Khayyam J. Math. 1 (2015), no. 2, 174–184
DOI: 10.22034/kjm.2015.13164

Khayyam Journal of Mathematics
emis.de/journals/KJM
kjm-math.org

EXPONENTIAL STABILITY AND INSTABILITY IN MULTIPLE DELAYS DIFFERENCE EQUATIONS

S. ALMUTAIRI[1], M. ALSHAMMARI[1] AND Y. RAFFOUL[1*]

This paper is dedicated to Father Hanna Al Bacha

Communicated by H.R. Ebrahimi Vishki

Abstract. We use Lyapunov functionals and obtain sufficient conditions that guarantee exponential stability of the zero solution of the difference equation with multiple delays

$$x(t+1) = a(t)x(t) + \sum_{j=1}^{k} b_j(t)x(t-h_j).$$

The novelty of our work is the relaxation of the condition $|a(t)| < 1$, in spite of the presence of multiple delays. Using a slightly modified Lyapunov functional, we obtain necessary conditions for the unboundedness of all solutions and for the instability of the zero solution. We provide an example as an application to our obtained results.

1. Introduction

In this paper we consider the scalar linear difference equation with multiple delays

$$x(t+1) = a(t)x(t) + \sum_{j=1}^{k} b_j(t)x(t-h_j) \tag{1.1}$$

where for $j = 1, ..., k$, $0 < h_j \leq h^*$ for some positive constant h^* and $a, b_j : \mathbb{Z}^+ \to \mathbb{R}$. Throughout the paper $\mathbb{R}$ and $\mathbb{Z}^+$ denote the set of real numbers and the set of positive integers, respectively. We will use Lyapunov functionals and obtain some inequalities regarding the solutions of (1.1) from which we can deduce exponential asymptotic stability of the zero solution. Also, we will provide a criteria for the

Date: Received: 09 July 2014; Accepted: 30 July 2015
[3] Corresponding author.
2010 *Mathematics Subject Classification.* Primary 34D20; Secondary 34D40, 34K20.
Key words and phrases. Exponential stability, instability, Lyapunov functional.

instability of the zero solution of (1.1) by means of Lyapunov functional.
In [8], Raffoul used a similar notion and obtained results regarding exponential stability and instability for the zero solution of Volterra difference system

$$x(t+1) = a(t)x(t) + \sum_{s=t-r}^{t-1} b(t,s)x(s).$$

Also, in [9] Raffoul considered the single delay difference equation

$$x(t+1) = a(t)x(t) + b(t)x(t-h).$$

It is clear that our considered system is different from the one in [8] and of more general nature of the system that was considered in [9], and as a consequence more suitable Lyapunov functional will have to be constructed and dealt carefully with. In [2] the author make use of the sign of eigenvalues and obtained results concerning stability of the the zero solution of an equation that is similar to (1.1). Our approach is totally different and in addition we obtain stronger stability. For more recent results on boundedness, stability and periodicity in difference systems, we refer to [10]-[12] and the results therein.

The novelty of this research is that the constructed Lyapunov functional will allow us to offset the size of $a(t)$ by the coefficients $\sum_{j=1}^{k} b_j(t+h_j)$. As a direct consequence, we are able to allow $|a(t)| \geq 1$.

Let $\psi : [-h^*, 0] \to (-\infty, \infty)$ be a given bounded initial function with

$$||\psi|| = \max_{-h \leq s \leq 0} |\psi(s)|.$$

It should cause no confusion to denote the norm of a function $\varphi : [-h^*, \infty) \to (-\infty, \infty)$ with

$$||\varphi|| = \sup_{-h \leq s < \infty} |\varphi(s)|.$$

The notation x_t means that $x_t(\tau) = x(t+\tau), \tau \in [-h^*, 0]$ as long as $x(t+\tau)$ is defined. Thus, x_t is a function mapping an interval $[-h^*, 0]$ into $\mathbb{R}$. We say $x(t) \equiv x(t, t_0, \psi)$ is a solution of (1.1) if $x(t)$ satisfies (1.1) for $t \geq t_0$ and $x_{t_0} = x(t_0 + s) = \psi(s)$, $s \in [-h^*, 0]$.
In preparation for our main results, we notice that (1.1) is equivalent to

$$\triangle x(t) = [a(t) - 1 + \sum_{j=1}^{k} b_j(t)(t+h_j)]x(t) - \sum_{j=1}^{k} \triangle_t \sum_{s=t-h_j}^{t-1} b_j(s+h_j)x(s). \quad (1.2)$$

We have the following definition.

