f is quasicontinuous at every point of X , we say that f is quasicontinuous.

The notion of quasicontinuity for real functions of a real variable was introduced by Kempisty in his paper [38], however the property of quasicontinuity was perhaps the first time used by R. Baire in [2] in the study of points of separately continuous functions. There is a rich literature concerning the study of quasicontinuity, see for example [2, 7, 15, 26, 34, 38, 39, 41, 43, 44, 54].

We say that a (single-valued) function $f : X \rightarrow Y$ is subcontinuous [19] at $x \in X$ if, for every net $\{x_\sigma : \sigma \in \Sigma\}$ in X converging to x , there is a convergent subnet of $\{f(x_\sigma) : \sigma \in \Sigma\}$. A function f is subcontinuous if it is subcontinuous at every point of X .

For each function f from X to Y denote by

$$C(f) = \{x \in X : f \text{ is continuous at } x\}.$$

3. MINIMAL USCO MAPS

In this part we give new characterizations of minimal usco maps from X to Y . Minimal usco maps were studied by Drewnowski and Labuda in [16]. In their paper they gave an interesting characterization of minimal usco maps. We extend some results of [16].

A map F is said to be minimal usco [16] if it is a minimal element in the family of all usco maps (with domain X and range Y); that is, if it is usco and does not properly contain any other usco map from X into Y . By an easy application of the Kuratowski-Zorn principle we can guarantee that every usco map from X to Y contains a minimal usco map from X to Y (see [8, 9, 16]).

A natural question arises which functions $f : X \rightarrow Y$ do have the property that the closures of their graphs \bar{f} in $X \times Y$ are the graphs of minimal usco maps.

If $f : [0, 1] \rightarrow [0, 1]$ is a function with the property that $\bar{f} = [0, 1] \times [0, 1]$, then \bar{f} is a usco map which is not minimal. (It is very easy to define such a function f .)

Proposition 3.1. *Let X and Y be topological spaces, and let Y be Hausdorff. Let F be a minimal usco map from X to Y . If f is a selection of F , then $F = \bar{f}$.*

Proof. F is a closed subset of $X \times Y$, thus $\bar{f} \subset F$. By Proposition 3.2 in [16] \bar{f} is a usco map. The minimality of F implies that $\bar{f} = F$. \square

Proposition 3.2. *Let X and Y be topological spaces, and let Y be Hausdorff. Let F be a usco map from X to Y . If $\bar{f} = F$ for every selection f of F , then F is a minimal usco map.*

Proof. Suppose, by way of contradiction, that F is not a minimal usco map. Let G be a minimal usco map which is contained properly in F . Let $(x, y) \in F \setminus G$. Let g be a selection of G . Then $\bar{g} \subset G$, since G is a closed set in $X \times Y$. Thus $(x, y) \notin \bar{g}$, a contradiction since g is also a selection of F . \square

Proposition 3.3. *Let X and Y be topological spaces. Let F be a usco map from X to Y . Then every selection f of F is subcontinuous.*

Proof. Suppose there is a selection f of F which is not subcontinuous. Thus there is a net $\{x_\sigma : \sigma \in \Sigma\} \subset X$ convergent to a point x such that $\{f(x_\sigma) : \sigma \in \Sigma\}$ has no cluster point in Y . The compactness of $F(x)$ implies that there is an open set $O \supset F(x)$ and $\sigma_0 \in \Sigma$ with $f(x_\sigma) \notin O$ for every $\sigma \geq \sigma_0$. The upper semi-continuity of F at x implies that there is a $V \in \mathcal{U}(x)$ with $F(z) \subset O$ for every $z \in V$. There is $\sigma_1 \geq \sigma_0$ such that $x_\sigma \in V$ for every $\sigma \geq \sigma_1$; i.e. $f(x_\sigma) \in F(x_\sigma) \subset O$ for every $\sigma \geq \sigma_1$, a contradiction. \square

The following theorem extends the Proposition 4.5 in [16].

Theorem 3.4. *Let X and Y be topological spaces. Let F be a usco map from X to Y . Then the following are equivalent.*

- (1) F is minimal;
- (2) F maps isolated points into singletons, and every selection f of F is quasicontinuous.

Proof. (1). \Rightarrow (2) Let F be a minimal usco map from X to Y . It is easy to verify that F maps isolated points into singletons. Let f be a selection of F . Suppose f is not quasicontinuous at x_0 ; of course, x_0 cannot be an isolated point. Thus, there are open sets O_{x_0} and $O_{f(x_0)}$ in X and Y , respectively, such that $x_0 \in O_{x_0}$, $f(x_0) \in O_{f(x_0)}$ and such that for every nonempty open set $V \subset O_{x_0}$ there is $z \in V$ with $f(z) \notin O_{f(x_0)}$. By Proposition 3.3, f is subcontinuous. The subcontinuity of f guarantees that, for every $x \in O_{x_0}$, $F(x) \cap (Y \setminus O_{f(x_0)}) \neq \emptyset$. Thus $G = F \setminus (O_{x_0} \times O_{f(x_0)})$ is the graph of a usco map and $G \subset F$, a contradiction with the minimality of F .

(2) \Rightarrow (1) Suppose F is not minimal. Let $G \subset F$ be a minimal usco map, and let $(x_0, y_0) \in F \setminus G$. Let g be any selection of G . Define function h from X to Y as follows:

$$h(x) = \begin{cases} y_0, & x = x_0; \\ g(x) & x \neq x_0. \end{cases}$$

Then of course h is a selection of F which is not quasicontinuous. (There are open sets U and V in X and Y , respectively, such that $x_0 \in U$, $y_0 \in V$ and $(U \times V) \cap G = \emptyset$, i.e., $(U \times V) \cap (h \setminus \{(x_0, h(x_0))\}) = \emptyset$.) \square

To present our answer we need some more notions.

Notice that the notion of subcontinuity can be extended for so-called densely defined functions.

Let A be a dense subset of a topological space X and Y be a topological space. Let $f : A \rightarrow Y$ be a function. We say that f is densely defined. Further we say that $f : A \rightarrow Y$ is subcontinuous at $x \in X$ [42] if for every net $(x_i) \subset A$ convergent to $x \in X$, $(f(x_i))$ has a convergent subnet. It is easy to verify that $f : A \rightarrow Y$ is subcontinuous at $x \in X$ if and only if

- (*) for every open cover \mathcal{H} of Y there is a finite subset \mathcal{F} of \mathcal{H}
and there is $U \in \mathcal{U}(x)$ such that $f(U \cap A) \subset \cup \mathcal{F}$

(a slight modification of Theorem 2.1 in [55]).

A function $f : A \rightarrow Y$ is subcontinuous if it is subcontinuous at every $x \in X$.

A densely defined function f is a densely defined quasicontinuous selection of a set-valued map F , if $f(x) \in F(x)$ for every $x \in \text{dom}f$, the domain of f and $f : \text{dom}f \rightarrow Y$ is quasicontinuous with respect to the induced topology on $\text{dom}f$.

Theorem 3.5. *Let X and Y be topological spaces and let Y be a regular space. Let F be a map from X to Y . Then the following are equivalent:*

- (1) F is a minimal usco map;
- (2) There exists a quasicontinuous and subcontinuous function f from X to Y such that $\bar{f} = F$;
- (3) Every selection f of F is quasicontinuous, subcontinuous and $\bar{f} = F$;
- (4) There is a densely defined quasicontinuous subcontinuous selection f of F such that $\bar{f} = F$.

Proof. (1) \Rightarrow (3) by Propositions 3.1, 3.3 and Theorem 3.4 (3) \Rightarrow (2) and (2) \Rightarrow (4) are clear.

To prove (4) \Rightarrow (1) let f be a densely defined quasicontinuous subcontinuous selection of F . Thus $\text{dom}f$, the domain of f is a dense set in X . We show that the subcontinuity of f implies that, $\bar{f}(x)$ is a nonempty compact set for every $x \in X$. Let $x \in X$. Of course $\bar{f}(x) \neq \emptyset$. Let \mathcal{H} be an open cover of $\bar{f}(x)$. Let \mathcal{H}' be a refinement of \mathcal{H} such that for every $H' \in \mathcal{H}'$ there is $H \in \mathcal{H}$ with $\overline{H'} \subset H$ and $\bar{f}(x) \subset \cup \mathcal{H}'$. For every $y \in Y \setminus \bar{f}(x)$ let O_y be an open neighborhood of y such that $\overline{O_y} \cap \bar{f}(x) = \emptyset$. Then the family $\mathcal{H}' \cup \{O_y : y \in Y \setminus \bar{f}(x)\}$ is an open cover of Y . By (*) there is $U \in \mathcal{U}(x)$, $H'_1, H'_2, \dots, H'_n \in \mathcal{H}'$ and a finite indexed set I such that $f(U \cap \text{dom}f) \subset \cup \{H'_i : i = 1, 2, \dots, n\} \cup \cup \{O_{y_i} : i \in I\}$. Thus

$$\bar{f}(x) \subset \overline{f(U \cap \text{dom}f)} \subset (\overline{H'_1} \cup \overline{H'_2} \cup \dots \cup \overline{H'_n}) \cup \cup \{\overline{O_{y_i}} : i \in I\}.$$

Thus $\bar{f}(x) \subset H_1 \cup H_2 \cup \dots \cup H_n$, where $H_i \in \mathcal{H}$ for $i = 1, 2, \dots, n$.

Now we will show that \bar{f} is upper semi-continuous. Suppose there is $x \in X$ such that \bar{f} is not upper semi-continuous at x . Let V be an open set in Y with $\bar{f}(x) \subset V$ such that for every $U \in \mathcal{U}(x)$ there are $x_U \in U$ and $y_U \in \bar{f}(x_U) \setminus V$. The regularity of Y implies that there is an open set G in Y such that $\bar{f}(x) \subset G \subset \overline{G} \subset V$. Thus for every $U \in \mathcal{U}(x)$ we have $(x_U, y_U) \in \bar{f} \cap (U \times (Y \setminus \overline{G}))$. For every $U \in \mathcal{U}(x)$ there is $a_U \in \text{dom}f \cap U$ such that $f(a_U) \in Y \setminus \overline{G}$. Since the net $(a_U)_{U \in \mathcal{U}(x)}$ converges to x , the subcontinuity of f at x implies that there is a cluster point $y \in Y \setminus G$ of the net $(f(a_U))_{U \in \mathcal{U}(x)}$, a contradiction, since $y \in \bar{f}(x) \subset G$.

