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LIPSCHITZ TENSOR PRODUCT

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ABSTRACT. Inspired by ideas of R. Schatten in his celebrated monograph [23] on a theory of cross-spaces, we introduce the notion of a Lipschitz tensor product $X \boxtimes E$ of a pointed metric space X and a Banach space E as a certain linear subspace of the algebraic dual of $\text{Lip}_0(X, E^*)$. We prove that $X \boxtimes E$ is linearly isomorphic to the linear space of all finite-rank continuous linear operators from $(X^\#, \tau_p)$ into E , where $X^\#$ denotes the space $\text{Lip}_0(X, \mathbb{K})$ and τ_p is the topology of pointwise convergence of $X^\#$. The concept of Lipschitz tensor product of elements of $X^\#$ and E^* yields the space $X^\# \boxtimes E^*$ as a certain linear subspace of the algebraic dual of $X \boxtimes E$. To ensure the good behavior of a norm on $X \boxtimes E$ with respect to the Lipschitz tensor product of Lipschitz functionals (mappings) and bounded linear functionals (operators), the concept of dualizable (respectively, uniform) Lipschitz cross-norm on $X \boxtimes E$ is defined. We show that the Lipschitz injective norm ε , the Lipschitz projective norm π and the Lipschitz p -nuclear norm d_p ($1 \leq p \leq \infty$) are uniform dualizable Lipschitz cross-norms on $X \boxtimes E$. In fact, ε is the least dualizable Lipschitz cross-norm and π is the greatest Lipschitz cross-norm on $X \boxtimes E$. Moreover, dualizable Lipschitz cross-norms α on $X \boxtimes E$ are characterized by satisfying the relation $\varepsilon \leq \alpha \leq \pi$. In addition, the Lipschitz injective (projective) norm on $X \boxtimes E$ can be identified with the injective (respectively, projective) tensor norm on the Banach-space tensor product of the Lipschitz-free space over X and E , but this identification does not hold for the Lipschitz 2-nuclear norm and the corresponding Banach-space tensor norm. In terms of the space $X^\# \boxtimes E^*$, we describe the spaces of Lipschitz compact (finite-rank, approximable) operators from X to E^* .

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INTRODUCTION

The Lipschitz space $\text{Lip}_0(X, E)$ is the Banach space of all Lipschitz maps f from a pointed metric space X to a Banach space E that vanish at the base point of X , under the Lipschitz norm given by

$$\text{Lip}(f) = \sup \left\{ \frac{\|f(x) - f(y)\|}{d(x, y)} : x, y \in X, x \neq y \right\}.$$

The elements of $\text{Lip}_0(X, E)$ are referred to as Lipschitz operators. If \mathbb{K} is the field of real or complex numbers, the space $\text{Lip}_0(X, \mathbb{K})$, denoted by $X^\#$, is called the Lipschitz dual of X . A comprehensive reference for the basic theory of the spaces of Lipschitz functions is the book [24] by N. Weaver.

The use of techniques of the theory of algebraic tensor product of Banach spaces to tackle the problem of the duality for Lipschitz operators from X to E goes back to the seventies with the works [18, 19] of J. A. Johnson. Recently, the second-named author [4] has adopted this approach to describe the duals of spaces of Lipschitz p -summing operators from X to E^* for $1 \leq p \leq \infty$. The notion of Lipschitz p -summing operators between metric spaces, a nonlinear generalization of p -summing operators, was introduced by J. D. Farmer and W. B. Johnson in [11], where a nonlinear version of the Pietsch factorization-domination theorem was established. The article [11] has motivated the study of Lipschitz versions of different types of bounded linear operators in the works [4, 5, 6, 7, 17].

The reading of the paper [4] invites to give a definition for the tensor product of X and E . In [18], J. A. Johnson proved that the dual of the closed linear subspace of $\text{Lip}_0(X, E^*)^*$ spanned by the functionals $\delta_x \boxtimes e$ on $\text{Lip}_0(X, E^*)$ with $x \in X$ and $e \in E$, defined by $(\delta_x \boxtimes e)(f) = \langle f(x), e \rangle$, is isometrically isomorphic to $\text{Lip}_0(X, E^*)$. It is well known that the dual of the projective tensor product of Banach spaces E and F can be identified with the space of all bounded linear operators from E to F^* , so the predual of $\text{Lip}_0(X, E^*)$ provided by Johnson's result plays the role of the projective tensor product in the linear theory and this fact suggests to call Lipschitz tensor product of X and E the linear subspace of the algebraic dual of $\text{Lip}_0(X, E^*)$ spanned by the functionals $\delta_x \boxtimes e$. In [4], this space is called the space of E -valued molecules on X , a generalization of the Arens–Eells space $\mathcal{A}E(X)$ of scalar-valued molecules on X (see [1, 24]).

Our purpose in the present paper is to formalize the notion of a Lipschitz tensor product between a metric space and a normed space, and to study its basic properties. We are also motivated by the problem of researching the spaces of Lipschitz compact (finite-rank, approximable) operators from X to E^* introduced in [17].

We now describe the contents of this paper. Sections 1 and 2 gather some preliminary results on the Lipschitz tensor product $X \boxtimes E$ between a metric space X and a normed space E , whereas Section 3 is devoted to the norms that can be defined on it that will be useful for our purposes (we have called them Lipschitz cross-norms, in accordance to the classic terminology). The section 4 is a first approach to the issue of duality, showing that Lipschitz maps are in duality with one of these Lipschitz cross-norms. Sections 5, 6 and 7 study important specific examples of Lipschitz cross-norms, which are Lipschitz versions of the classical injective, projective and Chevet–Saphar cross-norms. In Section 8, we deal with the space of all Lipschitz finite-rank operators

from X to E^* and its closure in the Lipschitz norm topology, the space of Lipschitz approximable operators from X to E^* .

Notation. Given two metric spaces (X, d_X) and (Y, d_Y) , let us recall that a map $f: X \rightarrow Y$ is said to be Lipschitz if there exists a real constant $C \geq 0$ such that $d_Y(f(x), f(y)) \leq C d_X(x, y)$ for all $x, y \in X$. The least constant C for which the preceding inequality holds will be denoted by $\text{Lip}(f)$, that is,

$$\text{Lip}(f) = \sup \left\{ \frac{d_Y(f(x), f(y))}{d_X(x, y)} : x, y \in X, x \neq y \right\}.$$

A pointed metric space X is a metric space with a base point in X , that is, a designated special point, which we will always denote by 0 . As usual, \mathbb{K} denotes the field of real or complex numbers. We will consider a normed space E over \mathbb{K} as a pointed metric space with the distance defined by its norm and the zero vector as the base point. As is customary, B_E and S_E stand for the closed unit ball of E and the unit sphere of E , respectively.

Given two pointed metric spaces X and Y , we denote by $\text{Lip}_0(X, Y)$ the set of all base-point preserving Lipschitz maps from X to Y . If E is a Banach space, then $\text{Lip}_0(X, E)$ is a Banach space under the Lipschitz norm given by

$$\text{Lip}(f) = \sup \left\{ \frac{\|f(x) - f(y)\|}{d(x, y)} : x, y \in X, x \neq y \right\}.$$

For two vector spaces E and F , $L(E, F)$ stands for the vector space of all linear operators from E into F . In the case that E and F are Banach spaces, $\mathcal{L}(E, F)$ represents the Banach space of all bounded linear operators from E to F endowed with the canonical norm of operators. In particular, the algebraic dual $L(E, \mathbb{K})$ and the topological dual $\mathcal{L}(E, \mathbb{K})$ are denoted by E' and E^* , respectively. For each $e \in E$ and $e^* \in E'$, we frequently will write $\langle e^*, e \rangle$ instead of $e^*(e)$.

Throughout this paper, unless otherwise stated, X will denote a pointed metric space with base point 0 and E a Banach space.

1. LIPSCHITZ TENSOR PRODUCTS

The Lipschitz tensor product of a pointed metric space X and a Banach space E , which we will denote from now on by $X \boxtimes E$, can be constructed as a space of linear functionals on $\text{Lip}_0(X, E^*)$.

Definition 1.1. Let X be a pointed metric space and E a Banach space. For each $x \in X$, let $\delta_{(x,0)}: \text{Lip}_0(X, E^*) \rightarrow E^*$ be the linear map defined by

$$\delta_{(x,0)}(f) = f(x) \quad (f \in \text{Lip}_0(X, E^*)).$$

For each $(x, y) \in X^2$, let $\delta_{(x,y)}: \text{Lip}_0(X, E^*) \rightarrow E^*$ be the linear map given by

$$\delta_{(x,y)} = \delta_{(x,0)} - \delta_{(y,0)}.$$

Let $\Delta(X, E^*)$ denote the linear subspace of $L(\text{Lip}_0(X, E^*), E^*)$ spanned by the set

$$\{\delta_{(x,y)}: (x, y) \in X^2\}.$$

For any $\gamma \in \Delta(X, E^*)$ and $e \in E$, let $\gamma \boxtimes e: \text{Lip}_0(X, E^*) \rightarrow \mathbb{K}$ be the linear functional given by

$$(\gamma \boxtimes e)(f) = \langle \gamma(f), e \rangle \quad (f \in \text{Lip}_0(X, E^*)).$$

In particular, for any $(x, y) \in X^2$ and $e \in E$, let $\delta_{(x,y)} \boxtimes e$ be the element of $\text{Lip}_0(X, E^*)'$ defined, for any $f \in \text{Lip}_0(X, E^*)$, by

$$(\delta_{(x,y)} \boxtimes e)(f) = \langle \delta_{(x,y)}(f), e \rangle = \langle \delta_{(x,0)}(f) - \delta_{(y,0)}(f), e \rangle = \langle f(x) - f(y), e \rangle.$$

The Lipschitz tensor product $X \boxtimes E$ is defined as the vector subspace of $\text{Lip}_0(X, E^*)'$ spanned by the set

$$\{\delta_{(x,y)} \boxtimes e: (x, y) \in X^2, e \in E\}.$$

The following properties of the Lipschitz tensor product can be checked easily.

Lemma 1.2. *Let $\lambda \in \mathbb{K}$, $(x, y), (x_1, y_1), (x_2, y_2) \in X^2$ and $e, e_1, e_2 \in E$.*

- (i) $\lambda(\delta_{(x,y)} \boxtimes e) = (\lambda\delta_{(x,y)}) \boxtimes e = \delta_{(x,y)} \boxtimes (\lambda e)$.
- (ii) $(\delta_{(x_1,y_1)} + \delta_{(x_2,y_2)}) \boxtimes e = \delta_{(x_1,y_1)} \boxtimes e + \delta_{(x_2,y_2)} \boxtimes e$.
- (iii) $\delta_{(x,y)} \boxtimes (e_1 + e_2) = \delta_{(x,y)} \boxtimes e_1 + \delta_{(x,y)} \boxtimes e_2$.
- (iv) $\delta_{(x,x)} \boxtimes e = \delta_{(x,y)} \boxtimes 0 = 0$.

We say that $\delta_{(x,y)} \boxtimes e$ is an elementary Lipschitz tensor. Note that each element u in $X \boxtimes E$ is of the form $u = \sum_{i=1}^n \lambda_i (\delta_{(x_i,y_i)} \boxtimes e_i)$, where $n \in \mathbb{N}$, $\lambda_i \in \mathbb{K}$, $(x_i, y_i) \in X^2$ and $e_i \in E$. This representation of u is not unique. It is worth noting that each element u of $X \boxtimes E$ can be represented as $u = \sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i$ since $\lambda(\delta_{(x,y)} \boxtimes e) = \delta_{(x,y)} \boxtimes (\lambda e)$. This representation of u admits the following refinement.

Lemma 1.3. *Every nonzero Lipschitz tensor $u \in X \boxtimes E$ has a representation in the form $\sum_{i=1}^m \delta_{(z_i,0)} \boxtimes d_i$, where*

$$m = \min \left\{ k \in \mathbb{N}: \exists z_1, \dots, z_k \in X, d_1, \dots, d_k \in E \mid u = \sum_{i=1}^k \delta_{(z_i,0)} \boxtimes d_i \right\}$$

and the points z_1, \dots, z_m in X are distinct from the base point 0 of X and pairwise distinct.

Proof. Let $u \in X \boxtimes E$ and let $\sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i$ be a representation of u . We have

$$\begin{aligned} u &= \sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i = \sum_{i=1}^n (\delta_{(x_i,0)} - \delta_{(y_i,0)}) \boxtimes e_i = \sum_{i=1}^n \delta_{(x_i,0)} \boxtimes e_i + \sum_{i=1}^n \delta_{(y_i,0)} \boxtimes (-e_i) \\ &= \sum_{i=1}^n \delta_{(x_i,0)} \boxtimes e_i + \sum_{i=n+1}^{2n} \delta_{(y_{i-n},0)} \boxtimes (-e_{i-n}) = \sum_{i=1}^{2n} \delta_{(z_i,0)} \boxtimes d_i, \end{aligned}$$

where

$$\delta_{(z_i,0)} \boxtimes d_i = \begin{cases} \delta_{(x_i,0)} \boxtimes e_i & \text{if } i = 1, \dots, n, \\ \delta_{(y_{i-n},0)} \boxtimes (-e_{i-n}) & \text{if } i = n+1, \dots, 2n. \end{cases}$$

Then, by the well-ordering principle of \mathbb{N} , there exists a smallest natural number m for which there is a representation of u in the form $\sum_{i=1}^m \delta_{(z_i,0)} \boxtimes d_i$. Since $u \neq 0$, it is clear that $z_i \neq 0$ for some $i \in \{1, \dots, m\}$. This implies that $z_j \neq 0$ for all $j \in \{1, \dots, m\}$. Otherwise, observe that $\sum_{i=1, i \neq j}^m \delta_{(z_i,0)} \boxtimes d_i$ would be a representation of u containing

$m - 1$ terms and this contradicts the definition of m . Moreover, if $z_j = z_k$ for some $j, k \in \{1, \dots, m\}$ with $j \neq k$, we would have

$$u = \sum_{i=1, j \neq i \neq k}^m \delta_{(z_i, 0)} \boxtimes d_i + \left(\delta_{(z_j, 0)} \boxtimes (d_j + d_k) \right),$$

and this is impossible. Hence the points z_i are pairwise distinct. \square

We can concatenate the representations of two elements of $X \boxtimes E$ to get a representation of their sum.

Lemma 1.4. *Let $u, v \in X \boxtimes E$ and let $\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i$ and $\sum_{i=1}^m \delta_{(x'_i, y'_i)} \boxtimes e'_i$ be representations of u and v , respectively. Then $\sum_{i=1}^{n+m} \delta_{(x''_i, y''_i)} \boxtimes e''_i$, where*

$$(x''_i, y''_i) = \begin{cases} (x_i, y_i) & \text{if } i = 1, \dots, n, \\ (x'_{i-n}, y'_{i-n}) & \text{if } i = n+1, \dots, n+m, \end{cases}$$

$$e''_i = \begin{cases} e_i & \text{if } i = 1, \dots, n, \\ e'_{i-n} & \text{if } i = n+1, \dots, n+m, \end{cases}$$

is a representation of $u + v$.

We now describe the action of a Lipschitz tensor u in $X \boxtimes E$ on a function f in $\text{Lip}_0(X, E^*)$.

Lemma 1.5. *Let $u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \in X \boxtimes E$ and $f \in \text{Lip}_0(X, E^*)$. Then*

$$u(f) = \sum_{i=1}^n \langle f(x_i) - f(y_i), e_i \rangle.$$

Our next aim is to characterize the zero Lipschitz tensor. For it we need the following Lipschitz operators.

