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STATISTICAL ERGODIC THEOREMS FOR MARKOV SEMIGROUPS IN SPACES WITH MIXED NORM

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Abstract. This paper describes the semigroups generated by the Markov processes in spaces with mixed norm and proves analogues of statistical ergodic theorems for such semigroups.

1. Introduction and preliminaries

Semigroups generated by the Markov process in some function spaces plays an important role in the theory of probability, economics, mathematical biology, molecular physics, quantum physics and mechanics, and so on. In monographs and textbooks I.I.Gikhman and A.V. Skorokhod [5], K. Yosida [16] M. Loeve [14], V. Feller [4] semigroups generated by the Markov process in L_p spaces and spaces of continuous functions and ergodic theorems for such semigroups studied in detail.

Markov process in space of measurable functions of variables generated by linear operators depends on measurable parameters. Linear operators depend on measurable parameters, that is random linear operators, appear in different areas of mathematics such as functional analysis, probability theory, mathematical physics and so on.

In [15], A.V. Skorokhod considered the theory of random linear operators in Hilbert spaces and investigated convergence of series of random operators. Also, spectral theory of random linear operators was described and applications to equations with random operators was given.

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In [3] G.P. Butsan investigated semigroups of operators on Hilbert spaces, depending on measurable parameters.

The partial integral operators, that is random integral operators in ideal spaces of measurable functions were considered in [1],[7]. In [10] K.K. Kydaybergenov showed that a partial integral operator acting on Hilbert-Kaplansky module $L_0[L_2]$ is a cyclic compact operator on $L_0[L_2]$.

In early 1990s A.E. Gutman [6] introduced measurable Banach bundles with lifting, axiomatically. He also established that every Banach-Kantorovich space over the ring of measurable functions can be expressed as measurable bundle of Banach spaces.

Banach spaces with mixed norm were introduced in [11] and operators in this spaces were studied in [12]. An ideal Banach space with mixed norm was considered in [2].

In this paper we consider the Markov process with invariant measure, generated by a contraction semigroup in a function space with mixed norm. We prove statistical ergodic theorems for such semigroup.

Let (Ω, Σ, μ) be a measure space with finite measure μ, and $L_0(\Omega)$ be the algebra of all measurable functions on Ω. Here the almost everywhere equal functions are identified.

Let $\mathcal{U}$ be a linear space over the real field $\mathbb{R}$. By $\|\cdot\|$ we denote an $L_0(\Omega)$-valued norm on $\mathcal{U}$. Then the pair $(\mathcal{U}, \|\cdot\|)$ is called a *lattice-normed space (LNS) over* $L_0(\Omega)$. An LNS, $\mathcal{U}$, is said to be *d-decomposable* if for every $x \in \mathcal{U}$ and the decomposition $\|x\| = f + g$ with f and g disjoint positive elements in $L_0(\Omega)$ there exist $y, z \in \mathcal{U}$ such that $x = y + z$ with $\|y\| = f$, $\|z\| = g$.

Suppose that $(\mathcal{U}, \|\cdot\|)$ is an LNS over $L_0(\Omega)$. A net $\{x_\alpha\}$ of elements of $\mathcal{U}$ is said to be (bo)-*converging* to $x \in E$ (in this case we write $x = (bo)\text{-}\lim x_\alpha$), if the net $\{\|x_\alpha - x\|\}$ (o)-converges to zero in $L_0(\Omega)$ (written as $(o)\text{-}\lim \|x_\alpha - x\| = 0$). A net $\{x_\alpha\}_{\alpha \in A}$ is called (bo)-*fundamental* if $(x_\alpha - x_\beta)_{(\alpha,\beta)\in A\times A}$ (bo)-converges to zero.

An LNS in which every (bo)-fundamental net (bo)-converges is called (bo)-*complete*. A *Banach-Kantorovich space (BKS) over* $L_0(\Omega)$ is a (bo)-complete d-decomposable LNS over $L_0(\Omega)$. It is well known (see [12],[13]) that every BKS E over $L_0(\Omega)$ admits an $L_0(\Omega)$-module structure such that $\|fx\| = |f| \cdot \|x\|$ for every $x \in E$, $f \in L_0(\Omega)$, where $|f|$ is the modulus of a function $f \in L_0(\Omega)$.