To prove that \bar{f} is minimal usco, suppose, by way of contradiction, that \bar{f} is not a minimal usco map. Let L be a minimal usco map which is contained properly in \bar{f} . Let $(x, y) \in \bar{f} \setminus L$. Let $U \in \mathcal{U}(x)$ and V be an open neighborhood of y such that $(U \times V) \cap L = \emptyset$. Let G be an open neighborhood of y such that $y \in G \subset \overline{G} \subset V$. Since $(x, y) \in \bar{f}$, there is $(z, f(z)) \in (U \cap \text{dom}f) \times G$. The quasi-continuity of f at z implies that there is a nonempty open set H in X such

that $H \cap \text{dom} f \subset U \cap \text{dom} f$ and $f(H \cap \text{dom} f) \subset G$. The set $H \cap U$ is a nonempty open set contained in U and $\bar{f}(H \cap U) \subset \bar{G} \subset V$, a contradiction since $L \subset \bar{f}$. \square

Now we mention some applications of the above characterizations.

- The condition (2) in Theorem 3.5 gives a useful tool how to construct minimal usco maps.

- From the condition (2) in Theorem 3.5 and a result from [34] which claims that if X is a Baire space, Y is a p -space with a G_δ -diagonal and $f : X \rightarrow Y$ is a quasicontinuous function, then the set $C(f)$ is a dense G_δ -set, we can prove a result concerning the set of points of single-valuedness of minimal usco maps.

To remind the definition of a p -space [21], let Y be a topological space, $y \in Y$ and \mathcal{G} be a collection of subsets of Y . Then $st(y, \mathcal{G}) = \bigcup \{G \in \mathcal{G} : y \in G\}$.

A completely regular space Y is a p -space [Gr] if there exists a sequence (\mathcal{U}_n) of families of open subsets of βY such that

- (i) each \mathcal{U}_n covers Y ;
- (ii) for each $y \in Y$, $\bigcap_n st(y, \mathcal{U}_n) \subset Y$.

The notion of a p -space was introduced by Archangelskii in 1963. Every Čech-complete space is a p -space, every Moore space is a p -space.

A topological space Y has a G_δ -diagonal if the diagonal $\Delta = \{(y, y) : y \in Y\}$ of $Y \times Y$ is a G_δ -set in $Y \times Y$.

Proposition 3.6. *Let X be a Baire space and Y be a p -space with a G_δ -diagonal. If $F : X \rightsquigarrow Y$ is a minimal usco map, then the set of points at which F is single-valued contains a dense G_δ -set.*

Proof. Let $f : X \rightarrow Y$ be a selection of F . By Theorem 3.5 f is quasicontinuous and $\bar{f} = F$. By the above mentioned result from [34] the set $C(f)$ is a dense G_δ -set in X . It is easy to verify that $F(x) = \{f(x)\}$ for every $x \in C(f)$. \square

In the paper [16] the authors gave the example of a minimal usco map, which is two-valued at every point. The example shows that the condition on Y to be a p -space with a G_δ -diagonal in Proposition 3.6 is essential.

Example 3.7. A nowhere single-valued minimal usco map. For each $t \in T = (0, 1)$, the map $F_t : [0, 1] \rightsquigarrow [0, 1]$ defined by

$$F_t(x) = \begin{cases} \{0\}, & 0 \leq x < t; \\ \{0, 1\}, & x = t; \\ \{1\}, & t < x \leq 1. \end{cases}$$

is evidently minimal usco. By Proposition 3.6 in [16] the corresponding product map $F : [0, 1] \rightsquigarrow [0, 1]^T$ is usco.

Suppose $G : [0, 1] \rightsquigarrow [0, 1]^T$ is usco, $G \subset F$, and $G(s) \neq F(s)$ for some $s \in [0, 1]$. Let p_s be the projection of $[0, 1]^T$ onto the s th copy of $[0, 1]$ in this product. Then the map $G_s : [0, 1] \rightsquigarrow [0, 1]$, defined by $G_s(x) = p_s[G(x)]$, is easily seen to be usco. Moreover, $G_s \subset F_s$ and $G_s(s) \neq F_s(s)$, where the latter follows from the fact that $F_t(s)$ is a singleton for $t \neq s$. Since F_s is minimal usco,

we must have $G_s = F_s$, a contradiction. Then the restriction of F to $(0, 1)$ is a nowhere single-valued minimal usco map.

Holá in [23] showed that the condition of Baireness in Proposition 3.6 is essential.

Example 3.8. Let X be the space of rational numbers with the usual topology. Enumerate X by $\{q_n : n \in \mathbb{Z}^+\}$ and define the set-valued map $F : X \rightsquigarrow \mathbb{R}$ as $F(x) = \{\sum_{n:q_n < x} \frac{1}{2^n}, \sum_{n:q_n \leq x} \frac{1}{2^n}\}$. Then F is a minimal usco map which is nowhere single-valued.

To prove that F is upper semicontinuous, let $x \in X$ and $\epsilon > 0$. There is $n_0 \in \mathbb{Z}^+$ such that $\sum_{n > n_0} \frac{1}{2^n} < \epsilon$. Put $\delta_x = \min\{|q_i - x| : i \leq n_0, q_i \neq x\}$ and put $O_x = (x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2})$. Then, for every $z \in O_x$, we have $F(z) \subset S_\epsilon[F(x)]$, where $S_\epsilon[F(x)] = \{s \in \mathbb{R} : d(s, F(x)) < \epsilon\}$. (Let $z \in O_x$, $z < x$. Then $\sum_{n:q_n \leq z} \frac{1}{2^n} - \sum_{n:q_n \leq x} \frac{1}{2^n} = \sum_{x < q_n \leq z} \frac{1}{2^n} < \epsilon$.)

To prove that F is minimal, suppose there is an usco map G such that $G \subset F$ and there is q_n such that $F(q_n) \neq G(q_n)$. Suppose that $G(q_n) = \{\sum_{i:q_i < q_n} \frac{1}{2^i}\}$ (the other case is similar). The upper semicontinuity of G implies that there is a neighborhood O of q_n such that $G(z) \subset S_{\frac{1}{2^{q_n}}}[G(q_n)]$ for every $z \in O$, a contradiction, since for every $z \in O$, $z > q_n$, we have $G(z) > \sum_{i:q_i \leq q_n} \frac{1}{2^i}$.

Notice that a topological space Y is called Stegall space, if for every Baire topological space X and every minimal usco map $F : X \rightsquigarrow Y$ the map F is single-valued at points of a dense subset of X . We will mention here at least the paper of Kalenda [37].

- The following remark shows that the condition (4) in Theorem 3.5 gives a useful tool how to construct a minimal usco map in a given usco map.

Remark 3.9. Let X be a Baire space and $F : X \rightsquigarrow \mathbb{R}$ be usco. Let $f : X \rightarrow \mathbb{R}$ be a function defined as follows: $f(x) = \inf\{t \in \mathbb{R} : t \in F(x)\}$ for $x \in X$. Then f is a lower semi-continuous function. It is known (see [17]) that the set $C(f)$ is a dense G_δ -set in X . Thus by Theorem 3.5 the map $G = \overline{f[C(f)]}$ is a minimal usco map from X to \mathbb{R} and $G \subset F$.

Also if $h : X \rightarrow \mathbb{R}$ is defined as $h(x) = \sup\{t \in \mathbb{R} : t \in F(x)\}$ for $x \in X$, then h is upper semi-continuous and by [17] the set $C(h)$ of the points of continuity of h is a dense G_δ -set in X . Thus by Theorem 3.5 the map $H = \overline{h[C(h)]}$ is a minimal usco map from X to \mathbb{R} and $H \subset F$.

Proposition 5.1.24 in [9] gives a construction of a minimal usco map contained in a given usco map from a general topological space with values in \mathbb{R} .

Other approach to minimality of set-valued maps can be found in [39, 45, 46, 53].

A set-valued mapping $F : X \rightsquigarrow Y$ is said to be minimal if for each pair of open subsets U of X and W of Y such that $F(U) \cap W \neq \emptyset$ there exists a non-empty open subset $V \subseteq U$ such that $F(V) \subseteq W$. This definition is modeled on the characterizing property of minimality of usco mappings [53]; i.e. an usco mapping $F : X \rightsquigarrow Y$ is a minimal usco if and only if $F : X \rightsquigarrow Y$ is minimal.

4. DENSELY CONTINUOUS FORMS

In this part of the paper we continue the study of so-called densely continuous forms introduced by McCoy and Hammer in [29] and then studied by Holá, McCoy, Holý, Vadovič in [22, 31, 35, 36]. We will also present characterizations of densely continuous forms.

Densely continuous forms from X to Y can be considered as maps (set-valued mappings) from X to Y which have a kind of minimality property found in the theory of minimalusco maps. In particular, every minimalusco map from a Baire space X into a metric space Y is a densely continuous form.

To define a densely continuous form from X to Y [29], let $DC(X, Y)$ be the set of all functions f from X to Y such that the $C(f)$ is dense in X . We call such functions densely continuous. Of course $DC(X, Y)$ contains the set $C(X, Y)$ of all continuous functions from X to Y . There are many other interesting subsets in $DC(X, Y)$. For example, if Y is a locally compact second countable space and X is a Baire space, then $DC(X, Y)$ contains all functions from X to Y with closed graphs [22].

If Y is the set \mathbb{R} of all real numbers and X is a Baire space, then also all upper and lower semi-continuous functions on X belongs to $DC(X, Y)$.

We define the set $D(X, Y)$ of densely continuous forms by

$$D(X, Y) = \{\overline{f \upharpoonright C(f)} : f \in DC(X, Y)\}.$$

The densely continuous forms from X to Y may be considered as maps (set-valued) mappings. For each $x \in X$ and $\Phi \in D(X, Y)$ define $\Phi(x) = \{y \in Y : (x, y) \in \Phi\}$.