Lemma 1.6. *Let $g \in X^\#$ and $\phi \in E^*$. The map $g \cdot \phi: X \rightarrow E^*$, given by $(g \cdot \phi)(x) = g(x)\phi$, belongs to $\text{Lip}_0(X, E^*)$ and $\text{Lip}(g \cdot \phi) = \text{Lip}(g) \|\phi\|$.*

Proof. Clearly, $g \cdot \phi$ is well defined. Let $x, y \in X$. For any $e \in E$, we obtain

$$\begin{aligned} | \langle (g \cdot \phi)(x) - (g \cdot \phi)(y), e \rangle | &= | \langle (g(x) - g(y)) \phi, e \rangle | \\ &= |g(x) - g(y)| | \langle \phi, e \rangle | \\ &\leq \text{Lip}(g) d(x, y) \|\phi\| \|e\|, \end{aligned}$$

and so $\| (g \cdot \phi)(x) - (g \cdot \phi)(y) \| \leq \text{Lip}(g) \|\phi\| d(x, y)$. Then we have $g \cdot \phi \in \text{Lip}_0(X, E^*)$ and $\text{Lip}(g \cdot \phi) \leq \text{Lip}(g) \|\phi\|$. For the converse inequality, note that

$$|g(x) - g(y)| \|\phi\| = \| (g \cdot \phi)(x) - (g \cdot \phi)(y) \| \leq \text{Lip}(g \cdot \phi) d(x, y)$$

for all $x, y \in X$, and therefore $\text{Lip}(g) \|\phi\| \leq \text{Lip}(g \cdot \phi)$. \square

Proposition 1.7. *If $u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \in X \boxtimes E$, then the following assertions are equivalent:*

- (i) $u = 0$.
- (ii) $\sum_{i=1}^n (g(x_i) - g(y_i)) \langle \phi, e_i \rangle = 0$ for every $g \in B_{X^\#}$ and $\phi \in B_{E^*}$.
- (iii) $\sum_{i=1}^n (g(x_i) - g(y_i)) e_i = 0$ for every $g \in B_{X^\#}$.

Proof. (i) implies (ii): If $u = 0$, then $u(f) = 0$ for all $f \in \text{Lip}_0(X, E^*)$. Since $u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i$, it follows that $\sum_{i=1}^n \langle f(x_i) - f(y_i), e_i \rangle = 0$ for all $f \in \text{Lip}_0(X, E^*)$ by Lemma 1.5. For any $g \in B_{X^\#}$ and $\phi \in B_{E^*}$, the function $g \cdot \phi$ is in $\text{Lip}_0(X, E^*)$ by Lemma 1.6, and therefore we have

$$\begin{aligned} \sum_{i=1}^n (g(x_i) - g(y_i)) \langle \phi, e_i \rangle &= \sum_{i=1}^n \langle (g(x_i) - g(y_i)) \phi, e_i \rangle \\ &= \sum_{i=1}^n \langle (g \cdot \phi)(x_i) - (g \cdot \phi)(y_i), e_i \rangle = 0. \end{aligned}$$

(ii) implies (iii): If (ii) holds, then $\langle \phi, \sum_{i=1}^n (g(x_i) - g(y_i)) e_i \rangle = 0$ for every $g \in B_{X^\#}$ and $\phi \in B_{E^*}$. Since B_{E^*} separates the points of E , it follows that $\sum_{i=1}^n (g(x_i) - g(y_i)) e_i = 0$ for all $g \in B_{X^\#}$.

(iii) implies (i): By Lemma 1.3, we can write $u = \sum_{i=1}^m \delta_{(z_i, 0)} \boxtimes d_i$, where the points z_i in X are pairwise distinct and different from the base point 0. It follows that

$$\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i + \sum_{i=1}^m \delta_{(z_i, 0)} \boxtimes (-d_i) = u - u = 0,$$

and by using the fact proved above that (i) implies (iii), we have

$$\sum_{i=1}^n (g(x_i) - g(y_i)) e_i + \sum_{i=1}^m (g(z_i) - g(0)) (-d_i) = 0$$

for all $g \in B_{X^\#}$. If (iii) holds, we get that

$$\sum_{i=1}^m g(z_i) d_i = \sum_{i=1}^n (g(x_i) - g(y_i)) e_i = 0$$

for all $g \in B_{X^\#}$. Set

$$r = \min \left(\left\{ d(z_i, z_j) : i, j \in \{1, \dots, m\}, i \neq j \right\} \cup \left\{ d(z_i, 0) : i \in \{1, \dots, m\} \right\} \right).$$

Clearly, $r > 0$. Given $j \in \{1, \dots, m\}$, define $g_j : X \rightarrow \mathbb{R}$ by

$$g_j(x) = \max \{0, r - d(x, z_j)\}.$$

It is easy to check that $g_j \in B_{X^\#}$, $g_j(z_j) = r$ and $g_j(z_i) = 0$ for all $i \in \{1, \dots, m\} \setminus \{j\}$. Hence $0 = \sum_{i=1}^m g_j(z_i) d_i = r d_j$, therefore $d_1 = d_2 = \dots = d_m = 0$ and thus $u = 0$. \square

According to Definition 1.1, $X \boxtimes E$ is a linear subspace of $\text{Lip}_0(X, E^*)'$. Furthermore, we have the next fact.

Theorem 1.8. $\langle X \boxtimes E, \text{Lip}_0(X, E^*) \rangle$ forms a dual pair, where the bilinear form $\langle \cdot, \cdot \rangle$ associated to the dual pair is given, for $u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \in X \boxtimes E$ and $f \in \text{Lip}_0(X, E^*)$, by

$$\langle u, f \rangle = \sum_{i=1}^n \langle f(x_i) - f(y_i), e_i \rangle.$$

Proof. Note that $\langle u, f \rangle = u(f)$ by Lemma 1.5. It is plain that $\langle \cdot, \cdot \rangle$ is a well-defined bilinear map from $(X \boxtimes E) \times \text{Lip}_0(X, E^*)$ to \mathbb{K} , and that $\text{Lip}_0(X, E^*)$ separates points of $X \boxtimes E$. Moreover, if $f \in \text{Lip}_0(X, E^*)$ and $\langle u, f \rangle = 0$ for all $u \in X \boxtimes E$, then $\langle f(x), e \rangle = \langle \delta_{(x,0)} \boxtimes e, f \rangle = 0$ for all $x \in X$ and $e \in E$. This implies that $f = 0$ and thus $X \boxtimes E$ separates points of $\text{Lip}_0(X, E^*)$. This completes the proof of the theorem. \square

Since $\langle X \boxtimes E, \text{Lip}_0(X, E^*) \rangle$ is a dual pair, $\text{Lip}_0(X, E^*)$ can be identified with a linear subspace of $(X \boxtimes E)'$ as follows.

Corollary 1.9. *For every map $f \in \text{Lip}_0(X, E^*)$, the functional $\Lambda(f): X \boxtimes E \rightarrow \mathbb{K}$, given by*

$$\Lambda(f)(u) = \sum_{i=1}^n \langle f(x_i) - f(y_i), e_i \rangle$$

for $u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \in X \boxtimes E$, is linear. We say that $\Lambda(f)$ is the linear functional on $X \boxtimes E$ associated to f . The map $f \mapsto \Lambda(f)$ is a linear monomorphism from $\text{Lip}_0(X, E^)$ into $(X \boxtimes E)'$.*

Proof. Let $f \in \text{Lip}_0(X, E^*)$. By Theorem 1.8, note that $\Lambda(f)(u) = \langle u, f \rangle$ for all u in $X \boxtimes E$. It is immediate that $\Lambda(f)$ is a well-defined linear functional on $X \boxtimes E$ and that $f \mapsto \Lambda(f)$ from $\text{Lip}_0(X, E^*)$ into $(X \boxtimes E)'$ is a well-defined linear map. Finally, let $f \in \text{Lip}_0(X, E^*)$ and assume that $\Lambda(f) = 0$. Then $\langle u, f \rangle = 0$ for all $u \in X \boxtimes E$. Since $X \boxtimes E$ separates points of $\text{Lip}_0(X, E^*)$, it follows that $f = 0$ and this proves that the map Λ is one-to-one. \square

We next show that $X \boxtimes E$ is linearly isomorphic to the linear space $\mathcal{F}((X^\#, \tau_p); E)$ of all finite-rank linear operators from $X^\#$ into E which are continuous from the topology of pointwise convergence τ_p of $X^\#$ to the norm topology of E .

Definition 1.10. Let X be a pointed metric space. For $f \in X^\#, x \in X$ and $\varepsilon \in \mathbb{R}^+$, we put

$$B(f, x, \varepsilon) = \{g \in X^\# : |g(x) - f(x)| < \varepsilon\}.$$

Let \mathcal{S} be the family of sets $\{B(f, x, \varepsilon) : f \in X^\#, x \in X, \varepsilon \in \mathbb{R}^+\}$. Then the topology of pointwise convergence τ_p on $X^\#$ is the topology generated by \mathcal{S} .

We can check that $(X^\#, \tau_p)$ is a locally convex space. Next we describe its dual space.

Lemma 1.11. *Let X be a pointed metric space. Then*

$$(X^\#, \tau_p)^* = \text{lin}(\{\delta_x : x \in X\}) \subset (X^\#)',$$

where δ_x is the functional on $X^\#$ defined by $\delta_x(g) = g(x)$.

Proof. Define the linear functional $T: X^\# \rightarrow \mathbb{K}$ by $T(g) = \sum_{i=1}^n \lambda_i g(x_i)$ for all $g \in X^\#$, where $n \in \mathbb{N}$, $\lambda_1, \dots, \lambda_n \in \mathbb{K}$ and $x_1, \dots, x_n \in X$. Put $r = 1 + \sum_{i=1}^n |\lambda_i|$ and let $\varepsilon > 0$ be arbitrary. If $g \in \bigcap_{i=1}^n B(0, x_i, \varepsilon/r)$, then $|T(g)| \leq \sum_{i=1}^n |\lambda_i| |g(x_i)| < \varepsilon$. This proves that T is continuous on $X^\#$ when it is equipped with the topology τ_p . Conversely, we need to show that every element S in $(X^\#, \tau_p)^*$ is of that form. Since S is continuous in the τ_p -topology, there is an open neighborhood V of 0 such that $|S(g)| < 1$ for all $g \in V$. We can suppose that $V = \bigcap_{i=1}^n B(0, x_i, \varepsilon)$ for suitable $n \in \mathbb{N}$, $x_1, \dots, x_n \in X$ and $\varepsilon > 0$. Take $f \in \bigcap_{i=1}^n \ker \delta_{x_i}$. Then $mf \in V$ for each $m \in \mathbb{N}$. By the linearity of S , it follows

that $|S(f)| < 1/m$ for all $m \in \mathbb{N}$ and so $S(f) = 0$. This shows that $\bigcap_{i=1}^n \ker \delta_{x_i} \subset \ker S$ and the lemma follows from a known fact of linear algebra. \square

Theorem 1.12. *The map $J: X \boxtimes E \rightarrow \mathcal{F}((X^\#, \tau_p); E)$, given by*

$$J(u)(g) = \sum_{i=1}^n (g(x_i) - g(y_i)) e_i$$

for $u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \in X \boxtimes E$ and $g \in X^\#$, is a linear isomorphism.

Proof. Let $u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \in X \boxtimes E$. It is immediate that $J(u): X^\# \rightarrow E$ is well defined and linear. Note that $J(u)(X^\#) \subset \text{lin}\{e_1, \dots, e_n\}$ and hence $J(u)$ has finite-dimensional range. In order to prove that $J(u)$ is continuous from $(X^\#, \tau_p)$ to E , it is sufficient to see that $J(u)$ is continuous at 0. Let $\varepsilon > 0$. Denote $r = 2(1 + \sum_{i=1}^n \|e_i\|)$ and put $z_i = x_i$ for $i = 1, \dots, n$ and $z_i = y_{i-n}$ for $i = n+1, \dots, 2n$. Take $U = \bigcap_{i=1}^{2n} B(0, z_i, \varepsilon/r)$. For any $g \in U$, we have

$$\|J(u)(g)\| \leq \sum_{i=1}^n (|g(x_i)| + |g(y_i)|) \|e_i\| \leq \sum_{i=1}^n 2 \frac{\varepsilon}{r} \|e_i\| < \varepsilon,$$

as required. By Proposition 1.7, the map $u \mapsto J(u)$ from $X \boxtimes E$ to $\mathcal{F}((X^\#, \tau_p); E)$ is well defined. We now show that J is linear. If $\lambda \in \mathbb{K}$, then $\lambda u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes (\lambda e_i)$ and so

$$J(\lambda u)(g) = \sum_{i=1}^n (g(x_i) - g(y_i)) (\lambda e_i) = \lambda \sum_{i=1}^n (g(x_i) - g(y_i)) e_i = \lambda J(u)(g)$$

for all $g \in X^\#$. Now, let $v = \sum_{i=1}^m \delta_{(x'_i, y'_i)} \boxtimes e'_i \in X \boxtimes E$ and take the representation $\sum_{i=1}^{n+m} \delta_{(x''_i, y''_i)} \boxtimes e''_i$ of $u + v$ given in Lemma 1.4. Then we have

$$\begin{aligned} J(u + v)(g) &= \sum_{i=1}^{n+m} (g(x''_i) - g(y''_i)) e''_i \\ &= \sum_{i=1}^n (g(x'_i) - g(y'_i)) e'_i + \sum_{i=n+1}^{n+m} (g(x''_i) - g(y''_i)) e''_i \\ &= \sum_{i=1}^n (g(x_i) - g(y_i)) e_i + \sum_{i=n+1}^{n+m} (g(x'_{i-n}) - g(y'_{i-n})) e'_{i-n} \\ &= \sum_{i=1}^n (g(x_i) - g(y_i)) e_i + \sum_{i=1}^m (g(x'_i) - g(y'_i)) e'_i \\ &= J(u)(g) + J(v)(g) \end{aligned}$$

for all $g \in X^\#$. It remains to show that J is bijective. On one hand, assume that $J(u) = 0$. Then $J(u)(g) = \sum_{i=1}^n (g(x_i) - g(y_i)) e_i = 0$ for all $g \in X^\#$, this implies that $u = 0$ by Proposition 1.7 and so J is one-to-one. On the other hand, if $T \in \mathcal{F}((X^\#, \tau_p); E)$, take a basis $\{e_1, \dots, e_n\}$ of $T(X^\#)$. For each $g \in X^\#$, there are unique $\lambda_1^{(g)}, \dots, \lambda_n^{(g)} \in \mathbb{K}$ such that $T(g) = \sum_{i=1}^n \lambda_i^{(g)} e_i$. For each $i \in \{1, \dots, n\}$, let $y^i: T(X^\#) \rightarrow \mathbb{K}$ be given by $y^i(T(g)) = \lambda_i^{(g)}$ for all $g \in X^\#$. The uniqueness of the representation of each element of $T(X^\#)$ implies that each y^i is linear. Hence y^i is continuous on $T(X^\#)$ since the linear space $T(X^\#)$ is finite-dimensional. Then each $T_i = y^i \circ T$ belongs to $(X^\#, \tau_p)^*$

and $T(g) = \sum_{i=1}^n T_i(g)e_i$ for all $g \in X^\#$. Since $(X^\#, \tau_p)^* = \text{lin}(\{\delta_x : x \in X\}) \subset (X^\#)'$ by Lemma 1.11, for each $i \in \{1, \dots, n\}$ there are $m(i) \in \mathbb{N}$, $\lambda_1^{(i)}, \dots, \lambda_{m(i)}^{(i)} \in \mathbb{K}$ and $x_1^{(i)}, \dots, x_{m(i)}^{(i)} \in X$ such that $T_i = \sum_{j=1}^{m(i)} \lambda_j^{(i)} \delta_{x_j^{(i)}}$. Then, for each $g \in X^\#$, we may write

$$T(g) = \sum_{i=1}^n T_i(g)e_i = \sum_{j=1}^m g(x_j)u_j = J \left(\sum_{j=1}^m \delta_{(x_j, 0)} \boxtimes u_j \right) (g)$$

for certain $m \in \mathbb{N}$, $x_1, \dots, x_m \in X$ and $u_1, \dots, u_m \in E$. This proves that J is onto. \square

Corollary 1.13. *Let X be a pointed metric space. If E is a finite-dimensional Banach space, then $X \boxtimes E$ is linearly isomorphic to $\mathcal{L}((X^\#, \tau_p); E)$. In particular, $X \boxtimes \mathbb{K}$ is linearly isomorphic to $(X^\#, \tau_p)^* = \text{lin}(\{\delta_x : x \in X\})$.*

2. LIPSCHITZ TENSOR PRODUCT FUNCTIONALS AND OPERATORS

We first introduce the concept of Lipschitz tensor product functional of a Lipschitz functional and a bounded linear functional.