A set $B \subset \mathcal{U}$ is called *bounded*, if the set $\{\|x\| : x \in B\}$ is order bounded in L_0.

An operator $T : \mathcal{U} \to \mathcal{U}$ is called L_0-linear, if $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for all $\alpha, \beta \in L_0, x, y \in \mathcal{U}$. An L_0-linear operator T is called L_0–bounded, if for any bounded set B in $\mathcal{U}$, the set $T(B)$ is bounded in $\mathcal{U}$. For an L_0-bounded operator T we will put

$$\|T\| = \sup\{\|T(x)\| : \|x\| \leq \mathbf{1}\},$$

where $\mathbf{1}$ is the identity element in L_0.

An ideal space on (Ω, Σ, μ) is a linear subset E in L_0 such that

$$(x \in L_0, y \in E; |x| \leq |y|) \Rightarrow (x \in E)$$

i.e., for every function $y \in E$, the set E contains its modulus and each function with smaller modulus. A norm on an ideal space E is said to be monotone if

$(x, y \in E; |x| \leq |y|) \Rightarrow \|x\|_E \leq \|y\|_E$. A sequence $\{b_n\}$ is said to be order convergent (or o-convergent) to b if there is a sequence $\{a_n\}$ in E satisfying $a_n \downarrow 0$ and $|b_n - b| \leq a_n$ for all n, written as $b_n \xrightarrow{(o)} b$ or $b = (o) - \lim\limits_n b_n$.

A Banach ideal space on (Ω, Σ, μ) is defined as an ideal space E endowed with a monotone norm making E into a Banach space.

A norm on a Banach ideal space E is said to be o-continuous or satisfies condition (A) in E if $(x_n \downarrow 0) \Rightarrow \|x_n\|_E \to 0$.

An even continuous convex function $M : R \to [0, \infty)$ is called an N-function, if $\lim\limits_{t\to 0} \frac{M(t)}{t} = 0$ and $\lim\limits_{t\to\infty} \frac{M(t)}{t} = \infty$. Every N-function M has the form $M(t) = \int\limits_0^{|t|} p(s)ds$, where $p(t)$ is a nondecreasing function that is positive for $t > 0$, right-continuous for $t \geq 0$, satisfying $p(0) = 0$ and $\lim\limits_{t\to\infty} p(t) = \infty$ (see [10]). Put $q(s) := \sup\{t : p(t) \leq s\}$, $s \geq 0$. The function $N(t) := \int\limits_0^{|t|} q(s)ds$ is an N-function which is called the complementary N-function to M (see [8]).

The Orlicz space $L_M(\Omega, \Sigma, \mu)$ is the set of measurable functions $f \in L_0$ such that there exists a $\lambda > 0$ for which $\int M(\frac{f}{\lambda})d\mu < \infty$, with norm given by

$$\|f\| = \inf\{\lambda > 0 : \int M\left(\frac{f}{\lambda}\right) d\mu \leq 1\}.$$

Orlicz spaces are examples of Banach ideal spaces. If the N-function M meets the $\triangle_2$-condition, i.e., $M(2t) \leq kM(t)$ for all $t \geq t_0 \geq 0$, for some $k > 0$ and $t_0 \geq 0$, then $L_M(\Omega, \Sigma, \mu)$ has condition (A).

Let E be an ideal space of measurable functions on (Ω, Σ, μ), $(S, \mathcal{B}, m)$ a measure space with complete measure m, and F a Banach ideal space of measurable functions on $(S, \mathcal{B}, m)$. Denote by $E[F]$ the space of all measurable functions K on $\Omega \times S$ satisfying the following two conditions:

a) the equivalence class of functions $x \mapsto K(\omega, x)$ belongs to F for almost all $\omega \in \Omega$;

b) the function $\omega \mapsto \|K(\omega, \cdot)\|_F$ is measurable and its equivalence class $\|K\|$ belongs to E.

It is known that $E[F]$ is a Banach-Kantorovich space over E with norm $\|K\|$ and $E[F]$ is an ideal space of measurable functions on $\Omega \times S$ (see.[12], pp. 67).