Define by $A(X, Y)$ the following set of functions

$$A(X, Y) = \{f : X \rightarrow Y : \text{for every } x \in X \text{ and for every neighborhood } U \text{ of } (x, f(x)) \text{ there exists } y \in C(f) \text{ such that } (y, f(y)) \in U\}.$$

Of course $A(X, Y) \subset DC(X, Y)$. The inclusion is proper. The function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(0) = 0$ and $f(x) = 1/x$ otherwise, belongs to $DC(X, Y)$ and not to $A(X, Y)$. For the densely continuous form $\Phi = \overline{f \upharpoonright C(f)}$ we have $\Phi(0) = \emptyset$.

It is very easy to verify that every function from $A(X, Y)$ is quasicontinuous. The following example shows that the opposite is not true.

Example 4.1. [57] Let $X = \mathbb{R}$ with the usual Euclidean topology, and let $Y = \mathbb{R}$ with the Sorgenfrey topology. Let $f : X \rightarrow Y$ be the identity function. Then f is quasicontinuous, but the set $C(f) = \emptyset$.

However if X is a Baire space and Y is a p -space with a G_δ -diagonal, then every quasicontinuous function $f : X \rightarrow Y$ has a dense set $C(f)$ of the points of continuity [34]; i.e., f belongs to $A(X, Y)$.

Clearly, if $f : X \rightarrow Y$, then $f \in A(X, Y)$ if and only if $\overline{f} = \overline{f \upharpoonright C(f)}$.

We have the following characterization of elements of $D(X, Y)$.

Proposition 4.2. *Let X and Y be topological spaces, Y regular and $F : X \rightsquigarrow Y$ such that $F(x) \neq \emptyset$ for every $x \in X$. Then the following are equivalent:*

- (1) $F \in D(X, Y)$;
- (2) *There is a function $f \in A(X, Y)$ such that $\bar{f} = F$;*
- (3) *Every selection f of F belongs to $A(X, Y)$ and $\bar{f} = F$.*

Proof. (1) \Rightarrow (3). Let f be a selection of F . There is $g \in DC(X, Y)$ such that $F = \overline{g[C(g)]}$. Of course, $F(x) = \{g(x)\}$ for every $x \in C(g)$; i.e., $f(x) = g(x)$ for every $x \in C(g)$. It is easy to verify that $C(g) \subset C(f)$. (Let $x \in C(g)$. Suppose $x \notin C(f)$. There is a $V \in \mathcal{V}(f(x))$ such that for every $U \in \mathcal{U}(x)$ there is an $x_U \in U$ with $f(x_U) \notin V$. Let $H \in \mathcal{V}(f(x))$ be such that $\overline{H} \subset V$. The continuity of g at x implies that there is an $O \in \mathcal{U}(x)$ such that $g(O) \subset H$. Then $O \times (Y \setminus \overline{H})$ is a neighborhood of $(x_O, f(x_O))$ which has an empty intersection with the graph of g , a contradiction, since $f \subset \overline{g[C(g)]}$.)

Thus, the set $C(f)$ of the points of continuity of f is dense in X ; i.e., $f \in DC(X, Y)$. Since $f \subset F = \overline{g[C(g)]} \subset \overline{f[C(f)]}$, we have that $f \in A(X, Y)$ and $F = \bar{f}$.

(3) \Rightarrow (2) is trivial. (2) \Rightarrow (1) is also trivial since if $f \in A(X, Y)$, then of course $f \in DC(X, Y)$ and by above, $\bar{f} = \overline{f[C(f)]}$. \square

Corollary 4.3. *Let X be a Baire space and Y be a p -space with a G_δ -diagonal. Let $F : X \rightsquigarrow Y$ be such that $F(x) \neq \emptyset$ for every $x \in X$. The following are equivalent:*

- (1) $F \in D(X, Y)$;
- (2) *There is a quasicontinuous function $f : X \rightarrow Y$ such that $\bar{f} = F$;*
- (3) *Every selection f of F is quasicontinuous and $\bar{f} = F$.*

Denote by $MU(X, Y)$ the space of all minimal usco maps from X to Y . A natural question arises under which conditions does $F \in D(X, Y)$ belong to $MU(X, Y)$. We have the following answer.

Proposition 4.4. *Let X and Y be topological spaces, Y regular and $F \in D(X, Y)$. Then the following are equivalent:*

- (1) $F \in MU(X, Y)$;
- (2) $F(x) \neq \emptyset$ for every $x \in X$ and every selection f of F is subcontinuous.

It is also interesting to know when does $F \in MU(X, Y)$ belong to $D(X, Y)$.

Proposition 4.5. *Let X and Y be topological spaces, Y regular and $F \in MU(X, Y)$. Then the following are equivalent:*

- (1) $F \in D(X, Y)$;
- (2) $\{x \in X : |F(x)| = 1\}$ is dense in X .

Proof. (1) \Rightarrow (2) is clear. To prove (2) \Rightarrow (1) put

$$H = \{x \in X : |F(x)| = 1\}.$$

Let f be a selection of F . Then $H \subset C(f)$. Thus $f \in DC(X, Y)$ and $\overline{f[C(f)]} \in D(X, Y)$. It is sufficient to show that $\overline{f[C(f)]} = F$. Since $f[C(f)]$ is a densely defined quasicontinuous subcontinuous selection of $\overline{f[C(f)]}$, from (4) Theorem 3.5 we know that $\overline{f[C(f)]}$ is a minimal usco map and it is contained in F . So we are done. \square

Corollary 4.6. *Let X be a Baire space and Y be a p -space with a G_δ -diagonal. Then $MU(X, Y) \subset D(X, Y)$.*

Proof. Let $F \in MU(X, Y)$. By Proposition 3.6 $\{x \in X : |F(x)| = 1\}$ contains a dense G_δ -set. Thus $F \in D(X, Y)$. \square

McCoy and Hammer in [29, 47] studied also the space $D^*(X, \mathbb{R})$, of locally bounded densely continuous forms with values in \mathbb{R} .

If Φ is a mapping from X to \mathbb{R} (single-valued or set-valued) and $A \subset X$, we say that Φ is bounded on A , provided that the set

$$\Phi(A) = \cup\{\Phi(x) : x \in A\}$$

is a bounded subset of \mathbb{R} . We say that Φ is locally bounded, provided that each point of X has a neighborhood on which Φ is bounded.

Now define $D^*(X, \mathbb{R})$ to be the set of all members of $D(X, \mathbb{R})$, that are locally bounded.

Remark 4.7. Let $U(X, \mathbb{R})$ be the set of all real-valued usco maps. Then $D^*(X, \mathbb{R}) \subset U(X, \mathbb{R})$. In fact, if $\Phi \in D^*(X, \mathbb{R})$, then for all $x \in X$, $\Phi(x)$ is a nonempty compact set. By a result of Berge ([4] page 112) any map with a closed graph which has a compact range is upper semi-continuous. Since upper semi-continuity is a local property, every $\Phi \in D^*(X, \mathbb{R})$ belongs to $U(X, \mathbb{R})$. By Proposition 3.3 every selection of $\Phi \in U(X, \mathbb{R})$ is subcontinuous, thus by Proposition 4.4 Φ is minimal usco and $D^*(X, \mathbb{R}) \subset MU(X, \mathbb{R})$.

If X is a Baire space, then $MU(X, \mathbb{R}) \subset D^*(X, \mathbb{R})$. In fact, by Corollary 4.6 $MU(X, \mathbb{R}) \subset D(X, \mathbb{R})$. It is easy to show that every usco map from X to \mathbb{R} is locally bounded. Thus $MU(X, \mathbb{R}) \subset D^*(X, \mathbb{R})$. Therefore, if X is a Baire space $MU(X, \mathbb{R}) = D^*(X, \mathbb{R})$.

5. MINIMAL CUSCO MAPS

A map F from a topological space X to a linear topological space Y is cusco if it is usco and $F(x)$ is convex for every $x \in X$.

We say that a map F from a topological space X to a linear topological space Y is said to be minimal cusco if it is a minimal element in the family of all cusco maps (with domain X and range Y); that is, if it is cusco and does not contain properly any other cusco map from X into Y . By Kuratowski-Zorn principle we can guarantee that every cusco map from X to Y contains a minimal cusco map from X to Y (see [8, 9, 16]).

Let Y be a linear topological space and $B \subset Y$ be a set. By $\overline{\text{co}}B$ we denote the closed convex hull of the set B (see [1]).

The proof of the following Lemma is a folklore result, but a reasonably old proof of this appears in [56], Lemma 7.12.

Lemma 5.1. *Let X be a topological space and Y be a Hausdorff locally convex linear topological space. Let G be a usco map from X to Y and $\overline{\text{co}}G(x)$ be compact for every $x \in X$. Then the map F defined as $F(x) = \overline{\text{co}}G(x)$ for every $x \in X$ is a cusco map.*

Remark 5.2. There are three important cases when the closed convex hull of a compact set is compact. The first is when the compact set is a finite union of compact convex sets. The second is when the space is completely metrizable and locally convex. This includes the case of all Banach spaces with their norm topologies. The third case is a compact set in the weak topology on a Banach space. (see [1])

A set-valued map F from a topological space X to a linear topological space Y is hyperplane minimal [8] if for every open half-space W in Y and open set U in X with $F(U) \cap W \neq \emptyset$ there is a nonempty open subset $V \subset U$ such that $F(V) \subset W$. It is known [8] that a cusco map from a topological space X into Hausdorff locally convex linear topological space Y is minimal cusco if, and only if, it is hyperplane minimal.

If $f : X \rightarrow Y$ is a quasicontinuous function from a topological space to a linear topological space then f is hyperplane minimal. The following example shows that a hyperplane minimal function is not quasicontinuous in general.

Example 5.3. Let $X = Y = \mathbb{R}$ with the usual topology. Define $f : X \rightarrow Y$ as follows: $f(x) = -1$ if $x < 0$, $f(0) = 0$ and $f(x) = 1$ if $x > 0$.