Definition 2.1. Let X be a pointed metric space and E a Banach space. Let $g \in X^\#$ and $\phi \in E^*$. The map $g \boxtimes \phi : X \boxtimes E \rightarrow \mathbb{K}$, given by

$$(g \boxtimes \phi)(u) = \sum_{i=1}^n (g(x_i) - g(y_i)) \langle \phi, e_i \rangle$$

for $u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \in X \boxtimes E$, is called the Lipschitz tensor product functional of g and ϕ .

By Lemma 1.5, note that

$$(g \boxtimes \phi)(u) = \sum_{i=1}^n \langle (g \cdot \phi)(x_i) - (g \cdot \phi)(y_i), e_i \rangle = u(g \cdot \phi).$$

The following result which follows easily from this formula gathers some properties of these functionals.

Lemma 2.2. *Let $g \in X^\#$ and $\phi \in E^*$. The functional $g \boxtimes \phi : X \boxtimes E \rightarrow \mathbb{K}$ is a well-defined linear map satisfying $\lambda(g \boxtimes \phi) = (\lambda g) \boxtimes \phi = g \boxtimes (\lambda \phi)$ for any $\lambda \in \mathbb{K}$. Moreover, $(g_1 + g_2) \boxtimes \phi = g_1 \boxtimes \phi + g_2 \boxtimes \phi$ for all $g_1, g_2 \in X^\#$ and $g \boxtimes (\phi_1 + \phi_2) = g \boxtimes \phi_1 + g \boxtimes \phi_2$ for all $\phi_1, \phi_2 \in E^*$.*

Definition 2.3. Let X be a pointed metric space and E a Banach space. The space $X^\# \boxtimes E^*$ is defined as the linear subspace of $(X \boxtimes E)'$ spanned by the set

$$\{g \boxtimes \phi : g \in X^\#, \phi \in E^*\}.$$

This space is called the associated Lipschitz tensor product of $X \boxtimes E$.

From the aforementioned formula we also derive easily the following fact.

Lemma 2.4. *For any $\sum_{j=1}^m g_j \boxtimes \phi_j \in X^\# \boxtimes E^*$ and $\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \in X \boxtimes E$, we have*

$$\left(\sum_{j=1}^m g_j \boxtimes \phi_j \right) \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right) = \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right) \left(\sum_{j=1}^m g_j \cdot \phi_j \right).$$

Each element u^* in $X^\# \boxtimes E^*$ has the form $u^* = \sum_{j=1}^m \lambda_j (g_j \boxtimes \phi_j)$, where $m \in \mathbb{N}$, $\lambda_j \in \mathbb{K}$, $g_j \in X^\#$ and $\phi_j \in E^*$, but this representation is not unique. Since $\lambda(g \boxtimes \phi) = (\lambda g) \boxtimes \phi = g \boxtimes (\lambda \phi)$, each element of $X^\# \boxtimes E^*$ can be expressed as $\sum_{j=1}^m g_j \boxtimes \phi_j$. This representation can be improved as follows.

Lemma 2.5. *Every nonzero element u^* in $X^\# \boxtimes E^*$ has a representation $\sum_{j=1}^m g_j \boxtimes \phi_j$ such that the functions g_1, \dots, g_m in $X^\#$ are nonzero and the functionals ϕ_1, \dots, ϕ_m in E^* are linearly independent.*

Proof. Let $u^* \in X^\# \boxtimes E^*$, $u^* \neq 0$. Since $0 \boxtimes \phi = 0$, we can take a representation for u^* , $\sum_{i=1}^n h_i \boxtimes \phi_i$, where h_1, \dots, h_n are nonzero. If the vectors ϕ_1, \dots, ϕ_n are linearly independent, we have finished. Otherwise, take $F = \text{lin}(\{\phi_1, \dots, \phi_n\})$ and choose a subset of $\{\phi_1, \dots, \phi_n\}$, which is a basis for F , ϕ_1, \dots, ϕ_p (after reordering) for some $p < n$. For each $i \in \{p+1, \dots, n\}$ we can express the vector ϕ_i as a unique linear combination in the form $\phi_i = \sum_{k=1}^p \lambda_k^{(i)} \phi_k$, where $\lambda_1^{(i)}, \dots, \lambda_p^{(i)} \in \mathbb{K}$. Using Lemma 2.2, we can write

$$\begin{aligned} u^* &= \sum_{i=1}^p h_i \boxtimes \phi_i + \sum_{i=p+1}^n h_i \boxtimes \phi_i = \sum_{i=1}^p h_i \boxtimes \phi_i + \sum_{i=p+1}^n h_i \boxtimes \left(\sum_{k=1}^p \lambda_k^{(i)} \phi_k \right) \\ &= \sum_{i=1}^p h_i \boxtimes \phi_i + \sum_{i=p+1}^n \left(\sum_{k=1}^p \lambda_k^{(i)} (h_i \boxtimes \phi_k) \right) = \sum_{i=1}^p h_i \boxtimes \phi_i + \sum_{k=1}^p \left(\sum_{i=p+1}^n \lambda_k^{(i)} (h_i \boxtimes \phi_k) \right) \\ &= \sum_{i=1}^p h_i \boxtimes \phi_i + \sum_{k=1}^p \left(\sum_{i=p+1}^n \lambda_k^{(i)} h_i \right) \boxtimes \phi_k = \sum_{j=1}^p \left(h_j + \sum_{i=p+1}^n \lambda_j^{(i)} h_i \right) \boxtimes \phi_j. \end{aligned}$$

Denote $g_j = h_j + \sum_{i=p+1}^n \lambda_j^{(i)} h_i$ for each $j \in \{1, \dots, p\}$. Since $u^* \neq 0$, after reordering, we can take $m \leq p$ for which $g_j \neq 0$ for all $j \leq m$ and $g_j = 0$ for all $j > m$. Then $\sum_{j=1}^m g_j \boxtimes \phi_j$ is a representation of u^* satisfying the required conditions. \square

Our next aim is to show that the associated Lipschitz tensor product $X^\# \boxtimes E^*$ is linearly isomorphic to the space of Lipschitz finite-rank operators from X to E^* . This class of Lipschitz operators appears in [18, 17].

Let us recall that if X is a set and E is a vector space, then a map $f: X \rightarrow E$ is said to have finite-dimensional rank if the subspace of E generated by $f(X)$, $\text{lin}(f(X))$, is finite-dimensional in which case the rank of f , denoted by $\text{rank}(f)$, is defined as the dimension of $\text{lin}(f(X))$.

For a pointed metric space X and a Banach space E , we denote by $\text{Lip}_{0F}(X, E^*)$ the set of all Lipschitz finite-rank operators from X to E^* . Clearly, $\text{Lip}_{0F}(X, E^*)$ is a linear subspace of $\text{Lip}_0(X, E^*)$. For any $g \in X^\#$ and $\phi \in E^*$, we consider in Lemma 1.6 the elements $g \cdot \phi$ of $\text{Lip}_{0F}(X, E^*)$ defined by $(g \cdot \phi)(x) = g(x)\phi$ for all $x \in X$. Note that $\text{rank}(g \cdot \phi) = 1$ if $g \neq 0$ and $\phi \neq 0$. Now we prove that these elements generate linearly the space $\text{Lip}_{0F}(X, E^*)$.

Lemma 2.6. *Every element $f \in \text{Lip}_{0F}(X, E^*)$ has a representation in the form $f = \sum_{j=1}^m g_j \cdot \phi_j$, where $m = \text{rank}(f)$, $g_1, \dots, g_m \in X^\#$ and $\phi_1, \dots, \phi_m \in E^*$.*

Proof. Suppose $\text{lin}(f(X))$ is m -dimensional and let $\{\phi_1, \dots, \phi_m\}$ be a basis of $\text{lin}(f(X))$. Then, for each $x \in X$, the element $f(x) \in f(X)$ is expressible in a unique form as

$f(x) = \sum_{j=1}^m \lambda_j^{(x)} \phi_j$ with $\lambda_1^{(x)}, \dots, \lambda_m^{(x)} \in \mathbb{K}$. For each $j \in \{1, \dots, m\}$, define the linear map $y^j: \text{lin}(f(X)) \rightarrow \mathbb{K}$ by $y^j(f(x)) = \lambda_j^{(x)}$ for all $x \in X$. Let $g_j = y^j \circ f$. Clearly, $g_j \in X^\#$ and, given $x \in X$, we have $f(x) = \sum_{j=1}^m \lambda_j^{(x)} \phi_j = \sum_{j=1}^m g_j(x) \phi_j$. Hence $f = \sum_{j=1}^m g_j \cdot \phi_j$. \square

Theorem 2.7. *The map $K: X^\# \boxtimes E^* \rightarrow \text{Lip}_{0F}(X, E^*)$, defined by*

$$K \left(\sum_{j=1}^m g_j \boxtimes \phi_j \right) = \sum_{j=1}^m g_j \cdot \phi_j,$$

is a linear isomorphism.

Proof. The map K is well defined by applying Lemma 2.4 and Theorem 1.8. Clearly, K is linear. Moreover, it is surjective by Lemma 2.6 and injective by Lemma 2.4. \square

We next introduce the concept of Lipschitz tensor product operator of a Lipschitz operator and a bounded linear operator.

Definition 2.8. Let X, Y be pointed metric spaces and let E, F be Banach spaces. Let $h \in \text{Lip}_0(X, Y)$ and $T \in \mathcal{L}(E, F)$. The map $h \boxtimes T: X \boxtimes E \rightarrow Y \boxtimes F$, given by

$$(h \boxtimes T)(u) = \sum_{i=1}^n \delta_{(h(x_i), h(y_i))} \boxtimes T(e_i)$$

for $u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \in X \boxtimes E$, is called the Lipschitz tensor product operator of h and T .

Lemma 2.9. *Let $h \in \text{Lip}_0(X, Y)$ and $T \in \mathcal{L}(E, F)$. Then $h \boxtimes T: X \boxtimes E \rightarrow Y \boxtimes F$ is a well-defined linear operator.*

Proof. Let $u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i$ and $v = \sum_{i=1}^m \delta_{(x'_i, y'_i)} \boxtimes e'_i$ be in $X \boxtimes E$. If $u = v$, then Proposition 1.7 says us that $\sum_{i=1}^n (g(x_i) - g(y_i)) e_i = \sum_{i=1}^m (g(x'_i) - g(y'_i)) e'_i$ for all $g \in B_{X^\#}$. In particular, this holds for all functions in $B_{X^\#}$ of the form $(f \circ h)/(1 + \text{Lip}(h))$ with f varying in $B_{Y^\#}$. It follows that $\sum_{i=1}^n (f(h(x_i)) - f(h(y_i))) T(e_i) = \sum_{i=1}^m (f(h(x'_i)) - f(h(y'_i))) T(e'_i)$ for all $f \in B_{Y^\#}$, and this implies that $\sum_{i=1}^n \delta_{(h(x_i), h(y_i))} \boxtimes T(e_i) = \sum_{i=1}^m \delta_{(h(x'_i), h(y'_i))} \boxtimes T(e'_i)$ again by Proposition 1.7. Hence the map $h \boxtimes T$ is well defined.

We see that $h \boxtimes T$ is linear. Let $\lambda \in \mathbb{K}$. Then $\lambda u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes (\lambda e_i)$, and Definition 2.8 and Lemma 1.2 give

$$\begin{aligned} (h \boxtimes T)(\lambda u) &= \sum_{i=1}^n \delta_{(h(x_i), h(y_i))} \boxtimes T(\lambda e_i) = \sum_{i=1}^n \delta_{(h(x_i), h(y_i))} \boxtimes \lambda T(e_i) \\ &= \lambda \sum_{i=1}^n \delta_{(h(x_i), h(y_i))} \boxtimes T(e_i) = \lambda (h \boxtimes T)(u). \end{aligned}$$

Take $u + v = \sum_{i=1}^{n+m} \delta_{(x'_i, y'_i)} \boxtimes e''_i$ as in Lemma 1.4. Then we have

$$\begin{aligned}
(h \boxtimes T)(u + v) &= \sum_{i=1}^{n+m} \delta_{(h(x'_i), h(y'_i))} \boxtimes T(e''_i) \\
&= \sum_{i=1}^n \delta_{(h(x'_i), h(y'_i))} \boxtimes T(e''_i) + \sum_{i=n+1}^{n+m} \delta_{(h(x'_i), h(y'_i))} \boxtimes T(e''_i) \\
&= \sum_{i=1}^n \delta_{(h(x_i), h(y_i))} \boxtimes T(e_i) + \sum_{i=n+1}^{n+m} \delta_{(h(x'_{i-n}), h(y'_{i-n}))} \boxtimes T(e'_{i-n}) \\
&= \sum_{i=1}^n \delta_{(h(x_i), h(y_i))} \boxtimes T(e_i) + \sum_{i=1}^m \delta_{(h(x'_i), h(y'_i))} \boxtimes T(e'_i) \\
&= (h \boxtimes T)(u) + (h \boxtimes T)(v).
\end{aligned}$$

□

3. LIPSCHITZ CROSS-NORMS

We denote the linear space $X \boxtimes E$ endowed with a norm α by $X \boxtimes_{\alpha} E$, and its completion by $\widehat{X \boxtimes_{\alpha} E}$. We are looking for a norm on the linear space $X \boxtimes E$, and for our purposes it is convenient to work with norms that satisfy the following conditions.

Definition 3.1. Let X be a pointed metric space and E a Banach space. We say that a norm α on $X \boxtimes E$ is a Lipschitz cross-norm if

$$\alpha(\delta_{(x,y)} \boxtimes e) = d(x, y) \|e\|$$

for all $(x, y) \in X^2$ and $e \in E$.

A Lipschitz cross-norm α on $X \boxtimes E$ is said to be dualizable if given $g \in X^{\#}$ and $\phi \in E^*$, we have

$$\left| \sum_{i=1}^n (g(x_i) - g(y_i)) \langle \phi, e_i \rangle \right| \leq \text{Lip}(g) \|\phi\| \alpha \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right)$$

for all $\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \in X \boxtimes E$.

A Lipschitz cross-norm α on $X \boxtimes E$ is called uniform if given $h \in \text{Lip}_0(X, X)$ and $T \in \mathcal{L}(E, E)$, we have

$$\alpha \left(\sum_{i=1}^n \delta_{(h(x_i), h(y_i))} \boxtimes T(e_i) \right) \leq \text{Lip}(h) \|T\| \alpha \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right)$$

for all $\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \in X \boxtimes E$.

The dualizable Lipschitz cross-norms on $X \boxtimes E$ may be characterized by the boundedness of the Lipschitz tensor product functionals.

Proposition 3.2. A Lipschitz cross-norm α on $X \boxtimes E$ is dualizable if and only if, for each $g \in X^{\#}$ and $\phi \in E^*$, the linear functional $g \boxtimes \phi: X \boxtimes_{\alpha} E \rightarrow \mathbb{K}$ is bounded and $\|g \boxtimes \phi\| = \text{Lip}(g) \|\phi\|$.

Proof. Let α be a Lipschitz cross-norm on $X \boxtimes E$. Given $g \in X^\#$ and $\phi \in E^*$, if α is dualizable, we have

$$\left| (g \boxtimes \phi) \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right) \right| = \left| \sum_{i=1}^n (g(x_i) - g(y_i)) \langle \phi, e_i \rangle \right| \leq \text{Lip}(g) \|\phi\| \alpha \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right)$$

for all $\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \in X \boxtimes E$. Hence the linear functional $g \boxtimes \phi$ is bounded on $X \boxtimes_\alpha E$ and $\|g \boxtimes \phi\| \leq \text{Lip}(g) \|\phi\|$. The opposite inequality $\text{Lip}(g) \|\phi\| \leq \|g \boxtimes \phi\|$ is deduced from the fact that

$$|g(x) - g(y)| |\langle \phi, e \rangle| = |(g \boxtimes \phi)(\delta_{(x, y)} \boxtimes e)| \leq \|g \boxtimes \phi\| \alpha(\delta_{(x, y)} \boxtimes e) = \|g \boxtimes \phi\| d(x, y) \|e\|$$

for all $x, y \in X$ and $e \in E$.