If E is a Banach ideal space, then the equality $\|K\|_{E[F]} =\| \, \|K\| \, \|_E$ makes $E[F]$ a Banach space named a space with mixed norm.

Let $L_0(\Omega, F) := L_0(\Omega, \Sigma, \mu, F)$ be the space of equivalence classes of all Bochner μ-measurable vector valued functions from Ω to F. As usual, the vector valued functions are equivalent if they accept the same values almost everywhere on Ω. If $\widetilde{u} \in L_0(\Omega, F)$ is the equivalence class of a measurable vector valued function $u : \Omega \to F$, then the scalar function $\omega \mapsto \|u(\omega)\|_F$ is measurable and the corresponding equivalence class, denoted by $\|\widetilde{u}\|$, belongs to $L_0(\Omega)$.

Put

$$E(F) := \{\widetilde{u} \in L_0(\Omega, F) : \|\widetilde{u}\| \in E\}.$$

Then $(E(F), \|\cdot\|)$ is a Banach-Kantorovich space over E (see [11], pp. 95).

Theorem 1.1. *([12], pp. 68). Let F be a Banach ideal space on $(S, \mathcal{B}, m)$ with order semicontinuous norm. For any measurable vector $f : \Omega \to F$, there exists a unique, up to equivalence measurable function $K(\omega, x)$ on $\Omega \times S$, such that for almost all $x \in S$ the equality $K(\omega, x) = (f(x))(\omega)$ holds for almost all $\omega \in \Omega$. Comparison of $\widetilde{f} \mapsto \widetilde{K}$ is an isometric isomorphism of $E(F)$ to the closed subspace and sublattice $E[F]$.*

Theorem 1.2. *([12], pp. 68). If the measure μ is not purely atomic, the lattice - normed spaces $E(F)$ and $E[F]$ are the same if and only if F is a Banach ideal space with order continuous norm.*

Let $F = L_p(S, \mathcal{B}, m)$ be a Banach ideal space with norm $\|f\|_p = \left(\int\limits_S |f(x)|^p dm\right)^{\frac{1}{p}}$, $p \in [1; +\infty)$.

Theorem 1.3. *([16], pp. 523). If $P(t, x, A)$ is a Markov process with an invariant measure m, the equality*

$$(T_t f)(x) = \int\limits_S f(y) P(t, x, dy), \tag{1.1}$$

defines a positive bounded linear operator on $L_p(S, \mathcal{B}, m)$, such that $T_{t+s} = T_t T_s$ $(t, s > 0)$, with $T_t \cdot \mathbf{1} = \mathbf{1}$ and $\|T_t f\|_p \le \|f\|_p$ for all $f \in L_p(S, \mathcal{B}, m)$, $p \in [1; +\infty)$.

Theorem 1.4. *([16], pp. 532) (Statistical ergodic theorem). If $P(t, x, A)$ is a Markov process with invariant measure m, then for any function $f \in L_p(S, \mathcal{B}, m)$ there exists a limit $f^* \in L_p(S, \mathcal{B}, m)$ of sequence $\frac{1}{n} \sum\limits_{k=1}^{n} T_k f$, with $T_1 f^* = f^*$.*

2. Markov semigroups in space $E[L_p]$

In this section we consider the Markov semigroups in $E[L_p]$. First we consider the case $E = L_0$ and $F = L_p(S, \mathcal{B}, m)$, $p \ge 1$.

Let H be the subset of $L_0[L_p]$ of all functions of the form $\sum\limits_{i=1}^{n} \chi_{A_i} f_i$, where $A_i \in \sum$, $A_i \bigcap A_j = 0$ when $i \ne j$, $f_i \in L_p(S, \mathcal{B}, m)$.

Lemma 2.1. *The set H is (bo)-dense in $L_0[L_p]$.*

Proof. By Theorem 1.2 we have $L_0[L_p] = L_0(\Omega, L_p)$.