Notice that all known characterizations of minimal cusco maps are given in the class of cusco maps (see [8, 20]). So the following characterization of minimal cusco maps in the class of all set-valued maps can be of some interest:

Theorem 5.4. *Let X be a topological space and Y be a Hausdorff locally convex (linear topological) space. Let F be a map from X to Y . Then the following are equivalent:*

- (1) F is a minimal cusco map;
- (2) F is nonempty compact valued and there is a quasicontinuous, subcontinuous selection f of F such that $\overline{\text{co}} \bar{f}(x) = F(x)$ for every $x \in X$;
- (3) F is nonempty compact valued and there is a hyperplane minimal, subcontinuous selection f of F such that $\overline{\text{co}} \bar{f}(x) = F(x)$ for every $x \in X$;
- (4) F is nonempty compact valued and every selection f of F is hyperplane minimal, subcontinuous and $\overline{\text{co}} \bar{f}(x) = F(x)$ for every $x \in X$.

Proof. (1) \Rightarrow (2) Let $G \subset F$ be a minimal usco map contained in F . Let f be a selection of G . By Theorem 3.5 f is a quasicontinuous and subcontinuous selection of G such that $\bar{f} = G$. So f is also a selection of F . By Proposition 2.7 in [8] we have $\overline{\text{co}} \bar{f}(x) = F(x)$ for every $x \in X$.

(2) \Rightarrow (3) is trivial, since every quasicontinuous function from X to Y is hyperplane minimal.

(3) \Rightarrow (1) Let f be a hyperplane minimal, subcontinuous selection of F . Since f is subcontinuous, \bar{f} is usco by [33]. Since $\overline{\text{co}} \bar{f}(x) = F(x)$ for every $x \in X$ and $F(x)$ is compact for every $x \in X$, F is cusco by Lemma 5.1. Thus it is sufficient to show that F is minimal. Suppose, by way of contradiction, that F is not minimal. Thus there is a minimal cusco map $L \subset F$ such that there is a point $(x_0, y_0) \in F \setminus L$. Since $L(x_0)$ is a convex set and $\overline{\text{co}} \bar{f}(x_0) = F(x_0)$, without loss of generality we can suppose that $y_0 \in \bar{f}(x_0) \setminus L(x_0)$. Since $L(x_0)$ is a closed convex set and $y_0 \notin L(x_0)$, there is a nonzero continuous linear functional strongly separating $L(x_0)$ and y_0 . So let $h : Y \rightarrow R$ be a continuous linear functional and $\lambda \in R$ such that

$$L(x_0) \subset \{y \in Y : h(y) < \lambda\} \text{ and } h(y_0) > \lambda.$$

Since the map L is upper semicontinuous there is $U \in \mathcal{U}(x_0)$ such that $L(U) \subset \{y \in Y : h(y) < \lambda\}$ and since $y_0 \in \bar{f}(x_0)$ and f is hyperplane minimal, there is a nonempty open set $V \subset U$ such that $f(V) \subset \{y \in Y : h(y) > \lambda\}$. Thus $\bar{f}(V) \subset \{y \in Y : h(y) \geq \lambda\}$. For every $x \in V$ we have $\overline{\text{co}} \bar{f}(x) \cap L(x) = \emptyset$, a contradiction.

Since (4) \Rightarrow (3) is trivial, it is sufficient to prove that (1) \Rightarrow (4). Let f be a selection of F . Since every selection of a usco map is subcontinuous, f must be subcontinuous. \bar{f} is usco and $\bar{f} \subset F$ implies that $\overline{\text{co}} \bar{f}(x)$ is compact for every $x \in X$. By Lemma 5.1 the map G defined as $G(x) = \overline{\text{co}} \bar{f}(x)$ for every $x \in X$ is cusco. Since $G \subset F$ and F is minimal, we have $\overline{\text{co}} \bar{f}(x) = F(x)$ for every $x \in X$. It is easy to verify from Theorem 2.6 in [8] that f is hyperplane minimal. \square

We have the following variant of Theorem 5.4:

Theorem 5.5. *Let X be a topological space and Y be a Hausdorff locally convex (linear topological) space in which the closed convex hull of a compact set is compact. Let F be a map from X to Y . The following are equivalent:*

- (1) F is minimal cusco map;
- (2) There is a quasicontinuous subcontinuous function $f : X \rightarrow Y$ such that $\overline{\text{co}} \bar{f}(x) = F(x)$ for every $x \in X$;
- (3) There is a hyperplane minimal subcontinuous function $f : X \rightarrow Y$ such that $\overline{\text{co}} \bar{f}(x) = F(x)$ for every $x \in X$;
- (4) Every selection f of F is hyperplane minimal and subcontinuous and $\overline{\text{co}} \bar{f}(x) = F(x)$ for every $x \in X$.

Notice that Theorem 5.5 gives us a rule how to construct minimal cusco maps with values in Hausdorff locally convex (linear topological) spaces in which the closed convex hull of a compact set is compact.

It is interesting to note that Theorem 5.4 (and also Theorem 5.5) implies the well-known result that every convex function on an open convex subset of a finite dimensional normed linear space is Frechet differentiable on a dense G_δ -subset of its domain. Let f be a convex function defined on an open convex subset A of a finite dimensional normed linear space X . It is known that the subdifferential mapping $x \rightsquigarrow \partial f(x)$ is a minimal cusco map from A into X [56]. Further f is

Frechet differentiable at $x \in A$ if and only if the subdifferential mapping $x \rightsquigarrow \partial f(x)$ is single-valued. By Theorem 5.4 (2) there is a quasicontinuous selection h of the subdifferential mapping such that $\text{co}\bar{h}(x) = \partial f(x)$. It is easy to verify that if x is a point of continuity of h , then $\text{co}\bar{h}(x) = \{h(x)\}$. It is known (see [34]) that the set of points of continuity of a quasicontinuous function defined on a Baire space with values in a p -space with a G_δ -diagonal is a dense G_δ -set.

We have the following extension of Theorem 2.18 in [8]:

Theorem 5.6. *Let X be a topological space and Y be a Hausdorff locally convex (linear topological) space. Let F be a map from X to Y . The following are equivalent:*

- (1) F is minimal cusco map;
- (2) F has a nonempty compact values and there is a densely defined quasi-continuous, subcontinuous selection f of F such that $\overline{\text{co}}\bar{f}(x) = F(x)$ for every $x \in X$;
- (3) F has a nonempty compact values and there is a densely defined hyperplane minimal, subcontinuous selection f of F such that $\overline{\text{co}}\bar{f}(x) = F(x)$ for every $x \in X$.

Proof. (1) \Rightarrow (2) is clear from the Theorem 5.4. (2) \Rightarrow (3) is trivial.

(3) \Rightarrow (1) Let f be a densely defined hyperplane minimal, subcontinuous selection of F such that $\overline{\text{co}}\bar{f}(x) = F(x)$ for every $x \in X$. As in the proof of Theorem 3.5 we can show that \bar{f} is usco. Since F has compact values, the map $x \rightsquigarrow \overline{\text{co}}\bar{f}(x)$ is cusco (by Lemma 5.1). To prove that F is minimal cusco we can use the same argument as in the proof of (3) \Rightarrow (1) of Theorem 5.4. \square

To see that Theorem 5.6 is an extension of Theorem 2.18 in [8] we need the following comment:

Let X be a topological space and Y be a Hausdorff locally convex (linear topological) space. If f is densely defined subcontinuous function such that $\overline{\text{co}}\bar{f}(x)$ is compact for every $x \in X$, then $CSC(f)(x) = \overline{\text{co}}\bar{f}(x)$ for every $x \in X$, where

$$CSC(f)(x) = \cap\{\overline{\text{co}}f(V) : V \in \mathcal{U}(x)\} \quad [8].$$

Notice that in [8] only densely defined selections of cusco maps are considered.

However the condition of subcontinuity of f is essential as the following example shows. (The inclusion $\overline{\text{co}}\bar{f}(x) \subset CSC(f)(x)$ can be proper.)

Example 5.7. Let $X = \mathbb{R} = Y$ with the usual topology. Let $f : X \rightarrow Y$ be defined as follows: $f(x) = 0$ for every $x \leq 0$ and $f(x) = 1/x$ for every $x > 0$. Then $\overline{\text{co}}\bar{f}(x) = \{f(x)\}$ for every $x \in X$, $CSC(f)(0) = [0, \infty)$ and $CSC(f)(x) = \{f(x)\}$ otherwise. Of course f is not subcontinuous at 0.

We have the following variant of Theorem 5.6:

Theorem 5.8. *Let X be a topological space and Y be a Hausdorff locally convex (linear topological) space in which the closed convex hull of a compact set is compact. Let F be a map from X to Y . The following are equivalent:*

- (1) F is minimal cusco map;

(2) There is a densely defined quasicontinuous subcontinuous function f with values in Y such that $\overline{co}f(x) = F(x)$ for every $x \in X$;

(3) There is a densely defined hyperplane minimal subcontinuous function f with values in Y such that $\overline{co}f(x) = F(x)$ for every $x \in X$.

Notice that Theorem 2.14 in [8] is an easy consequence of Theorem 5.8. The function $f : G \rightarrow \mathbb{R}$ from Lemma 2.13 in [8] is defined on a dense G_δ -set G of a topological space T . It is easy to verify that f is subcontinuous. Since f is continuous on G , by Theorem 5.8 the map $x \rightsquigarrow cof(x)$ (for every $x \in T$) is minimal cusco. Of course for Φ in Theorem 2.14 in [8] we have $\Phi(x) = \overline{co}f(x)$ for every $x \in T$.

Remark 5.9. Let X be a Baire space and $F : X \rightarrow \mathbb{R}$ be cusco. Let $f : X \rightarrow \mathbb{R}$ be defined as $f(x) = \inf\{t \in \mathbb{R} : t \in F(x)\}$ for $x \in X$. Using Remark 3.9 and Theorem 5.8 we see that the map $x \rightsquigarrow cof[\overline{C}(f)](x)$ is a minimal cusco map contained in F .

Similarly, if $h : X \rightarrow \mathbb{R}$ is defined as $h(x) = \sup\{t \in \mathbb{R} : t \in F(x)\}$ for $x \in X$ then the map $x \rightsquigarrow coh[\overline{C}(h)](x)$ is a minimal cusco map contained in F .

6. MINIMAL CUSCO MAPS AND EXTREME FUNCTIONS

Let B be a subset of a linear topological space. By $\mathcal{E}(B)$ we denote the set of all extreme points of B .