Definition 1.1. The zero solution of (1.1) is said to be exponentially stable if any solution $x(t, t_0, \psi)$ of (1.1) satisfies

$$|x(t, t_0, \psi)| \leq C\Big(||\psi||, t_0\Big)\zeta^{\gamma(t-t_0)}, \quad \text{for all } t \geq t_0,$$

where ζ is constant with $0 < \zeta < 1$, $C : \mathbb{R}^+ \times \mathbb{Z}^+ \to \mathbb{R}^+$, and γ is a positive constant. The zero solution of (1.1) is said to be uniformly exponentially stable if C is independent of t_0.

2. Exponential Stability

Now we turn our attention to the exponential stability of the zero solution of equation (1.1). For simplicity, we let

$$Q(t) = a(t) - 1 + \sum_{j=1}^{k} b_j(t+h_j).$$

We begin with the following proposition.

Proposition 2.1. *Suppose x is a solution of (1.2), then*

$$\begin{aligned}\triangle\Big[x(t) + \sum_{j=1}^{k}\sum_{s=t-h_j}^{t-1} b_j(s+h_j)x(s)\Big]^2 &= [Q^2(t)+Q(t)]x^2(t)\\ &+ Q(t)[x^2(t) + 2x(t)\sum_{j=1}^{k}\sum_{s=t-h_j}^{t-1} b_j(s+h_j)x(s)]\big].\end{aligned}$$

Proof. We use that fact that if $u(t)$ is a sequence, then $\triangle u^2(t) = u(t+1)\triangle u(t) + u(t)\triangle u(t)$. For more on the calculus of difference equations we refer the reader to [4] and [7]. Let $x(t) = x(t, t_0, \psi)$ be a solution of (1.2), then

$$\begin{aligned}\triangle\Big[x(t) &+ \sum_{j=1}^{k}\sum_{s=t-h_j}^{t-1} b_j(s+h_j)x(s)\Big]^2\\ &= \big[x(t+1) + \sum_{j=1}^{k}\sum_{s=t-h_j+1}^{t} b_j(s+h_j)x(s)\big]\triangle\big[x(t) + \sum_{j=1}^{k}\sum_{s=t-h_j}^{t-1} b_j(s+h_j)x(s)\big]\\ &+ \big[x(t) + \sum_{j=1}^{k}\sum_{s=t-h_j}^{t-1} b_j(s+h_j)x(s)\big]\triangle\big[x(t) + \sum_{j=1}^{k}\sum_{s=t-h_j}^{t-1} b_j(s+h_j)x(s)\big].\end{aligned}$$

Now using (1.1), we have

$$\begin{aligned}x(t+1) &+ \sum_{j=1}^{k}\sum_{s=t-h_j+1}^{t} b_j(s+h_j)x(s) = a(t)x(t) + \sum_{j=1}^{k} b_j(t)x(t-h_j)\\ &- \sum_{j=1}^{k} b_j(t)x(t-h_j) + \sum_{j=1}^{k}\sum_{s=t-h_j}^{t} b_j(s+h_j)x(s)\\ &= a(t)x(t) + \sum_{j=1}^{k}\sum_{s=t-h_j}^{t-1} b_j(s+h_j)x(s) + \sum_{j=1}^{k} b_j(t+h_j)x(t)\\ &= \big(Q(t)+1\big)x(t) + \sum_{j=1}^{k}\sum_{s=t-h_j}^{t-1} b_j(s+h_j)x(s).\end{aligned}$$