If $K \in L_0[L_p]$, then there exists a sequence $K_n \in L_0(\Omega, L_p)$ of the form $\sum\limits_{i=1}^{n} \chi_{A_i}(\omega) f_i$, where $f_i \in L_p$, such that $\|K_n - K\|(\omega) = \|K_n(\omega) - K(\omega)\|_{L_p} \to 0$ for almost all $\omega \in \Omega$ as $n \to \infty$. This means that $\|K_n - K\| \xrightarrow{(o)} 0$ in L_0, i.e. the set $\{\sum\limits_{i=1}^{n} \chi_{A_i}(\omega) f_i\}$ is (bo)-dense in $L_0[L_p]$. $\square$

Theorem 2.2. *Let $P(t,x,A)$ be a Markov process with invariant measure m. Then*

$$T_t(K)(\omega,x)=\int\limits_S K(\omega,y)P(t,x,dy)$$

defines L_0-linear and L_0- bounded positive linear operator in $L_0[L_p]$ such that $T_{t+s}=T_tT_s$, with $T_t\mathbf{1}=\mathbf{1}$ and $\|T_tK\|\leq\|K\|$.

Proof. As $K(\omega,x)\in L_p$ for almost all $\omega\in\Omega$, by Theorem 1.3 we have

$$\left\|\int\limits_S K(\omega,y)P(t,x,dy)\right\|_{L_p}\leq\|K(\omega,x)\|_{L_p} \tag{2.1}$$

for almost all $\omega\in\Omega$.

We will show that $\int\limits_S K(\omega,y)P(t,x,dy)\in L_0[L_p]$ for any $K\in L_0[L_p]$. First, let $K\in H$. Then $T_t(K)(\omega,x)=\int\limits_S K(\omega,y)P(t,x,dy)=\sum\limits_{i=1}^{n}\chi_{A_i}(\omega)\int\limits_S f_i(y)P(t,x,dy)\in L_0(\Omega,L_p)$ and $\|T_tK\|\leq\|K\|$.

In addition, $T_t(gK)=gT_t(K)$ for any simple function $g=\sum\limits_{i=1}^{m}\lambda_i\chi_{A_i}\in L_0$ and any $K\in H$.

Let now $K\in L_0[L_p]$. By Lemma 2.1 there exists a sequence $\{K_n\}\subset H$ such that $\|K_n-K\|\stackrel{(o)}{\to}0$ in L_0 as $n\to\infty$. Hence $\|K_n-K_m\|\stackrel{(o)}{\to}0$ in L_0 as $n,m\to\infty$. Since $\|T_tK_n-T_tK_m\|\leq\|K_n-K_m\|\stackrel{(o)}{\to}0$, as $n,m\to\infty$, then the sequence $\{T_tK_n\}$ is (bo)-fundamental in $L_0[L_p]$. Since $L_0[L_p]$ is (bo)-complete, then $\{T_tK_n\}$ (bo)-converges in $L_0[L_p]$. The limit $(bo)-\lim\limits_{n\to\infty}T_tK_n$ is denoted by T_tK, i.e.

$$T_tK=(bo)-\lim_{n\to\infty}T_tK_n=(bo)-\lim_{n\to\infty}\int\limits_S K_n(\omega,y)P(t,x,dy). \tag{2.2}$$

T_t is a continuation of T_t from H to $L_0[L_p]$. It is clear that T_t is well defined and $\|T_tK\|\leq\|K\|$ for any $K\in L_0[L_p]$.

We will show that T_t is L_0-linear. In fact, let $\alpha\in L_0$ and let $\{\alpha_n\}$ be a sequence of simple functions, such that $\alpha_n\stackrel{(o)}{\to}\alpha$. Then $\|T_t(\alpha K)-T_t(\alpha_nK)\|\leq\|\alpha K-\alpha_nK\|==|\alpha-\alpha_n|\|K\|\stackrel{(o)}{\to}0$. This means that $T_t(\alpha K)=(bo)-\lim\limits_{n\to\infty}T_t(\alpha_nK)$ for any $K\in L_0[L_p]$ and $\alpha\in L_0$. So

$$T_t(\alpha K)=(bo)-\lim_{n\to\infty}T_t(\alpha_nK)=(bo)-\lim_{n\to\infty}\alpha_nT_t(K)=\alpha T_t(K),$$

i.e. T_t is L_0-linear.