Let X be a topological space and Y be a Hausdorff locally convex (linear topological) space. Let $F : X \rightsquigarrow Y$ be a map with nonempty compact values. Then a selection f of F such that $f(x) \in \mathcal{E}(F(x))$ for every $x \in X$ is called an extreme function of F . (By Corollary 7.66 in [1] every nonempty compact subset of a Hausdorff locally convex (linear topological) space has an extreme point. The hypothesis of local convexity cannot be dispensed. [1], page 298)

Lemma 6.1. *Let X be a topological space and Y be a Hausdorff locally convex (linear topological) space. Let F be a minimal cusco map from X to Y and G be a minimal usco map from X to Y such that $G \subset F$. Then $\mathcal{E}(F(x)) \subset G(x)$ for every $x \in X$.*

Proof. Let $x \in X$. By Proposition 2.7 in [8] we have that $F(x) = \overline{co}G(x)$ for every $x \in X$. By Theorem 2.10.15 in [48] which was proved by D.P. Milman in [49] every extreme point of $\overline{co}G(x)$ is contained in $G(x)$. Thus $\mathcal{E}(F(x)) \subset G(x)$ for every $x \in X$. \square

Theorem 6.2. *Let X be a topological space and Y be a Hausdorff locally convex (linear topological) space. Let F be a map from X to Y . The following are equivalent:*

- (1) F is a minimal cusco map;
- (2) F is nonempty compact, convex valued, F has a closed graph, every extreme function of F is quasicontinuous, subcontinuous and any two extreme functions of F have the same closures of their graphs;
- (3) F is nonempty compact valued, every extreme function f of F is quasicontinuous, subcontinuous and $F(x) = \overline{co}f(x)$ for every $x \in X$.

Proof. (1) \Rightarrow (2) Of course, F has to have nonempty compact, convex values and F has to have a closed graph. Let f be an extreme function of F . We will show that f is quasicontinuous and subcontinuous. Let G be a minimal usco map contained in F (there is a unique minimal usco map contained in F by Theorem 7.3). By Lemma 6.1 we have $\mathcal{E}(F(x)) \subset G(x)$ for every $x \in X$. Since $f(x) \in \mathcal{E}(F(x))$ for every $x \in X$, f is a selection of G . By Theorem 3.5 f must be quasicontinuous, subcontinuous and $\bar{f} = G$. Thus every two extreme functions have to have the same closures of their graphs.

(2) \Rightarrow (3) Let f be an extreme function of F . (Such a function exists, for $F(x)$ is a nonempty compact set for every $x \in X$.) Since f is quasicontinuous and subcontinuous, \bar{f} is a minimal usco map by Theorem 3.5 and $\bar{f} \subset F$. We claim that $F(x) = \overline{co \bar{f}(x)}$ for every $x \in X$.

Suppose there is $(x, y) \in X \times Y$ such that $y \in F(x) \setminus \overline{co \bar{f}(x)}$. Without loss of generality we can suppose that $y \in \mathcal{E}(F(x))$, since by Krein-Milman theorem a compact convex set is the closed convex hull of its extreme points. Since $y \notin \overline{co \bar{f}(x)}$, there are two open and disjoint sets O_1, O_2 in Y such that

$$\overline{co \bar{f}(x)} \subset O_1 \text{ and } y \in O_2.$$

Let $U \in \mathcal{U}(x)$ be such that $\bar{f}(U) \subset O_1$. Let g be an extreme function of F such that $g(x) = y$, a contradiction with the fact that every two extreme functions of F have the same closures of their graphs.

(3) \Rightarrow (1) To prove that F is a minimal cusco map, let f be an extreme function of F . Since f is quasicontinuous and subcontinuous, by Theorem 3.5 \bar{f} is minimal usco and by Lemma 5.1 and Proposition 2.7 in [8] a map $x \rightsquigarrow \overline{co \bar{f}(x)}$ is minimal cusco. Since $F(x) = \overline{co \bar{f}(x)}$ for every $x \in X$, we are done. \square

Let $F \subset X \times \mathbb{R}$ such that $F(x)$ is a nonempty bounded set for every $x \in X$. Then there are two real-valued functions $\sup F$ and $\inf F$ defined on X by $\sup F(x) = \sup\{t \in \mathbb{R} : t \in F(x)\}$ and $\inf F(x) = \inf\{t \in \mathbb{R} : t \in F(x)\}$.

Theorem 6.3. *Let X be a topological space and F be a map from X to \mathbb{R} . The following are equivalent:*

- (1) F is a minimal cusco map;
- (2) F is nonempty compact, convex valued, F has a closed graph, $\sup F$ and $\inf F$ are quasicontinuous, subcontinuous functions and $\overline{\sup F} = \overline{\inf F}$;
- (3) F is nonempty compact valued, $\sup F$ and $\inf F$ are quasicontinuous, subcontinuous functions and $F(x) = co \overline{\sup F}(x) = co \overline{\inf F}(x)$.

Proof. (1) \Rightarrow (2) is clear from the above Theorem. (2) \Rightarrow (3) We will prove that $F(x) = co \overline{\sup F}(x)$ for every $x \in X$. Suppose there is $(x, y) \in X \times \mathbb{R}$ such that $y \in F(x) \setminus co \overline{\sup F}(x)$. Let $\epsilon > 0$ be such that

$$(y - 2\epsilon, y + 2\epsilon) \cap co \overline{\sup F}(x) = \emptyset.$$

The upper semicontinuity of $z \rightsquigarrow co \overline{\sup F}(z)$ at $x \in X$ implies that there is $U \in \mathcal{U}(x)$ such that $co \overline{\sup F}(z) \subset (y + \epsilon, \infty)$ for every $z \in U$. Since $\overline{\sup F} = \overline{\inf F}$ and $\inf F(x) \leq y < y + \epsilon$, we have a contradiction.

(3) \Rightarrow (1) By Theorem 3.5 $\overline{\sup F}$ is minimal usco. By Lemma 5.1 and Proposition 2.7 in [8] the map $x \rightsquigarrow c\overline{\sup F}(x)$ is minimal cusco, so we are done. \square

It is interesting to note that Theorem 2.14 in [8] follows also from Theorem 6.3. In fact, let f be a function from Lemma 2.13 in [8]. Let $H = \overline{f}$ be the closure of the graph of f . Then for Φ in Theorem 2.14 in [8] we have $\Phi(t) = [\inf H(t), \sup H(t)]$. Since $\overline{\inf H} = \overline{f} = \overline{\sup H}$ and $\inf H, \sup H$ are quasicontinuous and subcontinuous, Φ is minimal cusco. It is clear that $H(x) = \{f(x)\}$ at every $x \in G$.

7. MINIMAL USCO AND MINIMAL CUSCO MAPS

In this part we give a characterization of such maps which are minimal usco and minimal cusco simultaneously. We also show that there is a bijection between the space of minimal usco maps and the space of minimal cusco maps.

Let X be a topological space and Y be a Hausdorff locally convex (linear topological) space. Denote by $MU(X, Y)$ the set of all minimal usco maps from X to Y (see part 4) and by $MC(X, Y)$ the set of all minimal cusco maps from X to Y . Of course $MU(X, Y) \cap MC(X, Y) \neq \emptyset$. It follows from the next example that $MU(X, Y) \setminus MC(X, Y) \neq \emptyset$ and also $MC(X, Y) \setminus MU(X, Y) \neq \emptyset$.

Example 7.1. Let $X = [-1, 1]$ with the usual Euclidean topology. Consider the maps F and G from X to \mathbb{R} defined by:

$$F(x) = \begin{cases} 1, & x \in [-1, 0); \\ \{-1, 1\}, & x = 0; \\ -1, & x \in (0, 1]. \end{cases}$$

$$G(x) = \begin{cases} 1, & x \in [-1, 0); \\ [-1, 1], & x = 0; \\ -1, & x \in (0, 1]. \end{cases}$$

Definition 7.2. Let X be a topological space and Y be a Hausdorff locally convex (linear topological) space. We say that f is $*$ -quasicontinuous at x if for every $y \in \overline{c\overline{f}}(x)$, for every $V \in \mathcal{U}(y)$ and every $U \in \mathcal{U}(x)$ there is a nonempty open set $W \subset U$ such that $f(W) \subset V$. If f is $*$ -quasicontinuous at every point of X , we say that f is $*$ -quasicontinuous.

Example 7.3. Consider the function f from \mathbb{R} to \mathbb{R} defined by:

$$f(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0; \\ 0 & x = 0. \end{cases}$$

The function f is not continuous at $x = 0$, but it is $*$ -quasicontinuous at 0.

Theorem 7.4. *Let X be a topological space and Y be a Hausdorff locally convex (linear topological) space. Let f be a function from X to Y . Then f is $*$ -quasicontinuous at x if and only if every selection of \overline{f} is quasicontinuous at x and $f(x) = \overline{c\overline{f}}(x)$.*

It follows from the previous Theorem that every $*$ -quasicontinuous function is quasicontinuous.

Theorem 7.5. *Let X be a topological space and Y be a Hausdorff locally convex (linear topological) space. Let F be a map from X to Y . Then the following are equivalent:*

- (1) $F \in MU(X, Y) \cap MC(X, Y)$;
- (2) *There exists a $*$ -quasicontinuous and subcontinuous function f from X to Y such that $\bar{f} = F$;*
- (3) *Every selection f of F is $*$ -quasicontinuous, subcontinuous and $\bar{f} = F$.*

Proof. (1) \Rightarrow (3) Let f be a selection of F . Since $F \in MU(X, Y)$, by Theorem 3.5 f is quasicontinuous, subcontinuous and $\bar{f} = F$. Since $F \in MC(X, Y)$, $\overline{co}f(x) = F(x)$. So by Theorem 7.4 f is $*$ -quasicontinuous.

(3) \Rightarrow (2) is trivial.

(2) \Rightarrow (1) Let f be a $*$ -quasicontinuous and subcontinuous function from X to Y such that $\bar{f} = F$. Thus f is quasicontinuous and by Theorem 7.4 $F(x) = \bar{f}(x) = \overline{co}f(x)$. By Theorem 3.5 F is minimal usco map and since by Lemma 5.1 F is cusco, by Proposition 2.7 in [8] F is minimal cusco. \square

Denote by $F(X, Y)$ the set of all maps with closed values from a topological space X to a Hausdorff locally convex (linear topological) space Y . Define the function $\varphi : MU(X, Y) \rightarrow F(X, Y)$ as follows: $\varphi(F)(x) = \overline{co}F(x)$.