Thus,

$$\begin{aligned}\triangle\Big[x(t)+\sum_{j=1}^{k}\sum_{s=t-h_j}^{t-1}b_j(s+h_j)x(s)\Big]^2 &= \Big[\big(Q(t)+1\big)x(t)\\ &+ \sum_{j=1}^{k}\sum_{s=t-h_j}^{t-1}b_j(s+h_j)x(s)\Big]Q(t)x(t)\\ &+ \Big[x(t)+\sum_{j=1}^{k}\sum_{s=t-h_j}^{t-1}b_j(s+h_j)x(s)\Big]Q(t)x(t),\end{aligned}$$

and hence the result follows. □

Lemma 2.2. *Assume for* $\delta > 0$,

$$-\frac{\delta}{(1+\delta)h^*} \le Q(t) \le -\delta h^* \sum_{j=1}^{k} b_j^2(t+h_j) - Q^2(t), \tag{2.1}$$

hold. If

$$\begin{aligned} V(t) &= \big[x(t)+\sum_{j=1}^{k}\sum_{s=t-h_j}^{t-1}b_j(s+h_j)x(s)\big]^2\\ &+ \delta\sum_{j=1}^{k}\sum_{s=-h_j}^{-1}\sum_{z=t+s}^{t-1}b_j^2(z+h_j)x^2(z) \end{aligned}\tag{2.2}$$

then, along the solutions of (1.2) we have $\triangle V(t) \le Q(t)V(t)$.

Proof. First we note that due to condition (2.1), $Q(t) < 0$ for all $t \ge 0$. Define $V(t)$ by (2.2). Then along solutions of (1.2) we have by using Proposition 1 that

$$\begin{aligned}\triangle V(t) &= \triangle\Big[x(t)+\sum_{j=1}^{k}\sum_{s=t-h_j}^{t-1}b_j(s+h_j)x(s)\Big]^2\\ &+ \delta\sum_{j=1}^{k}\sum_{s=-h_j}^{-1}\triangle_t\big(\sum_{z=t+s}^{t-1}b_j^2(z+h_j)x^2(z)\big)\\ &= \big[Q(t)x(t)+2x(t)+2\sum_{j=1}^{k}\sum_{s=t-h_j}^{t-1}b_j(s+h_j)x(s)\big]Q(t)x(t)\\ &+ \delta\Big[\sum_{j=1}^{k}h_jb_j^2(t+h_j)x^2(t)-\sum_{j=1}^{k}\sum_{s=-h_j}^{-1}b_j^2(t+s+h_j)x^2(t+s)\Big]\\ &= [Q^2(t)+Q(t)]x^2(t)+Q(t)[x^2(t)+2x(t)\sum_{j=1}^{k}\sum_{s=t-h_j}^{t-1}b_j(s+h_j)x(s)]\\ &+ \delta\Big[\sum_{j=1}^{k}h_jb_j^2(t+h_j)x^2(t)-\sum_{j=1}^{k}\sum_{s=-h_j}^{-1}b_j^2(t+s+h_j)x^2(t+s)\Big].\end{aligned}$$

Or,

$$\begin{aligned}\triangle V(t) &= [Q^2(t)+Q(t)]x^2(t)+Q(t)V(t)-Q(t)\Big[\sum_{j=1}^{k}\sum_{s=t-h_j}^{t-1} b_j(s+h_j)x(s)\Big]^2\\ &+ \delta\Big[\sum_{j=1}^{k} h_j b_j^2(t+h_j)x^2(t)-\sum_{j=1}^{k}\sum_{s=-h_j}^{-1} b_j^2(t+s+h_j)x^2(t+s)\Big]\\ &- Q(t)\delta\sum_{j=1}^{k}\sum_{s=-h_j}^{-1}\sum_{z=t+s}^{t-1} b_j^2(z+h_j)x^2(z) \qquad (2.3)\end{aligned}$$

In what to follow we perform some calculations to simplify (2.3). First, if we let $u = s+t$, then

$$\begin{aligned}-\delta\sum_{j=1}^{k}\sum_{s=-h_j}^{-1} b_j^2(t+s+h_j)x^2(t+s) &= -\delta\sum_{j=1}^{k}\sum_{u=t-h_j}^{t-1}[b_j^2(u+h_j)x^2(u)]\\ &= -\delta\sum_{j=1}^{k}\sum_{s=t-h_j}^{t-1}[b_j^2(s+h_j)x^2(s)].\end{aligned}$$