Conversely, if for any $g \in X^\#$ and $\phi \in E^*$, the linear functional $g \boxtimes \phi: X \boxtimes_\alpha E \rightarrow \mathbb{K}$ is bounded and $\|g \boxtimes \phi\| = \text{Lip}(g) \|\phi\|$, then

$$\begin{aligned} \left| \sum_{i=1}^n (g(x_i) - g(y_i)) \langle \phi, e_i \rangle \right| &= \left| (g \boxtimes \phi) \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right) \right| \\ &\leq \|g \boxtimes \phi\| \alpha \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right) = \text{Lip}(g) \|\phi\| \alpha \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right) \end{aligned}$$

for all $\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \in X \boxtimes E$, and so α is dualizable. \square

Similarly, the boundedness of the Lipschitz tensor product operators characterizes the uniform Lipschitz cross-norms on $X \boxtimes E$.

Proposition 3.3. *A Lipschitz cross-norm α on $X \boxtimes E$ is uniform if and only if, for each $h \in \text{Lip}_0(X, X)$ and $T \in \mathcal{L}(E, E)$, the linear operator $h \boxtimes T: X \boxtimes_\alpha E \rightarrow X \boxtimes_\alpha E$ is bounded and $\|h \boxtimes T\| = \text{Lip}(h) \|T\|$.*

Proof. Let α be a Lipschitz cross-norm on $X \boxtimes E$. If α is uniform, given $h \in \text{Lip}_0(X, X)$ and $T \in \mathcal{L}(E, E)$, we have

$$\begin{aligned} \alpha \left((h \boxtimes T) \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right) \right) &= \alpha \left(\sum_{i=1}^n \delta_{(h(x_i), h(y_i))} \boxtimes T(e_i) \right) \\ &\leq \text{Lip}(h) \|T\| \alpha \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right) \end{aligned}$$

for all $\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \in X \boxtimes E$. It follows that the linear operator $h \boxtimes T$ is bounded on $X \boxtimes_\alpha E$ and $\|h \boxtimes T\| \leq \text{Lip}(h) \|T\|$. For the reverse inequality, notice that

$$\begin{aligned} d(h(x), h(y)) \|T(e)\| &= \alpha(\delta_{(h(x), h(y))} \boxtimes T(e)) = \alpha((h \boxtimes T)(\delta_{(x, y)} \boxtimes e)) \\ &\leq \|h \boxtimes T\| \alpha(\delta_{(x, y)} \boxtimes e) = \|h \boxtimes T\| d(x, y) \|e\| \end{aligned}$$

for all $x, y \in X$ and $e \in E$, and therefore $\text{Lip}(h) \|T\| \leq \|h \boxtimes T\|$.

Conversely, if for each $h \in \text{Lip}_0(X, X)$ and $T \in \mathcal{L}(E, E)$, the linear map $h \boxtimes T$ from $X \boxtimes_\alpha E$ to $X \boxtimes_\alpha E$ is bounded and $\|h \boxtimes T\| = \text{Lip}(h) \|T\|$, then

$$\begin{aligned} \alpha \left(\sum_{i=1}^n \delta_{(h(x_i), h(y_i))} \boxtimes T(e_i) \right) &= \alpha \left((h \boxtimes T) \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right) \right) \\ &\leq \|h \boxtimes T\| \alpha \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right) = \text{Lip}(h) \|T\| \alpha \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right) \end{aligned}$$

for all $\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \in X \boxtimes E$, and so α is uniform. \square

Remark 3.4. A reading of the proofs of the two preceding propositions shows that a Lipschitz cross-norm α on $X \boxtimes E$ is dualizable (uniform) if for each $g \in X^\#$ and $\phi \in E^*$, then $g \boxtimes \phi \in (X \boxtimes_\alpha E)^*$ and $\|g \boxtimes \phi\| \leq \text{Lip}(g) \|\phi\|$ (respectively, if for each $h \in \text{Lip}_0(X, X)$ and $T \in \mathcal{L}(E, E)$, then $h \boxtimes T \in \mathcal{L}(X \boxtimes_\alpha E, X \boxtimes_\alpha E)$ and $\|h \boxtimes T\| \leq \text{Lip}(h) \|T\|$).

As a consequence of this remark, note that if α is a dualizable Lipschitz cross-norm on $X \boxtimes E$, then $X^\# \boxtimes E^*$ is a linear subspace of $(X \boxtimes_\alpha E)^*$. Next we introduce the concept of Lipschitz cross-norm on $X^\# \boxtimes E^*$.

Definition 3.5. Let X be a pointed metric space and E a Banach space. We say that a norm β on $X^\# \boxtimes E^*$ is a Lipschitz cross-norm if $\beta(g \boxtimes \phi) = \text{Lip}(g) \|\phi\|$ for all $g \in X^\#$ and $\phi \in E^*$.

As a consequence of Proposition 3.2, we have the following Lipschitz cross-norm on $X^\# \boxtimes E^*$.

Corollary 3.6. Let α be a dualizable Lipschitz cross-norm on $X \boxtimes E$. The restriction to $X^\# \boxtimes E^*$ of the canonical norm of $(X \boxtimes_\alpha E)^*$, that is, the map $\alpha' : X^\# \boxtimes E^* \rightarrow \mathbb{R}$, given by

$$\alpha'(u^*) = \sup \left\{ \left| \left(\sum_{j=1}^m g_j \boxtimes \phi_j \right) \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right) \right| : \alpha \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right) \leq 1 \right\}$$

for $u^* = \sum_{j=1}^m g_j \boxtimes \phi_j \in X^\# \boxtimes E^*$, is a Lipschitz cross-norm on $X^\# \boxtimes E^*$.

Definition 3.7. Let X be a pointed metric space and E a Banach space. Let α be a dualizable Lipschitz cross-norm on $X \boxtimes E$. The norm α' on $X^\# \boxtimes E^*$ is called the associated Lipschitz norm of α . The vector space $X^\# \boxtimes E^*$ with the norm α' will be denoted by $X^\# \boxtimes_{\alpha'} E^*$ and its completion by $\widehat{X^\# \boxtimes_{\alpha'} E^*}$.

4. THE INDUCED LIPSCHITZ DUAL NORM

Definition 4.1. For each $u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \in X \boxtimes E$, define:

$$L(u) = \sup \left\{ \left| \sum_{i=1}^n \langle f(x_i) - f(y_i), e_i \rangle \right| : f \in \text{Lip}_0(X, E^*), \text{Lip}(f) \leq 1 \right\}.$$

Note that the supremum on the right side above exists and $L(u) \leq \sum_{i=1}^n d(x_i, y_i) \|e_i\|$ because

$$\begin{aligned} \left| \sum_{i=1}^n \langle f(x_i) - f(y_i), e_i \rangle \right| &\leq \sum_{i=1}^n |\langle f(x_i) - f(y_i), e_i \rangle| \\ &\leq \sum_{i=1}^n \|f(x_i) - f(y_i)\| \|e_i\| \leq \text{Lip}(f) \sum_{i=1}^n d(x_i, y_i) \|e_i\| \end{aligned}$$

for all $f \in \text{Lip}_0(X, E^*)$. Moreover, L defines a map from $X \boxtimes E$ to \mathbb{R} by Lemma 1.5.

Theorem 4.2. *The linear space $X \boxtimes E$ is contained in $\text{Lip}_0(X, E^*)^*$ and L is the dual norm of the norm Lip of $\text{Lip}_0(X, E^*)$ induced on $X \boxtimes E$. Moreover, L is a Lipschitz cross-norm on $X \boxtimes E$.*

Proof. Let $x, y \in X$ and $e \in E$. Since $\delta_{(x,y)} \boxtimes e$ is a linear map on $\text{Lip}_0(X, E^*)$ and

$$|(\delta_{(x,y)} \boxtimes e)(f)| = |\langle f(x) - f(y), e \rangle| \leq \|f(x) - f(y)\| \|e\| \leq \text{Lip}(f) d(x, y) \|e\|$$

for all $f \in \text{Lip}_0(X, E^*)$, then $\delta_{(x,y)} \boxtimes e \in \text{Lip}_0(X, E^*)^*$ and thus $X \boxtimes E \subset \text{Lip}_0(X, E^*)^*$. For every $u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \in X \boxtimes E$, we have

$$L(u) = \sup \{|u(f)| : f \in \text{Lip}_0(X, E^*), \text{Lip}(f) \leq 1\}$$

by Lemma 1.5, and therefore L is the dual norm of the norm Lip of $\text{Lip}_0(X, E^*)$ induced on $X \boxtimes E$. Finally, we prove that L is a Lipschitz cross-norm. By above-proved, we have $|(\delta_{(x,y)} \boxtimes e)(f)| \leq d(x, y) \|e\|$ for all $f \in \text{Lip}_0(X, E^*)$ with $\text{Lip}(f) \leq 1$, and hence $L(\delta_{(x,y)} \boxtimes e) \leq d(x, y) \|e\|$. For the reverse estimate, take $\phi \in E^*$ with $\|\phi\| = 1$ satisfying $|\langle \phi, e \rangle| = \|e\|$, and consider the map $f: X \rightarrow E^*$ given by

$$f(z) = (d(0, x) - d(z, x))\phi \quad (z \in X).$$

An easy verification shows that f is in $\text{Lip}_0(X, E^*)$ with $\text{Lip}(f) \leq 1$ and

$$|(\delta_{(x,y)} \boxtimes e)(f)| = |\langle f(x) - f(y), e \rangle| = |\langle d(x, y)\phi, e \rangle| = d(x, y) |\langle \phi, e \rangle| = d(x, y) \|e\|,$$

and therefore $d(x, y) \|e\| = |(\delta_{(x,y)} \boxtimes e)(f)| \leq L(\delta_{(x,y)} \boxtimes e)$. \square

The following result is essentially known. For completeness we include it here with an alternate proof.

Theorem 4.3. [18, Theorem 4.1] *Let X be a pointed metric space and let E be a Banach space. Then $\text{Lip}_0(X, E^*)$ is isometrically isomorphic to $(X \widehat{\boxtimes}_L E)^*$, via the map $\Lambda_0: \text{Lip}_0(X, E^*) \rightarrow (X \widehat{\boxtimes}_L E)^*$ given by*

$$\Lambda_0(f)(u) = \sum_{i=1}^n \langle f(x_i) - f(y_i), e_i \rangle$$

for $f \in \text{Lip}_0(X, E^*)$ and $u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \in X \boxtimes_L E$. Its inverse is the mapping $\Lambda_0^{-1}: (X \widehat{\boxtimes}_L E)^* \rightarrow \text{Lip}_0(X, E^*)$ defined by

$$\langle \Lambda_0^{-1}(\varphi)(x), e \rangle = \langle \varphi, \delta_{(x,0)} \boxtimes e \rangle$$

for $\varphi \in (X \widehat{\boxtimes}_L E)^*$, $x \in X$ and $e \in E$.

Proof. Let $f \in \text{Lip}_0(X, E^*)$ and let $\Lambda(f)$ be the linear functional on $X \boxtimes E$ defined in Corollary 1.9. Notice that $\Lambda(f) \in (X \boxtimes_L E)^*$ and $\|\Lambda(f)\| \leq \text{Lip}(f)$ since

$$|\Lambda(f)(u)| = \left| \sum_{i=1}^n \langle f(x_i) - f(y_i), e_i \rangle \right| \leq \text{Lip}(f)L(u)$$

for all $u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \in X \boxtimes E$. By the denseness of $X \boxtimes_L E$ in $X \widehat{\boxtimes}_L E$, there is a unique continuous extension $\Lambda_0(f)$ of $\Lambda(f)$ to $X \widehat{\boxtimes}_L E$. Let $\Lambda_0: \text{Lip}_0(X, E^*) \rightarrow (X \widehat{\boxtimes}_L E)^*$ be the map so defined. Since $\Lambda: \text{Lip}_0(X, E^*) \rightarrow (X \boxtimes E)'$ is a linear monomorphism by Corollary 1.9, it follows easily that so is Λ_0 .

In order to see that Λ_0 is a surjective isometry, let φ be an element of $(X \widehat{\boxtimes}_L E)^*$. Define $f: X \rightarrow E^*$ by

$$\langle f(x), e \rangle = \varphi(\delta_{(x,0)} \boxtimes e) \quad (x \in X, e \in E).$$

It is plain that $f(x)$ is a well-defined bounded linear functional on E and that f is well defined. Observe that $\langle f(x) - f(y), e \rangle = \varphi(\delta_{(x,y)} \boxtimes e)$ for all $x, y \in X$ and $e \in E$. Fix $x, y \in X$. It follows that

$$|\langle f(x) - f(y), e \rangle| = |\varphi(\delta_{(x,y)} \boxtimes e)| \leq \|\varphi\| L(\delta_{(x,y)} \boxtimes e) = \|\varphi\| d(x, y) \|e\|$$

for all $e \in E$, and so $\|f(x) - f(y)\| \leq \|\varphi\| d(x, y)$. Hence $f \in \text{Lip}_0(X, E^*)$ and $\text{Lip}(f) \leq \|\varphi\|$. For any $u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \in X \boxtimes_L E$, we get

$$\Lambda_0(f)(u) = \sum_{i=1}^n \langle f(x_i) - f(y_i), e_i \rangle = \sum_{i=1}^n \varphi(\delta_{(x_i, y_i)} \boxtimes e_i) = \varphi\left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i\right) = \varphi(u).$$

Hence $\Lambda_0(f) = \varphi$ on a dense subspace of $X \widehat{\boxtimes}_L E$ and, consequently, $\Lambda_0(f) = \varphi$. Moreover, $\text{Lip}(f) \leq \|\varphi\| = \|\Lambda_0(f)\|$. This completes the proof of the theorem. \square

5. THE LIPSCHITZ INJECTIVE NORM

We introduce the Lipschitz injective norm on $X \boxtimes E$, which bears this name because – as we will see later – it respects injections.

Definition 5.1. For each $u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \in X \boxtimes E$, let

$$\varepsilon(u) = \sup \left\{ \left| \sum_{i=1}^n (g(x_i) - g(y_i)) \langle \phi, e_i \rangle \right| : g \in B_{X^\#}, \phi \in B_{E^*} \right\}.$$

Notice that the supremum on the right side in the previous definition exists since

$$\begin{aligned} \left| \sum_{i=1}^n (g(x_i) - g(y_i)) \langle \phi, e_i \rangle \right| &\leq \sum_{i=1}^n |(g(x_i) - g(y_i)) \langle \phi, e_i \rangle| \\ &\leq \sum_{i=1}^n \text{Lip}(g) d(x_i, y_i) \|\phi\| \|e_i\| \leq \sum_{i=1}^n d(x_i, y_i) \|e_i\| \end{aligned}$$

for all $g \in B_{X^\#}$ and $\phi \in B_{E^*}$. Note that

$$\sum_{i=1}^n (g(x_i) - g(y_i)) \langle \phi, e_i \rangle = (g \boxtimes \phi) \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right) = (g \boxtimes \phi)(u),$$

and, consequently, $\varepsilon(u)$ does not depend on the representation of u by Lemma 2.2, so ε defines a map from $X \boxtimes E$ to \mathbb{R} .

We start by showing that there is a close relationship between the Lipschitz injective norm and the Banach-space injective norm. Before proceeding, we will need to introduce some terminology. We denote by $\mathcal{F}(X)$ the Lipschitz-free Banach space over a pointed metric space X , which is the closed linear subspace of $(X^\#)^*$ spanned by the set $\{\delta_x : x \in X\}$, where for each $x \in X$, δ_x is the evaluation functional at the point x defined on $X^\#$. We show below that the space $X \widehat{\boxtimes}_\varepsilon E$ can in fact be identified with the injective Banach-space tensor product $\mathcal{F}(X) \widehat{\otimes}_\varepsilon E$. Let us recall some fundamental properties of the space $\mathcal{F}(X)$.