To study properties of the mapping φ we start with the following result.

Theorem 7.6. *Let X be a topological space and Y be a Hausdorff locally convex (linear topological) space. Let $F : X \rightsquigarrow Y$ be a minimal cusco map. There is a unique minimal usco map contained in F .*

Proof. Let G, H be two minimal usco maps contained in F . It is sufficient to prove that $G(x) \cap H(x) \neq \emptyset$ for every $x \in X$. Then the map $L : X \rightsquigarrow Y$ defined as $L(x) = G(x) \cap H(x)$ for every $x \in X$ is usco and $L \subset G, L \subset H$. Thus $G = L = H$.

By Lemma 5.1 and the minimality of F , $\varphi(G) = \varphi(H) = F$. Now, by the Krein-Milman and Milman theorems we have that

$$\emptyset \neq \mathcal{E}(F(x)) \subset G(x) \cap H(x) \text{ for all } x \in X.$$

\square

Theorem 7.7. *Let X be a topological space and Y be a Hausdorff locally convex (linear topological) space in which the closed convex hull of a compact set is compact. The map φ is bijection from $MU(X, Y)$ to $MC(X, Y)$.*

Proof. Let $F \in MU(X, Y)$. To show that $\varphi(F) \in MC(X, Y)$ note that by Lemma 5.1 the map G defined as $G(x) = \overline{co}F(x)$ for every $x \in X$ is a cusco map and by Proposition 2.7 in [8] G is minimal cusco.

Next we show that φ maps $MU(X, Y)$ onto $MC(X, Y)$. Let $G \in MC(X, Y)$ and let F be a minimal usco map contained in G . By Lemma 5.1 the map $x \rightsquigarrow \overline{co}F(x)$ is a cusco map such that $\overline{co}F(x) \subset G(x)$ for every $x \in X$. Since G is minimal cusco, $G(x) = \overline{co}F(x)$ for every $x \in X$.

Finally, to show that φ is one-to-one, suppose that $F, G \in MU(X, Y)$ and $F \neq G$. Suppose, by way of contradiction, that $\varphi(F) = \varphi(G)$. So by Theorem 7.6 $F = G$, a contradiction. \square

8. TOPOLOGICAL PROPERTIES

In this part we will consider topologies on the space of minimal usco and minimal cusco maps and we will study topological properties of the mapping φ defined in the end of the section 7 and also further important mappings defined on the space of minimal usco maps (see [25, 28]).

Following [29] we will define topologies of pointwise convergence, uniform convergence on compacta and uniform convergence.

Let X be a Hausdorff topological space and (Y, d) be a metric space. As above denote by $F(X, Y)$ the set of all maps with closed values from X to Y .

By $\mathfrak{K}(X)$ and $\mathfrak{F}(X)$ we mean the family of all nonempty compact and finite subsets of X , respectively.

The open d -ball with center $z_0 \in Y$ and radius $\varepsilon > 0$ will be denoted by $S_\varepsilon(z_0)$ and the ε -parallel body $\bigcup_{a \in A} S_\varepsilon(a)$ for a subset A of Y will be denoted by $S_\varepsilon(A)$.

We denote by 2^Y the space of all closed subsets of Y , by $CL(Y)$ the space of all nonempty closed subsets of Y .

If $A \in CL(Y)$, the distance functional $d(\cdot, A) : Y \mapsto [0, \infty)$ is described by the familiar formula

$$d(z, A) = \inf\{d(z, a) : a \in A\}.$$

Let A and B be nonempty subsets of (Y, d) . The excess of A over B with respect to d is defined by the formula

$$e_d(A, B) = \sup\{d(a, B) : a \in A\}.$$

The Hausdorff (extended-valued) metric H_d on 2^Y [3] is defined by

$$H_d(A, B) = \max\{\sup\{d(a, B) : a \in A\}, \sup\{d(b, A) : b \in B\}\},$$

if A and B are nonempty. If $A \neq \emptyset$ take $H_d(A, \emptyset) = H_d(\emptyset, A) = \infty$.

We will often use the following equalities on $CL(Y)$ [3]:

$$H_d(A, B) = \inf\{\varepsilon > 0 : A \subset S_\varepsilon(B) \text{ and } B \subset S_\varepsilon(A)\},$$

$$H_d(A, B) = \max\{e_d(A, B), e_d(B, A)\}.$$

The topology generated by H_d is called the Hausdorff metric topology.

The topology τ_p of pointwise convergence on $F(X, Y)$ is induced by the uniformity \mathfrak{U}_p of pointwise convergence which has a base consisting of sets of the form

$$W(A, \varepsilon) = \{(\Phi, \Psi) : \forall x \in A \ H_d(\Phi(x), \Psi(x)) < \varepsilon\},$$

where $A \in \mathfrak{F}(X)$ and $\varepsilon > 0$. The general τ_p -basic neighborhood of $\Phi \in F(X, Y)$ will be denoted by $W(\Phi, A, \varepsilon)$, i.e. $W(\Phi, A, \varepsilon) = W(A, \varepsilon)[\Phi] = \{\Psi : H_d(\Phi(x), \Psi(x)) < \varepsilon \text{ for every } x \in A\}$. If $A = \{a\}$, we will write $W(\Phi, a, \varepsilon)$ instead of $W(\Phi, \{a\}, \varepsilon)$.

We will define the topology τ_{UC} of uniform convergence on compact sets on $F(X, Y)$. This topology is induced by the uniformity \mathfrak{U}_{UC} which has a base consisting of sets of the form

$$W(K, \varepsilon) = \{(\Phi, \Psi) : \forall x \in K \ H_d(\Phi(x), \Psi(x)) < \varepsilon\},$$

where $K \in \mathfrak{K}(X)$ and $\varepsilon > 0$. The general τ_{UC} -basic neighborhood of $\Phi \in F(X, Y)$ will be denoted by $W(\Phi, K, \varepsilon)$, i.e. $W(\Phi, K, \varepsilon) = W(K, \varepsilon)[\Phi]$.

Finally we will define the topology τ_U of uniform convergence on $F(X, Y)$ [29]. Let ϱ be the (extended-valued) metric on $F(X, Y)$ defined by

$$\varrho(\Phi, \Psi) = \sup\{H_d(\Phi(x), \Psi(x)) : x \in X\},$$

for each $\Phi, \Psi \in F(X, Y)$. Then the topology of uniform convergence for the space $F(X, Y)$ is the topology generated by the metric ϱ .

We will need the following lemma.

Lemma 8.1. *Let Y be a normed linear space. Let A, B be nonempty closed subsets of Y . Then $H_d(\overline{c\bar{o}A}, \overline{c\bar{o}B}) \leq H_d(A, B)$.*

Proof. At first we show that $e_d(\overline{c\bar{o}A}, \overline{c\bar{o}B}) \leq e_d(A, B)$. It is known (see [3] exercise 1.5.3. b), that if C is convex then $e_d(\overline{c\bar{o}A}, C) = e_d(A, C)$. So $e_d(\overline{c\bar{o}A}, \overline{c\bar{o}B}) = e_d(A, \overline{c\bar{o}B})$. Since $B \subset \overline{c\bar{o}B}$ we have that

$$e_d(\overline{c\bar{o}A}, \overline{c\bar{o}B}) = e_d(A, \overline{c\bar{o}B}) \leq e_d(A, B).$$

Similarly we can show that

$$e_d(\overline{c\bar{o}B}, \overline{c\bar{o}A}) = e_d(B, \overline{c\bar{o}A}) \leq e_d(B, A).$$

Since for every $C, D \in CL(Y)$

$$H_d(C, D) = \max\{e_d(C, D), e_d(D, C)\},$$

we are done. \square

Theorem 8.2. *Let X be a topological space and Y be a Banach space. The map φ from $(MU(X, Y), \tau)$ onto $(MC(X, Y), \tau)$ is continuous if τ is one of the following topologies $\tau_p, \tau_{UC}, \tau_U$.*

Proof. The proof follows from the above Lemma. \square

The following example shows that the map φ^{-1} from $(MC([-1, 1]), \tau_p)$ onto $(MU([-1, 1]), \tau_p)$ is not continuous.

Example 8.3. Let $X = [-1, 1]$ with the usual Euclidean topology. Let F and G are maps from Example 7.1. Then $F = \varphi^{-1}(G)$. We claim that φ^{-1} is not continuous at G . For every $n \in \mathbb{Z}^+$ let P_n be the map from $[-1, 1]$ to \mathbb{R} defined by

$$P_n(x) = \begin{cases} 1, & x \in [-1, 0); \\ [-1, 1], & x = 0; \\ \sin \frac{1}{x}, & x \in (0, \frac{2}{(4n-1)\pi}]; \\ -1, & x \in [\frac{2}{(4n-1)\pi}, 1]. \end{cases}$$

It is easy to see that for every $A \in \mathfrak{F}(X)$ and every $\varepsilon > 0$ there exists an $n_0 \in \mathbb{Z}^+$ such that $P_n \in W(G, A, \varepsilon)$ for every $n \geq n_0$. For every $n \in \mathbb{Z}^+$ we have that $H_d(F(0), \varphi^{-1}(P_n)(0)) = 1$. Then for every $n \in \mathbb{Z}^+$ $P_n \notin W(F, 0, \frac{1}{2})$ and so the map φ^{-1} is not continuous at G .