Using Holder's inequality, we have

$$\begin{aligned}-Q(t)\Big[\sum_{j=1}^{k}\sum_{s=t-h_j}^{t-1} b_j(s+h_j)x(s)\Big]^2 &\le -Q(t)\sum_{j=1}^{k}1^2\Big[\sum_{s=t-h_j}^{t-1} b_j(s+h_j)x(s)\Big]^2\\ &\le -Q(t)\sum_{j=1}^{k}1^2\sum_{s=t-h_j}^{t-1}1^2\sum_{s=t-h_j}^{t-1} b_j^2(s+h_j)x^2(s)\\ &= -Q(t)\sum_{j=1}^{k}h_j\sum_{s=t-h_j}^{t-1} b_j^2(s+h_j)x^2(s)\\ &\le -Q(t)h^*\sum_{j=1}^{k}\sum_{s=t-h_j}^{t-1} b_j^2(s+h_j)x^2(s). \qquad (2.4)\end{aligned}$$

One more term to reduce.

$$\begin{aligned}-Q(t)\delta\sum_{j=1}^{k}\sum_{s=-h_j}^{-1}\sum_{z=t+s}^{t-1} b_j^2(z+h_j)x^2(z) &\le -Q(t)\delta\sum_{j=1}^{k}\sum_{s=-h_j}^{-1}\sum_{z=t-h_j}^{t-1} b_j^2(z+h_j)x^2(z)\\ &= -Q(t)\delta\sum_{j=1}^{k}\sum_{l=-h_j}^{-1}\sum_{s=t-h_j}^{t-1} b_j^2(s+h_j)x^2(s)\\ &\le -Q(t)\delta h^*\sum_{j=1}^{k}\sum_{s=t-h_j}^{t-1} b_j^2(s+h_j)x^2(s).\end{aligned}$$

Substituting the above three inequalities into (2.3) and by invoking (2.1) we arrive at

$$\begin{aligned}\triangle V(t) &\leq Q(t)V(t) + \big[Q^2(t) + Q(t) + \delta h^* \sum_{j=1}^{k} b_j^2(t+h_j)\big]x^2(t)\\ &- \big[Q(t)h^* + Q(t)h^*\delta + \delta\big]\sum_{j=1}^{k}\sum_{s=t-h_j}^{t-1} b_j^2(s+h_j)x^2(s)\\ &\leq Q(t)V(t). \end{aligned} \tag{2.5}$$

□

Theorem 2.3. *Assume the hypothesis of Lemma 1. Then any solution $x(t) = x(t, t_0, \psi)$ of (1.1) satisfies the inequality*

$$|x(t)| \leq \sqrt{\frac{\delta + h^*}{\delta} V(t_0) \prod_{s=t_0}^{t-1} \big(a(s) + \sum_{j=1}^{k} b_j(s+h_j)\big)} \tag{2.6}$$

for $t \geq t_0$.

Proof. We changing the order of summation to get

$$\begin{aligned}\delta \sum_{j=1}^{k}\sum_{s=-h_j}^{-1}\sum_{z=t+s}^{t-1} b_j^2(z+h_j)x^2(z) &= \delta \sum_{j=1}^{k}\sum_{z=t-h_j}^{t-1}\sum_{s=-h_j}^{z-t} b_j^2(z+h_j)x^2(z)\\ &= \delta \sum_{j=1}^{k}\sum_{z=t-h_j}^{t-1} b_j^2(z+h_j)x^2(z)(z-t+h_j+1)\\ &\geq \delta \sum_{j=1}^{k}\sum_{z=t-h_j}^{t-1} b_j^2(z+h_j)x^2(z),\end{aligned}$$

where we have used the fact that when $t-h_j \leq z \leq t-1$ then $1 \leq z-t+h+1 \leq h$. By a similar argument as in (2.4) we have

$$\big(\sum_{j=1}^{k}\sum_{z=t-h_j}^{t-1} b_j(z+h_j)x(z)\big)^2 \leq h^* \sum_{j=1}^{k}\sum_{z=t-h_j}^{t-1} b_j^2(z+h_j)x^2(z).$$