Now we show that $T_t(K)(\omega,x)=\int\limits_S K(\omega,y)P(t,x,dy)$ for any $K\in L_0[L_p]$. Since $K\in L_0(\Omega,L_p)=L_0[L_p]$ then, by Lemma 2.1 there is a sequence $\{K_n\}\subset H$ such that $\|K_n-K\|\stackrel{(o)}{\to}0$, i.e. $\|K_n(\omega,x)-K(\omega,x)\|_{L_p}\to 0$ for almost all $\omega\in\Omega$.

As

$$\int_S K_n(\omega, y) P(t, x, dy) = \sum_{i=1}^{n} \chi_{A_i}(\omega) \int_S f_i(y) P(t, x, dy),$$

then $\int_S K_n(\omega, y)P(t, x, dy)$ is a simple measurable vector valued function with values in L_p, i.e. $\int_S K_n(\omega, y)P(t, x, dy) \in H$. Using (2.1), we have

$$\| \int_S K_n(\omega, y)P(t, x, dy) - \int_S K(\omega, y)P(t, x, dy)\|_{L_p} \leq \|K_n(\omega, x) - K(\omega, x)\|_{L_p} \to 0$$

for almost all $\omega \in \Omega$.

Consequently, for almost all $\omega \in \Omega$, $\int_S K(\omega, y)P(t, x, dy)$ is Bochner measurable, i.e.

$$\int_S K(\omega, y)P(t, x, dy) \in L_0(\Omega, L_p) = L_0[L_p],$$

with

$$\int_S K(\omega, y)P(t, x, dy) = (bo) - \lim_{n\to\infty} \int_S K_n(\omega, y)P(t, x, dy). \tag{2.3}$$

From (2.2) and (2.3) it follows that $T_t(K)(\omega, x) = \int_S K(\omega, y)P(t, x, dy)$ for almost all $\omega \in \Omega$. In the equation $T_t(K)(\omega, x) = \sum_{i=1}^{n} \chi_{A_i}(\omega) \int_S f_i(y)P(t, x, dy)$ for $K \in H$, put $\int_S f(y)P(t, x, dy) = S_t(f)$. Then $T_t(K)(\omega, x) = \sum_{i=1}^{n} \chi_{A_i}(\omega)S_t(f)(x)$.

Since for the operator $S_t(f)(x) = \int_S f(y)P(t, x, dy)$ (see (1.1)) we have $S_{t+s} = S_tS_s$ $(t, s > 0)$ (see [16], p. 523), then $T_{t+s}(K) = T_tT_s(K)$ for all $K \in H$. Hence we obtain that the family of operators $\{T_t\}$ is a semigroup in Banach-Kantorovich space $L_0[L_p]$. Finally, from the properties of the integral, we have $T_t(K) \geq 0$ for $K \geq 0$ and $T_t\mathbf{1} = \mathbf{1}$. □

Remark 2.3. From Theorem 2.2 it follows that for any ideal space E of measurable functions on (Ω, Σ, μ) the inclusion $T_t(E[L_p]) \subset E[L_p]$ is true. In in this case $\|T_t\|_{E[L_p]\to E[L_p]} \leq \mathbf{1}$ for all $t \in \mathbb{R}$.

Definition 2.4. Such a semigroups of linear operators $\{T_t\}$ is said to be a Markov semigroup in the space $E[L_p]$.

Let us consider some examples of Markov semigroups.

Example 2.5. Let Ω be a countable set. Then $L_0[L_p] = s[L_p]$, where s is the space of sequences. From the definition of the space $L_0[L_p]$ it follows that

$$s[L_p] = \left\{ K = (K_1(x), K_2(x), ...) : \int_S |K_i(x)|^p dm < \infty, i = 1, 2, ... \right\}$$

and

$$\|K\| = (\|K_1\|_{L_p}, \|K_2\|_{L_p}, ...) \in s,$$

i.e. $s[L_p]$ is a countably normed space. We define a family of linear operators $T_t : s[L_p] \to s[L_p]$ by $T_tK = (K_1(x+t), K_2(x+t), ...)$ for $t \geq 0$. Obviously, $\|T_t\| \leq \mathbf{1}$, $T_0 = I, T_{t+s} = T_tT_s$, i.e. the family $\{T_t\}$ is a semigroup of contractions in $s[L_p]$.