Theorem 8.4. *Let X be a locally compact space and Y be a Banach space. The map φ from $(MU(X, Y), \tau_{UC})$ onto $(MC(X, Y), \tau_{UC})$ is homeomorphism.*

Proof. By Theorem 7.7 φ is a bijection. By Theorem 8.2 it is sufficient to prove that φ^{-1} is continuous. Let $G \in MC(X, Y)$ and $F = \varphi^{-1}(G)$. Let $K \in \mathfrak{K}(X)$ and $\varepsilon > 0$. We show that there exist $K_1 \in \mathfrak{K}(X)$ and $\varepsilon_1 > 0$ such that $\varphi^{-1}(W(G, K_1, \varepsilon_1)) \subset W(F, K, \varepsilon)$. Let $K_1 \in \mathfrak{K}(X)$ be such that $K \subset \text{Int}K_1$. Put $\varepsilon_1 = \frac{\varepsilon}{3}$. Let $H \in W(G, K_1, \varepsilon_1)$ and $x \in K$. We show that $F(x) \subset S_\varepsilon(\varphi^{-1}(H)(x))$. Let $y \in F(x)$. By Proposition 3.6 and Theorem 3.5 for $\frac{\varepsilon}{3}$ and for every $U \in \mathcal{U}(x)$ there exists $x_U \in U \cap \text{Int}K_1$ such that $F(x_U)$ is single-valued and $F(x_U) \in S_{\frac{\varepsilon}{3}}(y)$. From the fact that $F(x_U)$ is single-valued it follows that $G(x_U)$ is single-valued too and consequently $F(x_U) = G(x_U)$. Since $H(x_U) \subset S_{\frac{\varepsilon}{3}}(G(x_U))$ we have that $\varphi^{-1}(H)(x_U) \subset S_{\frac{\varepsilon}{3}}(F(x_U))$ and hence there exists $y_U \in \varphi^{-1}(H)(x_U)$ such that $d(y_U, F(x_U)) < \frac{\varepsilon}{3}$. Hence $d(y, y_U) < \frac{2\varepsilon}{3}$. So there exists a subnet of the net $\{(x_U, y_U) : U \in \mathcal{U}(x)\}$ which converges to a point (x, z) , where $z \in \varphi^{-1}(H)(x)$. So $F(x) \subset S_\varepsilon(\varphi^{-1}(H)(x))$. The inclusion $\varphi^{-1}(H)(x) \subset S_\varepsilon(F(x))$ can be proved similarly. \square

The following Example shows that the condition of local compactness in Theorem 8.4 is essential.

Example 8.5. Let $X = [-1, 1]$ with the topology, where the open sets in X are all subsets of X not containing 0 and all subsets of X containing 0 that have countable complement. Every compact set in X is finite. Thus the topology τ_{UC} on $MU(X, \mathbb{R})$ and $MC(X, \mathbb{R})$ reduces to the topology τ_p . So we can use Example 8.3.

Theorem 8.6. *Let X be a Baire space and Y be a Banach space. The map φ from $(MU(X, Y), \tau_U)$ onto $(MC(X, Y), \tau_U)$ is homeomorphism.*

Proof. The proof is similar to the proof of Theorem 8.4. \square

We will consider also the Vietoris topology V on $MU(X, \mathbb{R})$ and on $MC(X, \mathbb{R})$. First we will consider the Vietoris topology V on the space $CL(X \times \mathbb{R})$ of nonempty closed subsets of $X \times \mathbb{R}$. The basic open subsets of $CL(X \times \mathbb{R})$ in V are the subsets of the form

$$W^+ \cap W_1^- \cap \dots \cap W_n^-,$$

where W, W_1, \dots, W_n are open subsets of $X \times \mathbb{R}$, $W^+ = \{F \in CL(X \times \mathbb{R}) : F \subset W\}$, and each $W_i^- = \{F \in CL(X \times \mathbb{R}) : F \cap W_i \neq \emptyset\}$.

Under the identification of every element of $MU(X, \mathbb{R})$ and $MC(X, \mathbb{R})$ with its graph, we can consider $MU(X, \mathbb{R})$ and $MC(X, \mathbb{R})$ as subsets of $CL(X \times \mathbb{R})$. We will consider the induced Vietoris topology V on $MU(X, \mathbb{R})$ and on $MC(X, \mathbb{R})$.

Theorem 8.7. *Let X be a locally connected space. The map φ from $(MU(X, \mathbb{R}), V)$ onto $(MC(X, \mathbb{R}), V)$ is continuous.*

Proof. Let $F \in MU(X, \mathbb{R})$ and $W^+ \cap W_1^- \cap \dots \cap W_n^-$ be a basic open set in $(MC(X, \mathbb{R}), V)$ such that $\varphi(F) \in W^+ \cap W_1^- \cap \dots \cap W_n^-$.

Let $G = \varphi(F)$. By Lemma 4.1 in [24] there is an open set $H \subset X \times \mathbb{R}$ such that $G \subset H \subset W$ and $H(x)$ is connected for every $x \in X$. Without loss of generality we can also suppose that for every $i = 1, 2, \dots, n$, $W_i \subset H$ and $W_i = U_i \times V_i$, U_i open in X , V_i an open interval in \mathbb{R} .

For every $i \in \{1, 2, \dots, n\}$ we will define an open set \mathcal{H}_i as follows. Let $i \in \{1, 2, \dots, n\}$. Let $(x_i, y_i) \in G \cap W_i$. If $y_i = \inf F(x_i)$ or $y_i = \sup F(x_i)$, we will put $\mathcal{H}_i = W_i^-$. Otherwise, let C_i be a connected set in X such that $x_i \in \text{Int}C_i \subset C_i \subset U_i$ and $\epsilon > 0$ be such that $\inf F(x_i) + \epsilon < y_i < \sup F(x_i) - \epsilon$. Put $\mathcal{H}_i = (\text{Int}C_i \times (\inf F(x_i) - \epsilon, \inf F(x_i) + \epsilon))^- \cap (\text{Int}C_i \times (\sup F(x_i) - \epsilon, \sup F(x_i) + \epsilon))^-$.

It is easy to verify that $L \in MU(X, \mathbb{R}) \cap \mathcal{H}_i$ implies that $\varphi(L) \in \mathcal{H}_i$. Since $\varphi(L)$ is upper semi-continuous set-valued map with connected values, $\varphi(L)(\text{Int}C_i)$ must be a connected set ([3], Proposition 6.2.12); i.e. $y_i \in \varphi(L)(\text{Int}C_i)$. Thus $\varphi(L) \in \cap W_i^-$.

Put $\mathcal{G} = H^+ \cap \mathcal{H}_1 \cap \dots \cap \mathcal{H}_n$. Then $F \in \mathcal{G}$ and $\varphi(S) \in W^+ \cap W_1^- \cap \dots \cap W_n^-$ for every $S \in \mathcal{G}$. \square

The following Example shows that the condition of local connectedness in the above Theorem is essential.

Example 8.8. Let $X = [-1, 1] \setminus \{\frac{1}{n} : n \in \mathbb{Z}^+\}$ with the usual Euclidean topology. Consider the maps F and G from X to \mathbb{R} defined by

$$F(x) = \begin{cases} 1, & x \in X \cap [-1, 0); \\ \{-1, 1\}, & x = 0; \\ -1, & x \in X \cap (0, 1]. \end{cases}$$

$$G(x) = \begin{cases} 1, & x \in X \cap [-1, 0); \\ [-1, 1], & x = 0; \\ -1, & x \in X \cap (0, 1]. \end{cases}$$

Then $G = \varphi(F)$ and we claim that φ is not continuous at F . For every $n \in \mathbb{Z}^+$ let f_n be the function from X to \mathbb{R} defined by

$$f_n(x) = \begin{cases} 1, & x \in X \cap [-1, \frac{1}{n}); \\ -1, & x \in X \cap (\frac{1}{n}, 1]. \end{cases}$$

We have that $f_n = \varphi(f_n)$ for every $n \in \mathbb{Z}^+$. The sequence $\{f_n : n \in \mathbb{Z}^+\}$ converges in $(MU(X, \mathbb{R}), V)$ to F , but $\{f_n : n \in \mathbb{Z}^+\}$ does not converge to G in $(MC(X, \mathbb{R}), V)$.

The following example shows that the map φ^{-1} from $(MC([-1, 1], \mathbb{R}), V)$ onto $(MU([-1, 1], \mathbb{R}), V)$ is not continuous.

Example 8.9. Let $X = [-1, 1]$ with the usual Euclidean topology. Let F, G be maps from Example 7.3. Then $F = \varphi^{-1}(G)$ and we claim that φ^{-1} is not continuous at G . For every $n \in \mathbb{Z}^+$ let g_n be the function from $[-1, 1]$ to \mathbb{R} defined by

$$g_n(x) = \begin{cases} 1, & x \in [-1, 0]; \\ 1 - 2nx, & x \in (0, \frac{1}{n}); \\ -1, & x \in [\frac{1}{n}, 1]. \end{cases}$$

Evidently $g_n = \varphi^{-1}(g_n)$ for every $n \in \mathbb{Z}^+$. It is easy to see that the sequence $\{g_n : n \in \mathbb{Z}^+\}$ converges in $(MC([-1, 1], \mathbb{R}), V)$ to G , but $\{g_n : n \in \mathbb{Z}^+\}$ does not converges to F in $(MU([-1, 1], \mathbb{R}), V)$.

In the last part of the paper we will prove further results concerning interesting bijections and homeomorphisms from the space $MU(X, \mathbb{R})$.

Let F be a usco map from a topological space X to \mathbb{R} . Define the function f^F as follows:

$$f^F(x) = \sup\{y : y \in F(x)\}.$$

Then of course f^F is a selection of F and f^F is upper semi-continuous and locally bounded.

If F is a minimal usco map from a topological space X to \mathbb{R} , then by Theorem 3.5 f^F is also quasicontinuous.

In what follows denote by $Q(X, \mathbb{R})$ the space of all quasicontinuous real-valued functions defined on a topological space X and by $Q^*(X, \mathbb{R})$ ($A^*(X, \mathbb{R})$) the set of all locally bounded elements of $Q(X, \mathbb{R})$ ($A(X, \mathbb{R})$). By $USC(X, \mathbb{R})$ we denote the set of all upper semi-continuous functions.

Define a mapping $\Omega : MU(X, \mathbb{R}) \rightarrow Q^*(X, \mathbb{R}) \cap USC(X, \mathbb{R})$ by $\Omega(F) = f^F$.