Thus, the above two inequalities imply that In a similar fashion as in (2.4) we arrive at

$$\delta \sum_{j=1}^{k}\sum_{s=-h_j}^{-1}\sum_{z=t+s}^{t-1} b_j^2(z+h_j)x^2(z) \geq \frac{\delta}{h^*}\big(\sum_{j=1}^{k}\sum_{z=t-h_j}^{t-1} b_j(z+h_j)x(z)\big)^2.$$

Let $V(t)$ be given by (2.2). Then

$$V(t) = \big[x(t) + \sum_{j=1}^{k}\sum_{z=t-h_j}^{t-1} b_j(z+h_j)x(z)\big]^2 + \delta\big[\sum_{j=1}^{k}\sum_{s=-h_j}^{-1}\sum_{z=t+s}^{t-1} b_j^2(z+h_j)x^2(z)\big]$$

$$\geq \big[x(t) + \sum_{j=1}^{k}\sum_{z=t-h_j}^{t-1} b_j(z+h_j)x(z)\big]^2 + \frac{\delta}{h^*}\sum_{j=1}^{k}\Big(\sum_{z=t-h_j}^{t-1} b_j(z+h_j)x(z)\Big)^2$$

$$= \frac{\delta}{h^*+\delta}x^2(t) + \big[\sqrt{\frac{h^*}{h^*+\delta}}x(t) + \sqrt{\frac{h^*+\delta}{h^*}}\sum_{j=1}^{k}\sum_{z=t-h_j}^{t-1} b(z+h_j)x(z)\big]^2$$

$$\geq \frac{\delta}{h^*+\delta}x^2(t)$$

Consequently,

$$\frac{\delta}{h^*+\delta}x^2(t) \leq V(t).$$

From (2.5) we get

$$V(t) \leq V(t_0)\prod_{s=t_0}^{t-1}\big(a(s) + \sum_{j=1}^{k} b_j(s+h_j)\big).$$

Substituting the above $V(t)$ into the inequality

$$x^2(t) \leq \frac{h^*+\delta}{\delta}V(t).$$

□

For the next corollary, we observe that due to condition (2.1) there exists a positive constant $0 < \alpha < 1$ such that $|\sum_{j=1}^{k} b_j(t+h_j) + a(t)| < \alpha < 1$.

Corollary 2.4. *Assume the hypothesis of Theorem 2.1. Then the zero solution of (1.1) is exponentially stable.*

Proof. From inequality (2.6) we have that

$$|x(t)| \leq \sqrt{\frac{\delta+h^*}{\delta}V(t_0)\prod_{s=t_0}^{t-1}\big(a(s) + \sum_{j=1}^{k} b_j(s+h_j)\big)}$$

$$\leq \sqrt{\frac{\delta+h^*}{\delta}V(t_0)\alpha^{t-t_0}} \tag{2.7}$$

for $t \geq t_0$. The proof is complete since $\alpha \in (0,1)$. □

Next we give a simple example to show that condition (2.1) can be easily verified and moreover, we take $|a(t)| > 1$.

Example 1. Let $a = 1.2, b_1 = -0.2,\ b_2 = -0.088, h^* = 2,$ and $\delta = 0.5$. Then one can easily verify that (2.1) is satisfied. Hence the zero solution of the difference equation with multiple delays

$$x(t+1) = 1.2\,x(t) - 0.2\,x(t-1) - 0.088\,x(t-2).$$

is exponentially stable.
It is worth mentioning that in both papers [6] and [10] it was assumed that

$$\prod_{s=0}^{t-1} a(s) \to 0, \text{ as } t \to \infty$$

for the asymptotic stability. Of course our $a = 1.2$ does not satisfy such a condition, and yet we concluded exponential stability.

Remark 2.5. If for a positive constant M we have

$$V(t_0) \leq M, \text{ for all } t_0 \geq 0,$$

then the zero solution of (1.1) is uniformly exponentially stable. This follows from the exponential inequality (2.7).

So we end this paper by giving a criteria for instability via Lyapunov functional.