Example 2.6. Assume that a family of linear operators $\{T_t\}$ in $s[L_p]$ is defined by the equality

$$T_tK = \Big(\int\limits_S K_1(y)P(t,x,dy), \int\limits_S K_2(y)P(t,x,dy), ... \Big).$$

It is clear that we have the semigroup property $T_{t+s} = T_tT_s$, $(t, s > 0)$.

From the relation

$$\|T_tK\| = \Big(\Big\| \int\limits_S K_1(y)P(t,x,dy) \Big\|_{L_p}, \Big\| \int\limits_S K_2(y)P(t,x,dy) \Big\|_{L_p}, ... \Big) \leq$$
$$\leq (\|K_1\|_{L_p}, \|K_2\|_{L_p}, ...) = \|K\|$$

it follows that T_t is a contraction.

3. Statistical ergodic theorems for Markov semigroups in space with mixed norm

In this section we will prove statistical ergodic theorems for Markov semigroups in Banach-Kantorovich spaces $E[L_p]$ with respect to E valued norm and in spaces $E[L_p]$ with respect to mixed numerical norm, where E is a Banach ideal space.

Theorem 3.1. *For any function $K \in L_0[L_p]$, $p \geq 1$, the limit*

$$(bo) - \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} T_kK = K^*$$

exists in $L_0[L_p]$ such that $T_1K^* = K^*$.

Proof. We set $A_t(f) = \int\limits_S f(y)P(t,x,dy)$. If $K \in L_0[L_p]$, then $K(\omega, x) \in L_p(S, m)$ for almost all $\omega \in \Omega$ and by Theorem 1.4, the limit $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=1}^{n} A_k(K(\omega, x)) = K^*_\omega(x)$ exists in $L_p(S, m)$ for almost all $\omega \in \Omega$. We show that the function of two variables defined by $K^*(\omega, x) = K^*_\omega(x)$ belongs to $L_0[L_p] = L_0(\Omega, L_p)$ and $S_n(K) = \frac{1}{n} \sum\limits_{k=1}^{n} T_k(K)$ is (bo) convergent to K^* in $L_0[L_p]$.

Since $S_n(K) = \frac{1}{n} \sum\limits_{k=1}^{n} T_k(K) \in L_0(\Omega, L_p)$ and

$$\|S_n(K)(\omega, \cdot) - K^*(\omega, \cdot)\|_{L_p} = \|\frac{1}{n} \sum_{k=1}^{n} A_k(K(\omega, \cdot)) - K^*(\omega, \cdot)\|_{L_p} \to 0$$

as $n \to \infty$ for almost all $\omega \in \Omega$, then $K^*(\omega, x)$ is the limit of a sequence of Bochner measurable functions $S_n(K)$. Hence $K^* \in L_0[L_p]$ and in this case $(bo)-$

$\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=1}^{n} T_k K = K^*$. From the equality $A_1(K_\omega^*(x)) = K_\omega^*(x)$ for almost all $\omega \in \Omega$ it follows that $T_1 K^* = K^*$. □

Theorem 3.2. *For any function $K \in E[L_p]$ the limit*

$$(bo) - \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} T_k K = K^*,$$

exists in $E[L_p]$ and $T_1 K^* = K^*$.

Proof. If $K \in E[L_p]$, then $K \in L_0(\Omega, L_p)$ and by Theorem 3.1 there exists $K^* \in L_0(\Omega, L_p)$, such that $\{\frac{1}{n} \sum\limits_{k=1}^{n} T_k K\}$ (bo)-converges to K^* in $L_0[L_p]$.

As $\|S_n(K)\| = \|\frac{1}{n} \sum\limits_{k=1}^{n} T_k K\| \leq \frac{1}{n} \sum\limits_{k=1}^{n} \|T_k K\| \leq \|K\| \in E$ and $\|S_n(K)\| \overset{(o)}{\to} \|K^*\|$ in L_0 we have $\|K^*\| \leq \|K\|$ and $K^* \in E[L_p]$. Since

$$\|S_n(K) - K^*\| \leq 2\|K\|$$

for all n we get $\sup\limits_n \|S_n(K) - K^*\| \in E$. Hence $\|S_n(K) - K^*\| \overset{(o)}{\to} 0$ in E i.e. $S_n(K) \overset{(bo)}{\to} K^*$ in $E[L_p]$. □

We illustrate the statistical ergodic theorem in the following example.