Proposition 8.10. *The mapping $\Omega : MU(X, \mathbb{R}) \rightarrow Q^*(X, \mathbb{R}) \cap USC(X, \mathbb{R})$ is a bijection and $\Omega(D^*(X, \mathbb{R})) = A^*(X, \mathbb{R}) \cap USC(X, \mathbb{R})$.*

Proof. To show that Ω is one-to-one, let $F, G \in MU(X, \mathbb{R})$ be such that $F \neq G$ i.e., $\overline{f^F} \neq \overline{f^G}$. Without loss of generality, we can suppose that there is a point $(u, v) \in \overline{f^F}$ such that $(u, v) \notin \overline{f^G}$. Then there exists an open neighborhood U of (u, v) such that $U \cap \overline{f^G} = \emptyset$. There must exist a point $w \in X$ such that $(w, f^F(w)) \in U$ and so $(w, f^F(w)) \notin \overline{f^G}$. Since f^G is a selection of G , we have that $f^F \neq f^G$.

To show that the mapping Ω is onto, let $f \in Q^*(X, \mathbb{R}) \cap USC(X, \mathbb{R})$. By Theorem 3.5, \overline{f} is a minimal usco map. The upper semi-continuity of f guarantees the equality $f(x) = \sup\{y : (x, y) \in \overline{f}\}$ for every $x \in X$; i.e., $\Omega(\overline{f}) = f$. Thus Ω is onto.

If $F \in D^*(X, \mathbb{R})$, then by Proposition 4.2, $f^F \in A(X, \mathbb{R})$. Of course, f^F is upper semi-continuous and locally bounded; i.e., $\Omega(D^*(X, \mathbb{R})) \subset A^*(X, \mathbb{R}) \cap USC(X, \mathbb{R})$. Now we prove the equality. Let $f \in A^*(X, \mathbb{R}) \cap USC(X, \mathbb{R})$. By Proposition 4.2, $\overline{f} \in D(X, \mathbb{R})$ and since f is locally bounded $\overline{f} \in D^*(X, \mathbb{R})$. The upper semi-continuity of f guarantees the equality $f(x) = \sup\{y : (x, y) \in \overline{f}\}$ for every $x \in X$. Thus $\Omega(\overline{f}) = f$. \square

Of course, for a usco map $F : X \rightsquigarrow \mathbb{R}$ we can define also

$$f_F(x) = \inf\{y : y \in F(x)\}.$$

It is easy to see that f_F is lower semi-continuous selection of F , which is locally bounded. We can give a similar result for lower semi-continuous functions as we gave above for upper semi-continuous functions. The result for lower semi-continuous functions is dual.

Denote by $LSC(X, \mathbb{R})$ the set of all lower semi-continuous functions, and define the mapping $\mathcal{S} : MU(X, \mathbb{R}) \rightarrow Q^*(X, \mathbb{R}) \cap LSC(X, \mathbb{R})$ by $\mathcal{S}(F) = f_F$.

Proposition 8.11. *The mapping $\mathcal{S} : MU(X, \mathbb{R}) \rightarrow Q^*(X, \mathbb{R}) \cap LSC(X, \mathbb{R})$ is a bijection, and $\mathcal{S}(D^*(X, \mathbb{R})) = A^*(X, \mathbb{R}) \cap LSC(X, \mathbb{R})$.*

Remark 8.12. It is easy to see that if A and B are nonempty compact subsets of \mathbb{R} , then $d(\sup A, \sup B) \leq H_d(A, B)$.

Proposition 8.13. *Let X be a topological space. Then the mapping Ω from $(MU(X, \mathbb{R}), \mathfrak{U}_p)$ onto $(Q^*(X, \mathbb{R}) \cap USC(X, \mathbb{R}), \mathfrak{U}_p)$ is uniformly continuous.*

Proof. The proof follows from Remark 8.12. □

The following example shows that even τ_{UC} -convergence in $Q^*(X, \mathbb{R}) \cap USC(X, \mathbb{R})$ does not imply the convergence in $(MU(X, \mathbb{R}), \tau_p)$.

Example 8.14. Let W be the set of all ordinal numbers less than or equal to the first uncountable ordinal number ω_1 with the usual topology. Let L be the set of all limit ordinal numbers different from ω_1 . Put $X = W \setminus L$ and equip X with the induced topology from W .

If λ is a nonlimit number, there are a unique integer $I(\lambda) \in Z^+$ and a limit number β such that $\lambda = \beta + I(\lambda)$.

For every $n \in Z^+$ put $C_n = \{\lambda \in X \setminus \omega_1 : I(\lambda) = n\}$. Then $\omega_1 \in \overline{C_n}$ for every $n \in Z^+$. Further, for every $n \in Z^+$, let $f_n \in Q^*(X, \mathbb{R}) \cap USC(X, \mathbb{R})$ be defined as follows: $f_n(x) = 0$ if $x \in C_n$ and $f_n(x) = 1$ otherwise. It is easy to verify that $\{f_n\}$ τ_{UC} -converges to the function f identically equal to 1. However, the sequence $\{\Omega^{-1}(f_n)\}$ fails to converge to $\Omega^{-1}(f)$ in $(MU(X, \mathbb{R}), \tau_p)$ since $\Omega^{-1}(f_n) = \overline{f_n}$ takes the value $\{0, 1\}$ at ω_1 for every $n \in Z^+$ and $\Omega^{-1}(f)(\omega_1) = \{1\}$.

Theorem 8.15. *Let X be a topological space. Then the spaces $(MU(X, \mathbb{R}), \varrho)$ and $(Q^*(X, \mathbb{R}) \cap USC(X, \mathbb{R}), \varrho)$ are uniformly isomorphic. Also the spaces $(D^*(X, \mathbb{R}), \varrho)$ and $(A^*(X, \mathbb{R}) \cap USC(X, \mathbb{R}), \varrho)$ are uniformly isomorphic.*

Proof. As we proved above, the mapping Ω from $MU(X, \mathbb{R})$ to $Q^*(X, \mathbb{R}) \cap USC(X, \mathbb{R})$ is a bijection. By Remark 8.12 we have that $\Omega : (MU(X, \mathbb{R}), \varrho) \rightarrow (Q^*(X, \mathbb{R}) \cap USC(X, \mathbb{R}), \varrho)$ is uniformly continuous. To prove that also Ω^{-1} is uniformly continuous, it is sufficient to show that if for $f, g \in Q^*(X, \mathbb{R}) \cap USC(X, \mathbb{R})$, $d(f(x), g(x)) < \varepsilon$ for every $x \in X$, then $H_d(\overline{f}(x), \overline{g}(x)) \leq \varepsilon$ for every $x \in X$.

Suppose that this is not true. Then there exists an $x_0 \in X$ such that $H_d(\overline{f}(x_0), \overline{g}(x_0)) > \varepsilon$. There is an $r \in \overline{f}(x_0)$ such that $d(r, \overline{g}(x_0)) > \varepsilon$, or there is an $s \in \overline{g}(x_0)$ such that $d(s, \overline{f}(x_0)) > \varepsilon$. Suppose the first case occurs; the proof of the second one is analogous. Put $\beta = d(r, \overline{g}(x_0)) - \varepsilon$. Let $\{x_\sigma : \sigma \in \Sigma\}$ be a net in X converging to x_0 , such that the net $\{f(x_\sigma) : \sigma \in \Sigma\}$ converges to r . Then for $\frac{\beta}{4}$ there is $\sigma_0 \in \Sigma$ such that $f(x_\sigma) \in S_{\frac{\beta}{4}}(r)$ for all $\sigma > \sigma_0$. The

upper semi-continuity of \bar{g} at x_0 implies that there is a $U \in \mathcal{U}(x_0)$ such that $\bar{g}(x) \subset S_{\frac{\varepsilon}{4}}(\bar{g}(x_0))$ for all $x \in U$. Let $\sigma \in \Sigma$ be such that $\sigma > \sigma_0$ and $x_\sigma \in U$. Then of course $d(f(x_\sigma), g(x_\sigma)) > \varepsilon$, a contradiction.

Concerning the proof of the second statement of the theorem, by Proposition 8.10 we have that $\Omega(D^*(X, \mathbb{R})) = A^*(X, \mathbb{R}) \cap USC(X, \mathbb{R})$ and by the above we know that $\Omega : (MU(X, \mathbb{R}), \varrho) \rightarrow (Q^*(X, \mathbb{R}) \cap USC(X, \mathbb{R}), \varrho)$ is uniformly isomorphic. Thus, also the restriction of Ω on $D^*(X, \mathbb{R})$ to $A^*(X, \mathbb{R}) \cap USC(X, \mathbb{R})$ is uniformly isomorphic. \square

Theorem 8.16. *Let X be a locally compact topological space. The spaces $(MU(X, \mathbb{R}), \mathfrak{U}_{UC})$ and $(Q^*(X, \mathbb{R}) \cap USC(X, \mathbb{R}), \mathfrak{U}_{UC})$ are uniformly isomorphic. Also the spaces $(D^*(X, \mathbb{R}), \mathfrak{U}_{UC})$ and $(A^*(X, \mathbb{R}) \cap USC(X, \mathbb{R}), \mathfrak{U}_{UC})$ are uniformly isomorphic.*

Proof. As we proved above, the mapping Ω from $MU(X, \mathbb{R})$ to $Q^*(X, \mathbb{R}) \cap USC(X, \mathbb{R})$ is a bijection. By Remark 8.12 we have that $\Omega : (MU(X, \mathbb{R}), \mathfrak{U}_{UC}) \rightarrow (Q^*(X, \mathbb{R}) \cap USC(X, \mathbb{R}), \mathfrak{U}_{UC})$ is uniformly continuous. To prove that also Ω^{-1} is uniformly continuous let $K \in \mathfrak{K}(X)$ and $\varepsilon > 0$. The local compactness of X implies that there is an open set G in X such that $K \subset G$ and \bar{G} is compact. Let $f, g \in Q^*(X, \mathbb{R}) \cap USC(X, \mathbb{R})$ be such that $d(f(x), g(x)) < \varepsilon$ for every $x \in \bar{G}$. To prove that $H_d(\bar{f}, \bar{g}) \leq \varepsilon$ for every $x \in K$, we can use a similar idea as in the proof of Theorem 8.15. \square

Topological properties and the cardinal function properties of character, pseudo character, density, weight, net weight and cellularity on $(D^*(X, \mathbb{R}), \tau_p)$, $(D^*(X, \mathbb{R}), \tau_{UC})$ and $(D^*(X, \mathbb{R}), \tau_U)$ were studied in papers [22, 23, 25, 30, 31, 36].

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