3. Unbounded Solutions and Criteria For Instability

In this section, we use a non-negative definite Lyapunov functional and obtain criteria that can be easily applied to test for unboundedness of solutions and instability of the zero solution of (1.1).

Theorem 3.1. *Let $H > h^*$ be a constant. Assume $Q(t) > 0$ such that*

$$Q^2(t) + Q(t) - H\sum_{j=1}^{k} b_j^2(t+h_j) \geq 0. \tag{3.1}$$

If

$$\begin{aligned} V(t) &= \Big[x(t) + \sum_{j=1}^{k}\sum_{s=t-h}^{t-1} b_j(s+h_j)x(s)\Big]^2 \\ &- H\sum_{j=1}^{k}\sum_{s=t-h_j}^{t-1} b_j^2(s+h_j)x^2(s) \end{aligned} \tag{3.2}$$

then, along the solutions of (1.1) we have

$$\triangle V(t) \geq Q(t)V(t).$$

Proof. Let $x(t) = x(t, t_0, \psi)$ be a solution of (1.1) and define $V(t)$ by (3.2). Since the calculation is similar to the one in lemma 1, we arrive at

$$\begin{aligned}\triangle V(t) &= Q(t)V(t) + \big(Q^2(t) + Q(t) - H\sum_{j=1}^{k} b_j^2(t+h_j)\big)x^2(t)\\ &+ H\sum_{j=1}^{k} b_j^2(t)x^2(t-h_j) - Q(t)\big(\sum_{j=1}^{k}\sum_{s=t-h_j}^{t-1} b_j(s+h_j)x(s)\big)^2\\ &+ HQ(t)\sum_{j=1}^{k}\sum_{s=t-h_j}^{t-1} b_j^2(s+h_j)x^2(s)\\ &\geq Q(t)V(t) \qquad (3.3)\end{aligned}$$

where we have used (3.1) and

$$\begin{aligned}\big(\sum_{j=1}^{k}\sum_{s=t-h_j}^{t-1} b_j(s+h_j)x(s)\big)^2 &\leq h^*\sum_{j=1}^{k}\sum_{s=t-h_j}^{t-1} b_j^2(s+h_j)x^2(s) \qquad (3.4)\\ &\leq H\sum_{j=1}^{k}\sum_{s=t-h_j}^{t-1} b_j^2(s+h_j)x^2(s)\end{aligned}$$

and (3.1). This completes the proof. □

Theorem 3.2. *Suppose hypothesis of Theorem 3.1 hold. Then all solutions of* (1.1) *are unbounded and its zero solution is unstable*

Proof. From (3.3) we have

$$V(t) \geq V(t_0)\prod_{s=t_0}^{t-1}\big(a(s) + \sum_{j=1}^{k} b_j(s+h_j)\big). \qquad (3.5)$$

Let $V(t)$ be given by (3.2). Then

$$\begin{aligned}V(t) &= x^2(t) + 2[x(t)\sum_{j=1}^{k}\sum_{s=t-h_j}^{t-1} b_j(s+h_j)x(s)] + \Big[\sum_{j=1}^{k}\sum_{s=t-h_j}^{t-1} b_j(s+h_j)x(s)\Big]^2\\ &- H\sum_{j=1}^{k}\sum_{s=t-h_j}^{t-1} b_j^2(s+h_j)x^2(s). \qquad (3.6)\end{aligned}$$

Let $\beta = H - h^*$. Then from

$$\left(\frac{\sqrt{h^*}}{\sqrt{\beta}}a - \frac{\sqrt{\beta}}{\sqrt{h^*}}b\right)^2 \geq 0,$$

we have

$$2ab \leq \frac{h^*}{\beta}a^2 + \frac{\beta}{h^*}b^2.$$

With this in mind we arrive at,

$$\begin{aligned} 2[x(t)\sum_{j=1}^{k}\sum_{s=t-h_j}^{t-1} b_j(s+h_j)x(s)] &\leq 2|x(t)||\sum_{s=t-h_j}^{t-1} b_j(s+h_j)x(s)| \\ &\leq \frac{h^*}{\beta}x^2(t)+\frac{\beta}{h^*}\Big[\sum_{j=1}^{k}\sum_{s=t-h_j}^{t-1} b_j(s+h_j)x(s)\Big]^2 \\ &\leq \frac{h^*}{\beta}x^2(t)+\frac{\beta}{h^*}h^*\sum_{j=1}^{k}\sum_{s=t-h_j}^{t-1} b_j^2(s+h_j)x^2(s). \end{aligned}$$