Theorem 5.2. [1], [24, pp. 39–41] *Let X, Y be pointed metric spaces, and E a Banach space.*

- (i) *The dual of $\mathcal{F}(X)$ is (canonically) isometrically isomorphic to $X^\#$, with the duality pairing given by $\langle g, \delta_x \rangle = g(x)$ for all $g \in X^\#$ and $x \in X$. Moreover, on bounded subsets of $X^\#$, the weak* topology coincides with the topology of pointwise convergence.*
- (ii) *The map $\iota_X : x \mapsto \delta_x$ is an isometric embedding of X into $\mathcal{F}(X)$.*
- (iii) *For any Lipschitz map $f : X \rightarrow Y$ with $f(0) = 0$, there is a unique linear map $\tilde{f} : \mathcal{F}(X) \rightarrow \mathcal{F}(Y)$ such that $\tilde{f} \circ \iota_X = \iota_Y \circ f$. Furthermore, $\|\tilde{f}\| = \text{Lip}(f)$.*
- (iv) *For any Lipschitz map $f : X \rightarrow E$ with $f(0) = 0$, there is a unique linear map $\hat{f} : \mathcal{F}(X) \rightarrow E$ such that $\hat{f} \circ \iota_X = f$. Furthermore, $\|\hat{f}\| = \text{Lip}(f)$.*

It is because of the universal properties above that the space $\mathcal{F}(X)$ is called the Lipschitz-free space over X , or simply the free space over X . These spaces have been recently used as tools in nonlinear Banach space theory, see [14, 20] and the survey [15].

Proposition 5.3. *The map $I : X \boxtimes_\varepsilon E \rightarrow \mathcal{F}(X) \otimes_\varepsilon E$, defined by*

$$I(u) = \sum_{i=1}^n (\delta_{x_i} - \delta_{y_i}) \otimes e_i$$

for $u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \in X \boxtimes E$, is a linear isometry. As a consequence, $X \widehat{\boxtimes}_\varepsilon E$ is isometrically isomorphic to $\mathcal{F}(X) \widehat{\otimes}_\varepsilon E$.

Proof. Let $u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \in X \boxtimes E$. Since $\mathcal{F}(X)^* \equiv X^\#$, note that the norm of $\sum_{i=1}^n (\delta_{x_i} - \delta_{y_i}) \otimes e_i$ in $\mathcal{F}(X) \otimes_\varepsilon E$ is given by

$$\sup \left\{ \left| \sum_{i=1}^n \langle g, \delta_{x_i} - \delta_{y_i} \rangle \langle \phi, e_i \rangle \right| : g \in B_{X^\#}, \phi \in B_{E^*} \right\}.$$

Since $\langle g, \delta_{x_i} - \delta_{y_i} \rangle$ is precisely $g(x_i) - g(y_i)$, Proposition 1.7 shows that I is well defined (and thus linear) and moreover a quick glance at Definition 5.1 shows that I is an isometry.

Recall that the linear span of $\{\delta_x\}_{x \in X}$ is dense in $\mathcal{F}(X)$, hence the tensors of the form $\sum_{i=1}^n (\delta_{x_i} - \delta_{y_i}) \otimes e_i$, with $x_i, y_i \in X$ and $e_i \in E$, are dense in $\mathcal{F}(X) \otimes_\varepsilon E$. This shows that the map I has dense range, and thus $X \widehat{\boxtimes}_\varepsilon E$ is isometrically isomorphic to $\mathcal{F}(X) \widehat{\otimes}_\varepsilon E$. \square

From the identification in Proposition 5.3 and the properties of the injective tensor product of Banach spaces, we immediately obtain the following.

Theorem 5.4. ε is a uniform and dualizable Lipschitz cross-norm on $X \boxtimes E$.

Just as in the Banach space case, the injective norm is the least of the cross-norms with nice properties.

Theorem 5.5. ε is the least dualizable Lipschitz cross-norm on $X \boxtimes E$.

Proof. According to Theorem 5.4, ε is a dualizable Lipschitz cross-norm on $X \boxtimes E$. Let α be a dualizable Lipschitz cross-norm on $X \boxtimes E$ and assume, for contradiction, that

$$\alpha \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right) < \varepsilon \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right)$$

for some $\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \in X \boxtimes E$. By the definition of ε , there exist $g \in B_{X^\#}$ and $\phi \in B_{E^*}$ such that

$$\alpha \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right) < \left| \sum_{i=1}^n (g(x_i) - g(y_i)) \langle \phi, e_i \rangle \right|.$$

By Corollary 3.6, α' is a Lipschitz cross-norm on $X^\# \boxtimes E^*$, and we have

$$\left| \sum_{i=1}^n (g(x_i) - g(y_i)) \langle \phi, e_i \rangle \right| = \left| (g \boxtimes \phi) \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right) \right| \leq \alpha'(g \boxtimes \phi) \alpha \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right).$$

Hence $\alpha'(g \boxtimes \phi) > 1$ and thus $\text{Lip}(g) \|\phi\| < \alpha'(g \boxtimes \phi)$. This contradicts that α' is a Lipschitz cross-norm. Therefore $\alpha \geq \varepsilon$ and this proves the theorem. \square

The completion $X \widehat{\boxtimes}_\varepsilon E$ of $X \boxtimes_\varepsilon E$ will be called the injective Lipschitz tensor product of X and E . We now justify this terminology in the case $\mathbb{K} = \mathbb{R}$.

Theorem 5.6. Let X be a pointed metric space and let E be a Banach space over \mathbb{R} . Let $X_0 \subset X$ be a subset of X containing 0 , and let E_0 be a closed linear subspace of E . Then $X_0 \widehat{\boxtimes}_\varepsilon E_0$ is (isometrically) a linear subspace of $X \widehat{\boxtimes}_\varepsilon E$.

Proof. The identification from Proposition 5.3 allows us to reduce this to the corresponding property for the injective tensor product of Banach spaces. Since in the case of real scalars we have that $\mathcal{F}(X_0)$ is (isometrically) a subspace of $\mathcal{F}(X)$ [24, Proposition 2.2.5 (b)], the result follows. \square

We can identify $X \widehat{\boxtimes}_\varepsilon E$ with the space of all approximable bounded linear operators of $(X^\#, \tau_p)$ to E .

Proposition 5.7. The map $J: X \boxtimes_\varepsilon E \rightarrow \mathcal{F}((X^\#, \tau_p); E)$, defined by

$$J(u)(g) = \sum_{i=1}^n (g(x_i) - g(y_i)) e_i$$

for $u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \in X \boxtimes E$ and $g \in X^\#$, is an isometric isomorphism. As a consequence, $X \widehat{\boxtimes}_\varepsilon E$ is isometrically isomorphic to the closure in the operator norm topology of $\mathcal{F}((X^\#, \tau_p); E)$.

Proof. By Theorem 1.12, J is a linear bijection. If $u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \in X \boxtimes E$, we have

$$\begin{aligned} \|J(u)\| &= \sup \{ \|J(u)(g)\| : g \in B_{X^\#} \} \\ &= \sup \left\{ \left\| \phi \left(\sum_{i=1}^n (g(x_i) - g(y_i)) e_i \right) \right\| : g \in B_{X^\#}, \phi \in B_{E^*} \right\} \\ &= \sup \left\{ \left| \sum_{i=1}^n (g(x_i) - g(y_i)) \langle \phi, e_i \rangle \right| : g \in B_{X^\#}, \phi \in B_{E^*} \right\} = \varepsilon(u). \end{aligned}$$

The consequence is immediate. \square

6. THE LIPSCHITZ PROJECTIVE NORM

We now introduce the Lipschitz projective norm on $X \boxtimes E$, which (as the name suggests) respects projections.

Definition 6.1. For each $u \in X \boxtimes E$, define

$$\pi(u) = \inf \left\{ \sum_{i=1}^n d(x_i, y_i) \|e_i\| : u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right\},$$

the infimum being taken over all representations of u .

First we show that π is a Lipschitz cross-norm with good properties.

Theorem 6.2. π is a uniform and dualizable Lipschitz cross-norm on $X \boxtimes E$ such that $L \leq \pi$.

Proof. Let $u \in X \boxtimes E$ and let $\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i$ be a representation of u . Using Definition 4.1, we have seen that $L(u) \leq \sum_{i=1}^n d(x_i, y_i) \|e_i\|$. Since this holds for every representation of u , it follows that $L(u) \leq \pi(u)$. Suppose that $\pi(u) = 0$. Since $L(u) \leq \pi(u)$ and L is a norm on $X \boxtimes E$, then $u = 0$.

We check that $\pi(\lambda u) = |\lambda| \pi(u)$. If $\lambda \in \mathbb{K}$, then $\lambda u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes (\lambda e_i)$ and so

$$\pi(\lambda u) \leq \sum_{i=1}^n d(x_i, y_i) \|\lambda e_i\| = |\lambda| \sum_{i=1}^n d(x_i, y_i) \|e_i\|.$$

Since the representation of u is arbitrary, this implies that $\pi(\lambda u) \leq |\lambda| \pi(u)$. If $\lambda = 0$, we have $\pi(\lambda u) = 0 = |\lambda| \pi(u)$ since $\pi(u) \geq 0$ for all $u \in X \boxtimes E$. Assume that $\lambda \neq 0$. Similarly, we have $\pi(u) = \pi(\lambda^{-1}(\lambda u)) \leq |\lambda^{-1}| \pi(\lambda u)$, thus $|\lambda| \pi(u) \leq \pi(\lambda u)$ and hence $\pi(\lambda u) = |\lambda| \pi(u)$.

We show that $\pi(u + v) \leq \pi(u) + \pi(v)$ for all $u, v \in X \boxtimes E$. Let $\varepsilon > 0$. Then there are representations $u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i$ and $v = \sum_{i=1}^m \delta_{(x'_i, y'_i)} \boxtimes e'_i$ such that $\sum_{i=1}^n d(x_i, y_i) \|e_i\| < \pi(u) + \varepsilon/2$ and $\sum_{i=1}^m d(x'_i, y'_i) \|e'_i\| < \pi(v) + \varepsilon/2$. We can concatenate these representations to get a representation $\sum_{i=1}^{n+m} \delta_{(x''_i, y''_i)} \boxtimes e''_i$ for $u + v$ as in Lemma 1.4. By Definition 6.1,

it follows that

$$\begin{aligned}
\pi(u + v) &\leq \sum_{i=1}^{n+m} d(x'_i, y'_i) \|e'_i\| = \sum_{i=1}^n d(x'_i, y'_i) \|e'_i\| + \sum_{i=n+1}^{n+m} d(x'_i, y'_i) \|e'_i\| \\
&= \sum_{i=1}^n d(x_i, y_i) \|e_i\| + \sum_{i=n+1}^{n+m} d(x'_{i-n}, y'_{i-n}) \|e'_{i-n}\| \\
&= \sum_{i=1}^n d(x_i, y_i) \|e_i\| + \sum_{i=1}^m d(x'_i, y'_i) \|e'_i\| \\
&< \pi(u) + \pi(v) + \varepsilon.
\end{aligned}$$

By the arbitrariness of ε , we deduce that $\pi(u + v) \leq \pi(u) + \pi(v)$. Hence π is a norm on $X \boxtimes E$.

We now prove that π is a Lipschitz cross-norm. Let $(x, y) \in X^2$ and $e \in E$. It is immediate that $\pi(\delta_{(x,y)} \boxtimes e) \leq d(x, y) \|e\|$. Conversely, since L is a Lipschitz cross-norm on $X \boxtimes E$ and $L \leq \pi$, it follows that $d(x, y) \|e\| = L(\delta_{(x,y)} \boxtimes e) \leq \pi(\delta_{(x,y)} \boxtimes e)$.

Let $g \in X^\#$ and $\phi \in E^*$. For any $\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \in X \boxtimes E$, we have

$$\left| (g \boxtimes \phi) \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right) \right| = \left| \sum_{i=1}^n (g(x_i) - g(y_i)) \langle \phi, e_i \rangle \right| \leq \text{Lip}(g) \|\phi\| \sum_{i=1}^n d(x_i, y_i) \|e_i\|.$$

Since the value of $(g \boxtimes \phi) \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right)$ does not depend on the representation of $\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i$ by Lemma 2.2, it follows that

$$\left| (g \boxtimes \phi) \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right) \right| \leq \text{Lip}(g) \|\phi\| \pi \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right).$$

Therefore the Lipschitz cross-norm π is dualizable by Proposition 3.2 and Remark 3.4.

Similarly, by applying Proposition 3.3 and Remark 3.4, we see that the Lipschitz cross-norm π is uniform. Let $\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \in X \boxtimes E$. For every $h \in \text{Lip}_0(X, X)$ and $T \in \mathcal{L}(E, E)$, we have

$$\begin{aligned}
\pi \left((h \boxtimes T) \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right) \right) &= \pi \left(\sum_{i=1}^n \delta_{(h(x_i), h(y_i))} \boxtimes T(e_i) \right) \leq \sum_{i=1}^n d(h(x_i), h(y_i)) \|T(e_i)\| \\
&\leq \sum_{i=1}^n \text{Lip}(h) d(x_i, y_i) \|T\| \|e_i\| = \text{Lip}(h) \|T\| \sum_{i=1}^n d(x_i, y_i) \|e_i\|.
\end{aligned}$$

The value of the image $(h \boxtimes T) \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right)$ is independent of the chosen representation for $\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i$ by Lemma 2.9, and therefore we conclude that

$$\pi \left((h \boxtimes T) \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right) \right) \leq \text{Lip}(h) \|T\| \pi \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right).$$

□

The Lipschitz projective norm on $X \boxtimes E$ and the dual norm of the norm Lip of $\text{Lip}_0(X, E^*)$ induced on $X \boxtimes E$ coincide as we see next.

Corollary 6.3. *Let X be a pointed metric space and E a Banach space. Then $\pi = L$ on $X \boxtimes E$.*

Proof. By Theorem 6.2, $L \leq \pi$. To prove that $L \geq \pi$, suppose by contradiction that $L(u_0) < 1 < \pi(u_0)$ for some $u_0 \in X \boxtimes E$. Denote $B = \{u \in X \boxtimes E : \pi(u) \leq 1\}$. Clearly, B is a closed and convex set in $X \boxtimes_\pi E$. Applying the Hahn–Banach separation theorem to B and $\{u_0\}$, we obtain a functional $\eta \in (X \boxtimes_\pi E)^*$ such that

$$1 = \|\eta\| = \sup\{\operatorname{Re} \eta(u) : u \in B\} < \operatorname{Re} \eta(u_0).$$

Define $f: X \rightarrow E^*$ by $\langle f(x), e \rangle = \eta(\delta_{(x,0)} \boxtimes e)$ for all $e \in E$ and $x \in X$. It is easy to prove that f is well defined and $f \in \operatorname{Lip}_0(X, E^*)$ with $\operatorname{Lip}(f) \leq 1$. Moreover $u(f) = \eta(u)$ for all $u \in X \boxtimes E$. Therefore $L(u_0) \geq |u_0(f)| \geq \operatorname{Re} u_0(f) = \operatorname{Re} \eta(u_0)$, so $L(u_0) > 1$ and this is a contradiction. \square

As is the case with the projective tensor product of Banach spaces, π dominates all other Lipschitz cross-norms.

Theorem 6.4. *π is the greatest Lipschitz cross-norm on $X \boxtimes E$.*

Proof. We have seen in Theorem 6.2 that π is a Lipschitz cross-norm on $X \boxtimes E$. Now, let α be a Lipschitz cross-norm on $X \boxtimes E$ and let $u \in X \boxtimes E$. If $\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i$ is a representation of u , we have

$$\alpha(u) = \alpha\left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i\right) \leq \sum_{i=1}^n \alpha(\delta_{(x_i, y_i)} \boxtimes e_i) = \sum_{i=1}^n d(x_i, y_i) \|e_i\|.$$

Now the very definition of π gives $\alpha(u) \leq \pi(u)$. \square

Dualizable Lipschitz cross-norms on $X \boxtimes E$ are characterized by being between the Lipschitz injective and Lipschitz projective norms.