Example 3.3. Let $\{T_t\}$ be a semigroup of linear operators in $s[L_p]$, which was considered in Example 2.6. Then

$$\frac{1}{n} \sum_{k=1}^{n} T_k K = \left(\frac{1}{n} \sum_{k=1}^{n} \int_S K_1(y) P(k, x, dy), \frac{1}{n} \sum_{k=1}^{n} \int_S K_2(y) P(k, x, dy), \ldots \right).$$

The operators $A_k K_i = \int\limits_S K_i(y) P(k, x, dy)$ map $L_p(S, m)$ to $L_p(S, m)$ and we have $A_{k+m} = A_k A_m$.

By Theorem 1.4 $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=1}^{n} A_k K_i = K_i^*$ exists in in $L_p(S, m)$ and $A_1 K_i^* = K_i^*$, for any $i = 1, 2, \ldots$.

By K^* we denote the element $(K_1^*, K_2^*, \ldots)$ in $s[L_p]$. Since

$$\left\| \frac{1}{n} \sum_{k=1}^{n} T_k K - K^* \right\| =$$

$$= \left(\left\| \frac{1}{n} \sum_{k=1}^{n} A_k K_1 - K_1^* \right\|_{L_p(S,m)}, \left\| \frac{1}{n} \sum_{k=1}^{n} A_k K_2 - K_2^* \right\|_{L_p(S,m)}, \ldots \right) \overset{(o)}{\to} 0$$

as $n \to \infty$, then $(bo) - \lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=1}^{n} T_k K = K^*$. As $A_1 K_i^* = K_i^*$ for all $i = 1, 2, \ldots$, then

$$T_1 K = (A_1 K_1^*, A_1 K_2^*, \ldots) = (K_1^*, K_2^*, \ldots) = K^*.$$

Theorem 3.4. *If a Banach ideal space E has condition (A), then for any function $K \in E[L_p]$,*

$$\lim_{n\to\infty} \|\frac{1}{n}\sum_{k=1}^{n} T_k K - K^*\|_{E[L_p]} = 0,$$

and $T_1 K^* = K^*$.

Proof. By the Theorem 3.2 we have that for any $K \in E[L_p]$ the (bo)–limit

$$(bo) - \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} T_k K = K^*,$$

exists in $E[L_p]$ i.e.

$$\|\frac{1}{n}\sum_{k=1}^{n} T_k K - K^*\|_{L_p} \stackrel{(o)}{\to} 0$$

in E. That means that there is a sequence $\{a_n\}$ in E satisfying $a_n \downarrow 0$ and $\|\frac{1}{n}\sum\limits_{k=1}^{n} T_k K - K^*\|_{L_p} \leq a_n$ for all n. Since a Banach ideal space E has condition (A) we get

$$\|\frac{1}{n}\sum_{k=1}^{n} T_k K - K^*\|_{E[L_p]} \leq \|a_n\|_E \to 0$$

as $n \to \infty$. □

Corollary 3.5. *If the N-function M meets the $\triangle_2$-condition for any function $K \in L_M[L_p]$,*

$$\lim_{n\to\infty} \|\frac{1}{n}\sum_{k=1}^{n} T_k K - K^*\|_{L_M[L_p]} = 0,$$

$L_M[L_p]$ and $T_1 K^* = K^*$.

Proof. As the Banach ideal space $E = L_M$ has condition (A) by Theorem 3.4

$$\lim_{n\to\infty} \|\frac{1}{n}\sum_{k=1}^{n} T_k K - K^*\|_{L_M[L_p]} \to 0$$

as $n \to \infty$. □

Corollary 3.6. *For any function $K \in L_q[L_p]$, $q \geq 1$*

$$\lim_{n\to\infty} \|\frac{1}{n}\sum_{k=1}^{n} T_k K - K^*\|_{L_q[L_p]} = 0$$

in $L_q[L_p]$ and $T_1 K^* = K^*$.

Proof. As the Banach ideal space $E = L_q$ has condition (A) by Theorem 3.4

$$\lim_{n\to\infty} \|\frac{1}{n}\sum_{k=1}^{n} T_k K - K^*\|_{L_q[L_p]} \to 0$$

as $n \to \infty$. □

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