A substitution of the above inequality and (3.4) into (3.6) yields,

$$\begin{aligned} V(t) &\leq x^2(t)+\frac{h^*}{\beta}x^2(t)+(\beta+h^*-H)\sum_{j=1}^{k}\sum_{s=t-h_j}^{t-1} b_j^2(s+h_j)x^2(s) \\ &= \frac{\beta+h^*}{\beta}x^2(t) \\ &= \frac{H}{H-h^*}x^2(t) \end{aligned}$$

Using inequality (3.5), we get

$$\begin{aligned} |x(t)| &\geq \sqrt{\frac{H-h^*}{H}}\,V^{1/2}(t) \\ &= \sqrt{\frac{H-h^*}{H}}\,V^{1/2}(t_0)\Big(\prod_{s=t_0}^{t-1}\big(a(s)+\sum_{j=1}^{k}(b_j(s+h_j)\big)\Big)^{\frac{1}{2}}. \end{aligned}$$

Since $Q(t)>0$, there exists a positive constant $\alpha>1$ such that $b_j(t+s)+a(t)>\alpha>1$. Thus we have from the above inequality that

$$|x(t)|\geq\sqrt{\frac{H-h^*}{H}}\,V^{1/2}(t_0)\alpha^{\frac{t-t_0}{2}}\to\infty,\ \text{as } t\to\infty.$$

This completes the proof. □

References

1. L. Berezansky and E. Braverman, *Exponential stability of difference equations with several delays: Recursive approach*, Adv. Difference. Equ. **2009** (2009), Article ID 104310, 13 pages.
2. M. Buslowicz, *Simple stability conditions for linear positive discrete-time systems with delays*, Bull. Pol. Acad. Sci. Techn. **56** (2008), no. 4, 325-328.
3. H.A. El-Morshedy, *New explicit global asymptotic stability criteria for higher order difference equations*, J. Math. Anal. Appl. **336** (2007), no. 1, 262–276.
4. S. Elaydi, *An Introduction to Difference Equations*, Springer Verlag, New York, 3rd Edition, 2005.
5. M. Islam and Y.N. Raffoul, *Exponential stability in nonlinear difference equations*, J. Difference Equ. Appl. **9** (2003), no. 9, 819–825.
6. M. Islam and E. Yankson, *Boundedness and stability in nonlinear delay difference equations employing fixed point theory*, Electron. J. Qual. Theory Differ. Equ. **26** (2005), 19 pages.

7. W. Kelley and A. Peterson, *Difference Equations: An Introduction with Applications*, 2nd Edition, Academic Press, New York, 2001.
8. C. Kublik and Y. Raffoul, *Lyapunov functionals that lead to exponential stability and instability in finite delay Volterra difference equations*, Acta Math. Vietnam. 14 November 2014, 13 pages. DOI 10.1007/s40306-014-0098-4.
9. Y.N. Raffoul, *Inequalities that lead to exponential stability and instability in delay difference equations*, J. Inequal. Pure Appl. Math **10** (2009), no. 3, Article 70, 9 pages.
10. Y.N. Raffoul, *Stability and periodicity in completly discrete delay equations*, J. Math. Anal. Appl. **324** (2006), no. 2, 1356–1362.
11. Y.N. Raffoul, *Qualitative analysis of solutions of functional difference equations using fixed point theory*, Commun. Korean Math. Soc. **29** (2014), no. 1, 195–204.
12. Y.N. Raffoul, *Discrete population models with asymptotically constant or periodic solutions*, Int. J. Difference Equ. **6** (2011), no. 2, 143–152 .

[1] Department of Mathematics, University of Dayton, Dayton, OH 45469-2316 USA;

E-mail address: `almutairis5@udayton.edu; alshammarim6@udayton.edu; yraffoul1@udayton.edu.`