Proposition 6.5. *A norm α on $X \boxtimes E$ is a dualizable Lipschitz cross-norm if and only if $\varepsilon \leq \alpha \leq \pi$.*

Proof. If α is a dualizable Lipschitz cross-norm on $X \boxtimes E$, then $\varepsilon \leq \alpha \leq \pi$ by Theorems 5.5 and 6.4. Conversely, if α is a norm on $X \boxtimes E$ that lies between ε and π , then $\alpha(\delta_{(x, y)} \boxtimes e) = d(x, y) \|e\|$ follows immediately from the fact that ε and π are Lipschitz cross-norms. Let $g \in X^\#$ and $\phi \in E^*$. Then

$$\begin{aligned} \left| \sum_{i=1}^n (g(x_i) - g(y_i)) \langle \phi, e_i \rangle \right| &\leq \operatorname{Lip}(g) \|\phi\| \varepsilon \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right) \\ &\leq \operatorname{Lip}(g) \|\phi\| \alpha \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right) \end{aligned}$$

for all $\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \in X \boxtimes E$, and so the Lipschitz cross-norm α is dualizable. \square

Let us recall that a completion of a normed space E is a Banach space \widetilde{E} that includes a dense linear subspace isometric to E . Every normed space has a completion and the completion is unique up to isometric isomorphism. By [10, Lemma 3.100], every element \widetilde{e} of the completion \widetilde{E} of E can be written as $\widetilde{e} = \sum_{n=1}^{\infty} e_n$, where $e_n \in E$ and $\sum_{n=1}^{\infty} \|e_n\| < \infty$. Moreover, $\|\widetilde{e}\| = \inf \{\sum_{n=1}^{\infty} \|e_n\|\}$, where the infimum is taken over

all series in E summing up to \tilde{e} . Combining this with Definition 6.1, we obtain the following result.

Theorem 6.6. *Every element $u \in X \widehat{\boxtimes}_\pi E$ admits a representation*

$$u = \sum_{i=1}^{\infty} \delta_{(x_i, y_i)} \boxtimes e_i$$

such that $\sum_{i=1}^{\infty} d(x_i, y_i) \|e_i\| < \infty$. Moreover,

$$\pi(u) = \inf \left\{ \sum_{i=1}^{\infty} d(x_i, y_i) \|e_i\| : u = \sum_{i=1}^{\infty} \delta_{(x_i, y_i)} \boxtimes e_i, \sum_{i=1}^{\infty} d(x_i, y_i) \|e_i\| < \infty \right\}.$$

The dual pairing satisfies the formula

$$\left\langle \sum_{i=1}^{\infty} \delta_{(x_i, y_i)} \boxtimes e_i, f \right\rangle = \sum_{i=1}^{\infty} \langle f(x_i) - f(y_i), e_i \rangle$$

for all $f \in \text{Lip}_0(X, E^*)$.

Proof. We follow the proof of [10, Lemma 3.100]. Obviously, $\overline{B_{X \boxtimes_\pi E}} = B_{X \widehat{\boxtimes}_\pi E}$. We may assume, without loss of generality, that $u \in B_{X \widehat{\boxtimes}_\pi E}$. Fix $\varepsilon > 0$. There exists $u_1 \in B_{X \boxtimes_\pi E}$ such that $\pi(u - u_1) \leq \varepsilon$. Moreover, by Definition 6.1, we can take a representation of u_1 , $\sum_{i=1}^{m_1} \delta_{(x_i, y_i)} \boxtimes e_i$, such that

$$\sum_{i=1}^{m_1} d(x_i, y_i) \|e_i\| < \pi(u_1) + \varepsilon.$$

Since $(u - u_1)/\varepsilon \in B_{X \widehat{\boxtimes}_\pi E}$, there is $u_2 \in \varepsilon B_{X \boxtimes_\pi E}$ such that $\pi((u - u_1)/\varepsilon - u_2/\varepsilon) \leq 1/2$, that is, $\pi(u - u_1 - u_2) \leq \varepsilon/2$. As before, by Definition 6.1, we can take a representation of u_2 , $\sum_{i=m_1+1}^{m_2} \delta_{(x_i, y_i)} \boxtimes e_i$, such that

$$\sum_{i=m_1+1}^{m_2} d(x_i, y_i) \|e_i\| < \pi(u_2) + \frac{\varepsilon}{2}.$$

Find $u_3 \in (\varepsilon/2)B_{X \boxtimes_\pi E}$ such that $\pi((2/\varepsilon)(u - u_1 - u_2) - (2/\varepsilon)u_3) \leq 1/2$, which yields $\pi(u - u_1 - u_2 - u_3) \leq \varepsilon/2^2$, and take a representation of u_3 , $\sum_{i=m_2+1}^{m_3} \delta_{(x_i, y_i)} \boxtimes e_i$, such that

$$\sum_{i=m_2+1}^{m_3} d(x_i, y_i) \|e_i\| < \pi(u_3) + \frac{\varepsilon}{2^2}.$$

Proceed recursively to obtain sequences $\{u_n\}$ in $X \boxtimes_\pi E$, $\{m_n\}$ in \mathbb{N} , $\{x_n\}, \{y_n\}$ in X and $\{e_n\}$ in E verifying that $u_n \in (\varepsilon/2^{n-2})B_{X \boxtimes_\pi E}$, $u_n = \sum_{i=m_{n-1}+1}^{m_n} \delta_{(x_i, y_i)} \boxtimes e_i$ and

$$\pi\left(u - \sum_{j=1}^n u_j\right) \leq \frac{\varepsilon}{2^{n-1}}, \quad \sum_{i=m_{n-1}+1}^{m_n} d(x_i, y_i) \|e_i\| < \pi(u_n) + \frac{\varepsilon}{2^{n-1}}$$

for all $n \in \mathbb{N}$, $n \geq 2$. Clearly, $u = \sum_{n=1}^{\infty} u_n$. Furthermore, from $\pi(u_1) - \pi(u) \leq \varepsilon$, it follows that

$$\sum_{n=1}^{\infty} \pi(u_n) \leq \pi(u) + \varepsilon + \varepsilon \sum_{n=2}^{\infty} \frac{1}{2^{n-2}} = \pi(u) + 3\varepsilon.$$

Note that the sequence $\{m_n\}$ is strictly increasing and

$$\sum_{i=1}^n d(x_i, y_i) \|e_i\| \leq \sum_{i=1}^{m_n} d(x_i, y_i) \|e_i\| < \sum_{j=1}^n \pi(u_j) + \sum_{j=1}^n \frac{\varepsilon}{2^{j-1}} \leq \pi(u) + 3\varepsilon + 2\varepsilon = \pi(u) + 5\varepsilon$$

for all $n \in \mathbb{N}$. Then the series $\sum_{i \geq 1} \delta_{(x_i, y_i)} \boxtimes e_i$ is absolutely convergent. We calculate its sum. Denote, for each $n \in \mathbb{N}$, $S_n = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i$. We have that $\{S_{m_n}\}$ is a partial sequence of $\{S_n\}$ with $S_{m_n} = \sum_{j=1}^n u_j$ for all $n \in \mathbb{N}$. Since $\{S_{m_n}\}$ converges to $\sum_{n=1}^{\infty} u_n = u$, then $\{S_n\}$ converges to u , that is, $\sum_{i=1}^{\infty} \delta_{(x_i, y_i)} \boxtimes e_i = u$. Finally, from the inequality

$$\sum_{i=1}^{\infty} d(x_i, y_i) \|e_i\| \leq \pi(u) + 5\varepsilon$$

and the arbitrariness of ε , we obtain that

$$\inf \left\{ \sum_{i=1}^{\infty} d(x_i, y_i) \|e_i\| : u = \sum_{i=1}^{\infty} \delta_{(x_i, y_i)} \boxtimes e_i, \sum_{i=1}^{\infty} d(x_i, y_i) \|e_i\| < \infty \right\} \leq \pi(u).$$

The opposite inequality is obvious.

To check the dual pairing formula, consider a representation of u , $\sum_{i=1}^{\infty} \delta_{(x_i, y_i)} \boxtimes e_i$. Given $f \in \text{Lip}_0(X, E^*)$, we must see that the series $\sum_{i \geq 1} \langle f(x_i) - f(y_i), e_i \rangle$ converges to $u(f)$. Denote, for each $i \in \mathbb{N}$, $u_i = \sum_{j=1}^i \delta_{(x_j, y_j)} \boxtimes e_j \in X \boxtimes_{\pi} E$. Taking into account that $\sum_{j=1}^i \langle f(x_j) - f(y_j), e_j \rangle = u_i(f)$, we have

$$\left| u(f) - \sum_{j=1}^i \langle f(x_j) - f(y_j), e_j \rangle \right| = |u(f) - u_i(f)| \leq L(u - u_i) \text{Lip}(f) = \pi(u - u_i) \text{Lip}(f).$$

Since $\{\pi(u - u_i)\}$ converges to 0, we obtain $\sum_{i=1}^{\infty} \langle f(x_i) - f(y_i), e_i \rangle = u(f)$. \square

As in the injective case (Proposition 5.3), the Lipschitz projective norm is closely related to the projective tensor product of Banach spaces through the Lipschitz-free space: the Lipschitz projective norm on $X \boxtimes E$ can be identified with the projective norm on the tensor product of $\mathcal{F}(X)$ and E . The authors wish to thank Richard Haydon for suggesting that this might be true.

Proposition 6.7. *The map $I: X \boxtimes_{\pi} E \rightarrow \mathcal{F}(X) \otimes_{\pi} E$, defined by*

$$I(u) = \sum_{i=1}^n (\delta_{x_i} - \delta_{y_i}) \otimes e_i$$

for $u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \in X \boxtimes E$, is a linear isometry. Moreover, $X \widehat{\boxtimes}_{\pi} E$ is isometrically isomorphic to $\mathcal{F}(X) \widehat{\otimes}_{\pi} E$.

Proof. As in the proof of Proposition 5.3, Proposition 1.7 guarantees that the map I is well defined (and it is clearly linear). Letting $u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \in X \boxtimes E$, note that using the fact that the map $x \mapsto \delta_x$ is an isometry from X into $\mathcal{F}(X)$,

$$\left\| \sum_{i=1}^n (\delta_{x_i} - \delta_{y_i}) \otimes e_i \right\|_{\mathcal{F}(X) \otimes_{\pi} E} \leq \sum_{i=1}^n \|\delta_{x_i} - \delta_{y_i}\|_{\mathcal{F}(X)} \|e_i\| = \sum_{i=1}^n d(x_i, y_i) \|e_i\|.$$

Taking the infimum over all representations of u , we conclude that $\|I(u)\| \leq \pi(u)$. Let $\eta > 0$ be given. From Corollary 6.3 there exists $f \in \text{Lip}_0(X, E^*)$ with $\text{Lip}(f) \leq 1$ such that $|\langle u, f \rangle| > \pi(u) - \eta$, where the pairing is the one given in Theorem 6.6. By Theorem 5.2, the linear extension $\hat{f}: \mathcal{F}(X) \rightarrow E^*$ of f has norm at most one. From the properties of the projective tensor product of Banach spaces, the dual of $\mathcal{F}(X) \widehat{\otimes}_\pi E$ can be identified with $\mathcal{L}(\mathcal{F}(X), E^*)$, where the pairing is given by

$$\left\langle \sum_{i=1}^n \gamma_i \otimes e_i, T \right\rangle = \sum_{i=1}^n \langle T\gamma_i, e_i \rangle \quad (\gamma_i \in \mathcal{F}(X), e_i \in E, T \in \mathcal{L}(\mathcal{F}(X), E^*)).$$

In particular,

$$\|I(u)\| \geq \left| \langle u, \hat{f} \rangle \right| = \left| \sum_{i=1}^n \langle \hat{f}(\delta_{x_i} - \delta_{y_i}), e_i \rangle \right| = \left| \sum_{i=1}^n \langle f(x_i) - f(y_i), e_i \rangle \right| = |\langle u, f \rangle| > \pi(u) - \eta.$$

Letting η go to 0, we obtain that $\|I(u)\| \geq \pi(u)$, and thus I is a linear isometry. Since the sums of the form $\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i$ and $\sum_{i=1}^n (\delta_{x_i} - \delta_{y_i}) \otimes e_i$ are dense respectively in $X \widehat{\boxtimes}_\pi E$ and $\mathcal{F}(X) \widehat{\otimes}_\pi E$, I extends to an isometric isomorphism between $X \widehat{\boxtimes}_\pi E$ and $\mathcal{F}(X) \widehat{\otimes}_\pi E$. \square

Remark 6.8. Propositions 5.3 and 6.7 could give the impression that Lipschitz cross-norms on $X \boxtimes E$ always reduce to the corresponding Banach-space tensor norm on $\mathcal{F}(X) \boxtimes E$. That is indeed the case for the injective and projective norms because of their extreme positions among Lipschitz cross-norms, but it does not always happen. See Remark 7.4 for an example.

Proposition 6.7 implies in particular that given a pointed metric space X , there is a Banach space A such that $X \widehat{\boxtimes}_\pi E$ is isometric to $A \widehat{\otimes}_\pi E$ for every Banach space E . The authors would like to thank Jesús Castillo for pointing out a result in categorical Banach space theory that shows this was to be expected. Without going into all the details, let us outline the argument. First, a theorem of Fuks [12, Section 6] (a nice presentation can be found in [3, Proposition 5.6], where the reader can also find the definitions of the categorical terms we use below) states the following: if $\mathfrak{F}, \mathfrak{G}$ are two covariant Banach functors such that for any Banach spaces E and F , we have that $\mathcal{L}(\mathfrak{F}(E), F)$ is linearly isometric to $\mathcal{L}(E, \mathfrak{G}(F))$, then there exists a Banach space A such that for every Banach space E , $\mathfrak{F}(E)$ is linearly isometric to $A \widehat{\otimes}_\pi E$ and $\mathfrak{G}(F)$ is linearly isometric to $\mathcal{L}(A, F)$. Now consider a fixed pointed metric space X . Note that it induces two covariant Banach functors $X \widehat{\boxtimes}_\pi(\cdot)$ and $\text{Lip}_0(X, \cdot)$. Arguments closely related to those that led us to prove Corollary 6.3 show that, for any Banach spaces E and F , we have $\mathcal{L}(X \widehat{\boxtimes}_\pi E, F)$ is linearly isometric to $\mathcal{L}(E, \text{Lip}_0(X, F))$, so Fuks' result applies.

The identification in Proposition 6.7 will allow us to transfer several properties from the Banach space case to the Lipschitz one. First we consider the boundedness of the linear operator $h \boxtimes T: X \boxtimes E \rightarrow Y \boxtimes F$ for the Lipschitz projective norms.

Proposition 6.9. *Let X, Y be pointed metric spaces and E, F Banach spaces. Let $h \in \text{Lip}_0(X, Y)$ and $T \in \mathcal{L}(E, F)$. Then there exists a unique bounded linear operator $h \boxtimes_\pi T: X \widehat{\boxtimes}_\pi E \rightarrow Y \widehat{\boxtimes}_\pi F$ such that $(h \boxtimes_\pi T)(u) = (h \boxtimes T)(u)$ for all $u \in X \boxtimes E$. Furthermore, $\|h \boxtimes_\pi T\| = \text{Lip}(h) \|T\|$.*

Proof. The uniqueness part is clear, so we only need to show the boundedness. Let $\widetilde{h}: \mathcal{F}(X) \rightarrow \mathcal{F}(Y)$ be the induced map given in Theorem 5.2. By the properties of the projective tensor product of Banach spaces,

$$\|\widetilde{h} \otimes T: \mathcal{F}(X) \widehat{\otimes}_\pi E \rightarrow \mathcal{F}(Y) \widehat{\otimes}_\pi F\| = \|\widetilde{h}\| \|T\| = \text{Lip}(h) \|T\|.$$

The identification from Proposition 6.7 finishes the proof. \square

In fact, we have the following relationship between Lipschitz tensor product operators and usual tensor product operators.

Remark 6.10. Let X, Y be pointed metric spaces and E, F be Banach spaces. Let $h \in \text{Lip}_0(X, Y)$, $T \in \mathcal{L}(E, F)$ and $\widetilde{h}: \mathcal{F}(X) \rightarrow \mathcal{F}(Y)$ be the induced map given in Theorem 5.2. Then the diagram

$$\begin{array}{ccc} X \widehat{\otimes}_\pi E & \xrightarrow{h \boxtimes_\pi T} & Y \widehat{\otimes}_\pi F \\ \downarrow I_{X \widehat{\otimes}_\pi E} & & \downarrow I_{Y \widehat{\otimes}_\pi F} \\ \mathcal{F}(X) \widehat{\otimes}_\pi E & \xrightarrow{\widetilde{h} \otimes_\pi T} & \mathcal{F}(Y) \widehat{\otimes}_\pi F \end{array}$$

commutes, that is, $I_{Y \widehat{\otimes}_\pi F} \circ (h \boxtimes_\pi T) = (\widetilde{h} \otimes_\pi T) \circ I_{X \widehat{\otimes}_\pi E}$.

The space $X \widehat{\otimes}_\pi E$ is called the projective Lipschitz tensor product of X and E . This term derives from the following result. Before stating it, recall that a Lipschitz map $f: X \rightarrow Z$ is called C -co-Lipschitz if for every $x \in X$ and $r > 0$, $f(B(x, r)) \supset B(f(x), r/C)$. Moreover, it is called a Lipschitz quotient if it is surjective, Lipschitz and co-Lipschitz [2].

Theorem 6.11. *Let X, Z be pointed metric spaces and $q: X \rightarrow Z$ a Lipschitz quotient that is 1-Lipschitz and C -co-Lipschitz for every $C > 1$. Let E, F be Banach spaces and $Q: E \rightarrow F$ a quotient operator. Then $q \boxtimes_\pi Q: X \widehat{\otimes}_\pi E \rightarrow Z \widehat{\otimes}_\pi F$ is a quotient operator.*

Proof. Thanks to Proposition 6.7, Remark 6.10 and the behavior of the projective tensor norm with respect to quotients [21, Proposition 2.5], it suffices to prove that if $q: X \rightarrow Z$ is such a Lipschitz quotient then the induced map $\tilde{q}: \mathcal{F}(X) \rightarrow \mathcal{F}(Z)$ is a linear quotient operator. Notice that from Theorem 5.2, $\|\tilde{q}\| = \text{Lip}(q) = 1$. Now let $u \in \mathcal{F}(Z)$, and let $\varepsilon > 0$. From Proposition 6.7, $\mathcal{F}(X) \equiv \mathcal{F}(X) \widehat{\otimes}_\pi \mathbb{K} \equiv X \widehat{\otimes}_\pi \mathbb{K}$. Thus from Theorem 6.6, there exists a representation $u = \sum_{j=1}^{\infty} \delta_{(z_j, z'_j)} \boxtimes a_j$ such that $(1 + \varepsilon)\pi(u) \geq \sum_{j=1}^{\infty} |a_j| d(z_j, z'_j)$. For each j , choose $x_j, x'_j \in X$ such that $q(x_j) = z_j$, $q(x'_j) = z'_j$ and $d(x_j, x'_j) \leq (1 + \varepsilon)d(z_j, z'_j)$. Setting $u' = \sum_{j=1}^{\infty} \delta_{(x_j, x'_j)} \boxtimes a_j$, clearly $\tilde{q}(u') = u$ (so it follows that \tilde{q} is surjective) and

$$\pi(u') \leq \sum_{j=1}^{\infty} |a_j| d(x_j, x'_j) \leq (1 + \varepsilon) \sum_{j=1}^{\infty} |a_j| d(z_j, z'_j) \leq (1 + \varepsilon)^2 \pi(u).$$

Since this holds for all $\varepsilon > 0$, it follows that $\pi(u) = \inf \{\pi(u') : \tilde{q}(u') = u\}$. \square

In a similar manner, the projective norm respects complemented subspaces. We say that a subset $Z \subset X$ that contains the point 0 is a Lipschitz retract of X , or that it is Lipschitz complemented in X , if there exists a Lipschitz map (called a Lipschitz retraction) $r: X \rightarrow Z$ such that $r(z) = z$ for all $z \in Z$.

Proposition 6.12. *Let Z be a Lipschitz retract of X , and let F be a complemented subspace of E . Then $Z \widehat{\otimes}_\pi F$ is complemented in $X \widehat{\otimes}_\pi E$ and the norm on $Z \widehat{\otimes}_\pi F$ induced by the Lipschitz projective norm of $X \widehat{\otimes}_\pi E$ is equivalent to the Lipschitz projective norm on $Z \widehat{\otimes}_\pi F$. If Z is Lipschitz complemented with a Lipschitz retraction of Lipschitz constant one and F is complemented by a linear projection of norm one, then $Z \widehat{\otimes}_\pi F$ is a subspace of $X \widehat{\otimes}_\pi E$ and is also complemented by a projection of norm one.*

Proof. This follows from the corresponding result for the projective tensor product (see [21, Proposition 2.4]), after noting that a Lipschitz retraction $r: X \rightarrow Z$ (that in particular sends 0 to 0) extends to a linear projection $\tilde{r}: \mathcal{F}(X) \rightarrow \mathcal{F}(Z) \subset \mathcal{F}(X)$ with $\|\tilde{r}\| = \text{Lip}(r)$. \square

Calculating the projective norm of an element in a tensor product of Banach spaces is generally difficult, but there is a particular case where the calculation is relatively easy: for any Banach space E , $\ell_1 \widehat{\otimes}_\pi E$ is isometrically isomorphic to $\ell_1(E)$ (see [21, Example 2.6]). In the nonlinear setting, trees play a role analogous to that of ℓ_1 in the linear theory, so the following result is not surprising.

Proposition 6.13. *Let (X, \mathcal{E}) be a graph with finite vertex set X and edge set \mathcal{E} which is a tree, that is, it is connected and contains no cycles. Consider X as a pointed metric space, with distance function given by the shortest-path distance and a distinguished fixed point $0 \in X$. Let E be a Banach space. Then $X \boxtimes_\pi E$ is isometrically isomorphic to $\ell_1(\mathcal{E}; E)$.*

Proof. We say that a vertex $x \in X$ is positive (negative) if it is at an even (respectively, odd) distance from $0 \in X$. Note that, since (X, \mathcal{E}) is a tree, the endpoints of every edge in \mathcal{E} have different parities. Therefore every edge $\{x, y\}$ in \mathcal{E} will be written as (x, y) with x negative and y positive.

Consider $x, y \in X$. Let $n = d(x, y)$ and $\{x = z_0, z_1, \dots, z_n = y\}$ be the unique minimal-length path in (X, \mathcal{E}) joining x and y . Since, for each $v \in E$, $\|v\| d(x, y) = \sum_{i=1}^n \|v\| d(z_i, z_{i-1})$, in order to calculate $\pi(u)$ for $u \in X \boxtimes E$, it suffices to consider only representations involving $\delta_{(x_i, y_i)}$ with $(x_i, y_i) \in \mathcal{E}$. By the triangle inequality, in the representation we can consolidate all terms corresponding to the same edge $(x_i, y_i) \in \mathcal{E}$, so we can consider only representations of the form $u = \sum_{(x,y) \in \mathcal{E}} \delta_{(x,y)} \boxtimes v_{(x,y)}$. But, for each $u \in X \boxtimes E$, there is only one such representation. To see that we use the following claim, which can be proved by induction on the size of the tree: given $(x_0, y_0) \in \mathcal{E}$, there exists a function $g \in X^\#$ such that $g(x_0) - g(y_0) \neq 0$ and $g(x) - g(y) = 0$ for all $(x, y) \in \mathcal{E}$ with $(x, y) \neq (x_0, y_0)$.

Now let $\sum_{(x,y) \in \mathcal{E}} \delta_{(x,y)} \boxtimes e_{(x,y)}$ and $\sum_{(x,y) \in \mathcal{E}} \delta_{(x,y)} \boxtimes v_{(x,y)}$ be two representations of $u \in X \boxtimes E$. Given $(x_0, y_0) \in \mathcal{E}$, take the function $g \in X^\#$ of the previous claim. Then, by Proposition 1.7, we have that

$$0 = \sum_{(x,y) \in \mathcal{E}} (g(x) - g(y))(e_{(x,y)} - v_{(x,y)}) = (g(x_0) - g(y_0))(e_{(x_0,y_0)} - v_{(x_0,y_0)}),$$

and thus $e_{(x_0, y_0)} = v_{(x_0, y_0)}$. The arbitrariness of (x_0, y_0) shows that the representation of u is unique. If we define $J: X \boxtimes_\pi E \rightarrow \ell_1(\mathcal{E}; E)$ by $u \mapsto (v_{(x,y)})_{(x,y) \in \mathcal{E}}$, J is then clearly an isometric isomorphism between $X \boxtimes_\pi E$ and $\ell_1(\mathcal{E}; E)$. \square

More generally, in the linear case we have that $L_1(\mu) \widehat{\otimes}_\pi E$ is isometrically isomorphic to $L_1(\mu; E)$ for any measure μ (see [21, Example 2.19]). In our nonlinear setting, a possible analogue will be given by a generalization of Proposition 6.13 to a more general class of metric trees. This will depend heavily on the identification of the Lipschitz-free space over such trees carried out in [13]. Before stating the result, let us recall a definition. An \mathbb{R} -tree is a metric space X satisfying the following two conditions: (1) For any points a and b in X , there exists a unique isometry ϕ of the closed interval $[0, d(a, b)]$ into X such that $\phi(0) = a$ and $\phi(d(a, b)) = b$; (2) Any one-to-one continuous mapping $\varphi: [0, 1] \rightarrow X$ has the same range as the isometry ϕ associated to the points $a = \varphi(0)$ and $b = \varphi(1)$.

Corollary 6.14. *Let X be an \mathbb{R} -tree and E a Banach space. Then there exists a measure μ such that $X \widehat{\otimes}_\pi E$ is isometric to $L_1(\mu; E)$.*

Proof. By [13, Corollary 3.3], there exists a measure μ such that $\mathcal{F}(X)$ is isometrically isomorphic to $L_1(\mu)$. From Proposition 6.7, $X \widehat{\otimes}_\pi E$ is isometrically isomorphic to $\mathcal{F}(X) \widehat{\otimes}_\pi E$. Finally, from [21, Example 2.19], $L_1(\mu) \widehat{\otimes}_\pi E$ is isometric to $L_1(\mu; E)$. \square

We finish this section noting a universal property for $X \boxtimes_\pi E$ which follows from the universal property of the projective tensor product of Banach spaces, using Proposition 6.7.

Proposition 6.15. *Let X be a pointed metric space and E a Banach space. Then the map $(x, e) \mapsto \delta_{(x,0)} \boxtimes e$, from $X \times E$ into $X \boxtimes_\pi E$, satisfies:*

- (i) *For each $e \in E$, the function $x \mapsto \delta_{(x,0)} \boxtimes e$, from X into $X \boxtimes_\pi E$, belongs to $\text{Lip}_0(X, X \boxtimes_\pi E)$.*
- (ii) *Given $x \in X$, the map $e \mapsto \delta_{(x,0)} \boxtimes e$, from E into $X \boxtimes_\pi E$, is a bounded linear operator.*

Moreover, for each normed space F and for each map $\psi: X \times E \rightarrow F$ verifying (i) and (ii) (that is, ψ is a Lipschitz operator in the first variable and a bounded linear operator in the second one), there is a unique bounded linear map $\widetilde{\psi}: X \boxtimes_\pi E \rightarrow F$ such that $\widetilde{\psi}(\delta_{(x,0)} \boxtimes e) = \psi(x, e)$ for all $x \in X$ and $e \in E$.

7. THE LIPSCHITZ p -NUCLEAR NORMS

We now recall the Lipschitz p -nuclear norms d_p on $X \boxtimes E$ for $1 \leq p \leq \infty$. They are Lipschitz versions of the known tensor norms of Chevet [8] and Saphar [22]. The Lipschitz versions were introduced in [4] under a slightly different notation, where they were shown to be in duality with spaces of Lipschitz p' -summing maps.

Definition 7.1. Let $1 \leq p \leq \infty$. Let E be a Banach space and let $e_1, \dots, e_n \in E$. Define:

$$\|(e_1, \dots, e_n)\|_p = \begin{cases} \left(\sum_{i=1}^n \|e_i\|^p \right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ \max_{1 \leq i \leq n} \|e_i\| & \text{if } p = \infty. \end{cases}$$

Let X be a pointed metric space, $x_1, \dots, x_n, y_1, \dots, y_n \in X$ and $\lambda_1, \dots, \lambda_n \in \mathbb{R}^+$. Define:

$$\|(\lambda_1 \delta_{(x_1, y_1)}, \dots, \lambda_n \delta_{(x_n, y_n)})\|_p^{Lw} = \begin{cases} \sup_{g \in B_{X^\#}} \left(\sum_{i=1}^n \lambda_i^p |g(x_i) - g(y_i)|^p \right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ \sup_{g \in B_{X^\#}} \left(\max_{1 \leq i \leq n} \lambda_i |g(x_i) - g(y_i)| \right) & \text{if } p = \infty. \end{cases}$$

Let p' be the conjugate index of p defined by $p' = p/(p-1)$ if $p \neq 1$, $p' = \infty$ if $p = 1$, and $p' = 1$ if $p = \infty$.

For each $u \in X \boxtimes E$, define:

$$d_p(u) = \inf \left\{ \|(\lambda_1 \delta_{(x_1, y_1)}, \dots, \lambda_n \delta_{(x_n, y_n)})\|_{p'}^{Lw} \|(e_1, \dots, e_n)\|_p : u = \sum_{i=1}^n \lambda_i \delta_{(x_i, y_i)} \boxtimes e_i \right\},$$

the infimum being taken over all representations of u .

Theorem 7.2. For $1 \leq p \leq \infty$, d_p is a uniform and dualizable Lipschitz cross-norm on $X \boxtimes E$.

Proof. It follows from the calculations in [4, Theorem 4.1] that d_p is a norm on $X \boxtimes E$. Now, we claim that $\varepsilon(u) \leq d_p(u) \leq \pi(u)$. Indeed, we have

$$\begin{aligned} \left| \sum_{i=1}^n \lambda_i (g(x_i) - g(y_i)) \langle \phi, e_i \rangle \right| &\leq \sum_{i=1}^n \lambda_i |g(x_i) - g(y_i)| \|e_i\| \\ &\leq \|(\lambda_1 \delta_{(x_1, y_1)}, \dots, \lambda_n \delta_{(x_n, y_n)})\|_{p'}^{Lw} \|(e_1, \dots, e_n)\|_p \end{aligned}$$

for every $g \in B_{X^\#}$ and $\phi \in B_{E^*}$, where we have used Hölder's inequality in the case $1 < p < \infty$. Therefore $\varepsilon(u) \leq \|(\lambda_1 \delta_{(x_1, y_1)}, \dots, \lambda_n \delta_{(x_n, y_n)})\|_{p'}^{Lw} \|(e_1, \dots, e_n)\|_p$, and since it holds for each representation of u , we deduce that $\varepsilon(u) \leq d_p(u)$. Since ε is a Lipschitz cross-norm,

$$\|e\| d(x, y) = \varepsilon(\delta_{(x, y)} \boxtimes e) \leq d_p(\delta_{(x, y)} \boxtimes e)$$

for all $x, y \in X$ and $e \in E$. Moreover, Definition 7.1 gives

$$d_p(\delta_{(x, y)} \boxtimes e) \leq \|e\| \sup_{g \in B_{X^\#}} |g(x) - g(y)| = \|e\| d(x, y).$$

Hence d_p is a Lipschitz cross-norm. Then $d_p \leq \pi$ by Theorem 6.4 as we wanted. Now our claim implies that d_p is dualizable by Proposition 6.5.

Finally, to prove that d_p is uniform, take $h \in \text{Lip}_0(X, X)$ and $T \in \mathcal{L}(E, E)$. Let $u \in X \boxtimes E$ and pick a representation $\sum_{i=1}^n \lambda_i \delta_{(x_i, y_i)} \boxtimes e_i$ for u . We have

$$\begin{aligned} &\|(\lambda_1 \delta_{(h(x_1), h(y_1))}, \dots, \lambda_n \delta_{(h(x_n), h(y_n))})\|_{p'}^{Lw} \|(T(e_1), \dots, T(e_n))\|_p \\ &\leq \text{Lip}(h) \|(\lambda_1 \delta_{(x_1, y_1)}, \dots, \lambda_n \delta_{(x_n, y_n)})\|_{p'}^{Lw} \|T\| \|(e_1, \dots, e_n)\|_p. \end{aligned}$$

Taking infimum over all the representations of u , it follows that $d_p((h \boxtimes T)(u)) \leq \text{Lip}(h) \|T\| d_p(u)$. \square

Let us record the fact that the Lipschitz 1-nuclear norm d_1 is just the Lipschitz projective norm π , as shown in [4, Proposition 4.2].

Proposition 7.3. *For every $u \in X \boxtimes E$,*

$$d_1(u) = \inf \left\{ \sum_{i=1}^n d(x_i, y_i) \|e_i\| : u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right\}$$

taking the infimum over all representations of u .

Remark 7.4. We are now in a position to show that in the case of the Lipschitz p -nuclear norms d_p and assuming that X is a finite pointed metric space, the norm on $X \boxtimes E$ is not the one induced from $\mathcal{F}(X) \otimes E$ endowed with the Banach-space version of d_p . If that were the case, then by duality (see [4, Theorem 4.3] and [21, Proposition 6.11]) the Lipschitz p' -summing norm of a map $f: X \rightarrow E^*$ would coincide with the p' -summing norm of the induced linear map $\tilde{f}: \mathcal{F}(X) \rightarrow E^*$. In particular, consider the case of $X = D_n$, the discrete metric space with n points so that the distance between any two distinct points is one, $E^* = \mathcal{F}(D_n)$ and $f: D_n \rightarrow \mathcal{F}(D_n)$ the canonical injection. From [11], the Lipschitz 2-summing norm of f is $(2 - 2/n)^{1/2}$. However, \tilde{f} is the identity map on the $(n - 1)$ -dimensional Banach space $\mathcal{F}(D_n)$ and therefore its 2-summing norm is $\sqrt{n - 1}$ (see [9, Theorem 4.17]).

8. LIPSCHITZ APPROXIMABLE OPERATORS

The notions of Lipschitz compact operators and Lipschitz approximable operators from X to E were introduced in [17]. Let us recall that a mapping $f \in \text{Lip}_0(X, E)$ is said to be Lipschitz compact if its Lipschitz image

$$\left\{ \frac{f(x) - f(y)}{d(x, y)} : x, y \in X, x \neq y \right\}$$

is relatively compact in E , and f is said to be Lipschitz approximable if it is the limit in the Lipschitz norm Lip of a sequence of Lipschitz finite-rank operators from X to E .

We show that the spaces of Lipschitz finite-rank operators and Lipschitz approximable operators can be identified as spaces of continuous linear functionals.

Theorem 8.1. *The map $K: X^\# \boxtimes_{\pi'} E^* \rightarrow \text{Lip}_{0F}(X, E^*)$, defined by*

$$K \left(\sum_{j=1}^m g_j \boxtimes \phi_j \right) = \sum_{j=1}^m g_j \cdot \phi_j,$$

is an isometric isomorphism. As a consequence, the space of all Lipschitz approximable operators from X to E^ is isometrically isomorphic to $X^\# \widehat{\boxtimes}_{\pi'} E^*$.*

Proof. By Theorem 2.7, K is a linear bijection. For any $\sum_{j=1}^m g_j \boxtimes \phi_j \in X^\# \boxtimes E^*$, we have

$$\begin{aligned} \pi' \left(\sum_{j=1}^m g_j \boxtimes \phi_j \right) &= \sup \left\{ \left\| \left(\sum_{j=1}^m g_j \boxtimes \phi_j \right) \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right) \right\| : \pi \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right) \leq 1 \right\} \\ &= \sup \left\{ \left\| \sum_{i=1}^n \left(\left(\sum_{j=1}^m g_j \cdot \phi_j \right) (x_i) - \left(\sum_{j=1}^m g_j \cdot \phi_j \right) (y_i), e_i \right) \right\| : \pi \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right) \leq 1 \right\} \\ &= \sup \left\{ \left\| \Lambda_0 \left(\sum_{j=1}^m g_j \cdot \phi_j \right) \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right) \right\| : L \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right) \leq 1 \right\} \\ &= \left\| \Lambda_0 \left(\sum_{j=1}^m g_j \cdot \phi_j \right) \right\| \\ &= \text{Lip} \left(\sum_{j=1}^m g_j \cdot \phi_j \right). \end{aligned}$$

by using Corollary 3.6, Lemmas 2.4 and 1.5, Corollary 6.3 and Theorem 4.3. Hence K is an isometry. The consequence follows from a known result of Functional Analysis. \square

From Theorems 4.3 and 8.1 and Corollary 6.3, we infer the next consequence.

Corollary 8.2. *The space $X^\# \widehat{\boxtimes}_{\pi'} E^*$ is isometrically isomorphic to $(X \widehat{\boxtimes}_{\pi} E)^*$ if and only if $\text{Lip}_0(X, E^*)$ is isometrically isomorphic to the space of Lipschitz approximable operators from X to E^* .*

We recall that a Banach space E is said to have the approximation property if given a compact set $K \subset E$ and $\varepsilon > 0$, there is a finite-rank bounded linear operator $T : E \rightarrow E$ such that $\|Tx - x\| < \varepsilon$ for every $x \in K$. The approximation property was thoroughly studied by Grothendieck in [16]. In [17, Corollary 2.5], it was shown that $X^\#$ has the approximation property if and only if the space of all Lipschitz approximable operators from X to E is the space of all Lipschitz compact operators from X to E . Using this fact and Theorem 8.1, we derive the following result.

Corollary 8.3. *Let X be a pointed metric space such that $X^\#$ has the approximation property. Then, for any Banach space E , the space of all Lipschitz compact operators from X to E^* is isometrically isomorphic to $X^\# \widehat{\boxtimes}_{\pi'} E^*$.*

By Theorem 4.3 and Corollary 8.3, we have the following.

Corollary 8.4. *Let X be a pointed metric space such that $X^\#$ has the approximation property and let E be a Banach space. Then $X^\# \widehat{\boxtimes}_{\pi'} E^*$ is isometrically isomorphic to $(X \widehat{\boxtimes}_{\pi} E)^*$ if and only if $\text{Lip}_0(X, E^*)$ is isometrically isomorphic to the space of Lipschitz compact operators from X to E^* .*

We close this section with a new formula for the norm π' . By [17, Lemma 1.1], the closed unit ball of the Lipschitz-free Banach space $\mathcal{F}(X)$ coincides with the closure of

the convex balanced hull of the set $\{(\delta_x - \delta_y)/d(x, y) : x, y \in X, x \neq y\}$ in $(X^\#)^*$. It is well known that every element in the convex balanced hull of that set is of the form

$$\sum_{i=1}^n \lambda_i \frac{\delta_{x_i} - \delta_{y_i}}{d(x_i, y_i)}$$

for some $n \in \mathbb{N}$, $\lambda_1, \dots, \lambda_n \in \mathbb{K}$, $\sum_{i=1}^n |\lambda_i| \leq 1$ and $(x_1, y_1), \dots, (x_n, y_n) \in X^2$ with $x_i \neq y_i$ for $i \in \{1, \dots, n\}$.

Definition 8.5. For each $\sum_{j=1}^m g_j \boxtimes \phi_j \in X^\# \boxtimes E^*$, define:

$$\varepsilon \left(\sum_{j=1}^m g_j \boxtimes \phi_j \right) = \sup \left\{ \left| \sum_{j=1}^m \gamma(g_j) \langle \phi_j, e \rangle \right| : \gamma \in B_{\mathcal{F}(X)}, e \in B_E \right\}.$$

Note that this supremum exists since, for all $\gamma \in B_{\mathcal{F}(X)}$ and $e \in B_E$,

$$\left| \sum_{j=1}^m \gamma(g_j) \langle \phi_j, e \rangle \right| \leq \sum_{j=1}^m |\gamma(g_j) \langle \phi_j, e \rangle| \leq \sum_{j=1}^m \|\gamma\| \text{Lip}(g_j) \|\phi_j\| \|e\| \leq \sum_{j=1}^m \text{Lip}(g_j) \|\phi_j\|.$$

Theorem 8.6. The associated Lipschitz norm π' of π on $X \boxtimes E$ is ε on $X^\# \boxtimes E^*$.

Proof. Let $\sum_{j=1}^m g_j \boxtimes \phi_j \in X^\# \boxtimes E^*$. We have

$$\begin{aligned} \pi' \left(\sum_{j=1}^m g_j \boxtimes \phi_j \right) &= \sup \left\{ \left| \left(\sum_{j=1}^m g_j \boxtimes \phi_j \right) \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right) \right| : \pi \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right) \leq 1 \right\} \\ &= \sup \left\{ \left| \left(\sum_{j=1}^m g_j \boxtimes \phi_j \right) (\delta_{(x, y)} \boxtimes e) \right| : \pi (\delta_{(x, y)} \boxtimes e) \leq 1 \right\} \\ &= \varepsilon \left(\sum_{j=1}^m g_j \boxtimes \phi_j \right). \end{aligned}$$

In order to justify these equalities, denote by $\alpha(\sum_{j=1}^m g_j \boxtimes \phi_j)$ and $\beta(\sum_{j=1}^m g_j \boxtimes \phi_j)$ the first and the second supremum which appear above. The first equality follows from Corollary 3.6. To see that $\alpha(\sum_{j=1}^m g_j \boxtimes \phi_j) \leq \beta(\sum_{j=1}^m g_j \boxtimes \phi_j)$, we will use the easy fact that if $n \in \mathbb{N}$, $a_1, \dots, a_n \in \mathbb{R}_0^+$ and $b_1, \dots, b_n \in \mathbb{R}^+$, then

$$\frac{a_1 + \dots + a_n}{b_1 + \dots + b_n} \leq \max \left\{ \frac{a_1}{b_1}, \dots, \frac{a_n}{b_n} \right\}.$$

Fix $\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \in X \boxtimes E$, nonzero. If $\sum_{i=1}^p \delta_{(x'_i, y'_i)} \boxtimes e'_i = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i$, we have

$$\begin{aligned} \frac{\left| \left(\sum_{j=1}^m g_j \boxtimes \phi_j \right) \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right) \right|}{\sum_{i=1}^p d(x'_i, y'_i) \|e'_i\|} &= \frac{\left| \left(\sum_{j=1}^m g_j \boxtimes \phi_j \right) \left(\sum_{i=1}^p \delta_{(x'_i, y'_i)} \boxtimes e'_i \right) \right|}{\sum_{i=1}^p d(x'_i, y'_i) \|e'_i\|} \\ &\leq \frac{\sum_{i=1}^p \left| \left(\sum_{j=1}^m g_j \boxtimes \phi_j \right) \left(\delta_{(x'_i, y'_i)} \boxtimes e'_i \right) \right|}{\sum_{i=1}^p d(x'_i, y'_i) \|e'_i\|} \\ &\leq \max \left\{ \frac{\left| \left(\sum_{j=1}^m g_j \boxtimes \phi_j \right) \left(\delta_{(x'_i, y'_i)} \boxtimes e'_i \right) \right|}{d(x'_i, y'_i) \|e'_i\|} : 1 \leq i \leq p \right\} \\ &\leq \beta \left(\sum_{j=1}^m g_j \boxtimes \phi_j \right). \end{aligned}$$

By the definition of π , it follows that

$$\left| \left(\sum_{j=1}^m g_j \boxtimes \phi_j \right) \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right) \right| \leq \beta \left(\sum_{j=1}^m g_j \boxtimes \phi_j \right) \pi \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right).$$

This ensures that $\alpha(\sum_{j=1}^m g_j \boxtimes \phi_j) \leq \beta(\sum_{j=1}^m g_j \boxtimes \phi_j)$. The converse inequality is clearly certain.

Now, given any $\delta_{(x, y)} \boxtimes e \in X \boxtimes E$ with $0 < \pi(\delta_{(x, y)} \boxtimes e) \leq 1$, we obtain

$$\begin{aligned} \left| \left(\sum_{j=1}^m g_j \boxtimes \phi_j \right) \left(\delta_{(x, y)} \boxtimes e \right) \right| &= \left| \sum_{j=1}^m (g_j(x) - g_j(y)) \langle \phi_j, e \rangle \right| \\ &= \left| \sum_{j=1}^m \left(\frac{\delta_x - \delta_y}{d(x, y)} \right) (g_j) \langle \phi_j, d(x, y)e \rangle \right| \\ &\leq \varepsilon \left(\sum_{j=1}^m g_j \boxtimes \phi_j \right) \end{aligned}$$

since $(\delta_x - \delta_y)/d(x, y) \in S_{\mathcal{F}(X)}$ and $d(x, y)e \in B_E$. Passing to the supremum we arrive at $\beta(\sum_{j=1}^m g_j \boxtimes \phi_j) \leq \varepsilon(\sum_{j=1}^m g_j \boxtimes \phi_j)$.

Finally, we show that $\varepsilon(\sum_{j=1}^m g_j \boxtimes \phi_j) \leq \pi'(\sum_{j=1}^m g_j \boxtimes \phi_j)$. For any $n \in \mathbb{N}$, $\lambda_1, \dots, \lambda_n \in \mathbb{K}$, $\sum_{i=1}^n |\lambda_i| \leq 1$, $(x_i, y_i) \in X^2$ and $x_i \neq y_i$ for each $i \in \{1, \dots, n\}$, and $e \in B_E$, we have

$$\begin{aligned} \left| \sum_{j=1}^m \left(\sum_{i=1}^n \lambda_i \frac{\delta_{x_i} - \delta_{y_i}}{d(x_i, y_i)} \right) (g_j) \langle \phi_j, e \rangle \right| &= \left| \sum_{j=1}^m \left(\sum_{i=1}^n \lambda_i \frac{g_j(x_i) - g_j(y_i)}{d(x_i, y_i)} \langle \phi_j, e \rangle \right) \right| \\ &= \left| \sum_{j=1}^m \left(\sum_{i=1}^n \frac{\lambda_i}{d(x_i, y_i)} (g_j \boxtimes \phi_j)(\delta_{(x_i, y_i)} \boxtimes e) \right) \right| \\ &= \left| \left(\sum_{j=1}^m g_j \boxtimes \phi_j \right) \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes \frac{\lambda_i e}{d(x_i, y_i)} \right) \right| \\ &\leq \pi' \left(\sum_{j=1}^m g_j \boxtimes \phi_j \right) \end{aligned}$$

since

$$\begin{aligned} \pi \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes \frac{\lambda_i e}{d(x_i, y_i)} \right) &\leq \sum_{i=1}^n \pi \left(\delta_{(x_i, y_i)} \boxtimes \frac{\lambda_i e}{d(x_i, y_i)} \right) \\ &= \sum_{i=1}^n d(x_i, y_i) \frac{|\lambda_i| \|e\|}{d(x_i, y_i)} \\ &= \|e\| \sum_{i=1}^n |\lambda_i| \leq 1. \end{aligned}$$

By the density of the elements $\sum_{i=1}^n \lambda_i (\delta_{x_i} - \delta_{y_i}) / d(x_i, y_i)$ in $B_{\mathcal{F}(X)}$, we infer that

$$\left| \sum_{j=1}^m \gamma(g_j) \langle \phi_j, e \rangle \right| \leq \pi' \left(\sum_{j=1}^m g_j \boxtimes \phi_j \right)$$

for all $\gamma \in B_{\mathcal{F}(X)}$ and $e \in B_E$. Taking supremum over all such γ and e , we conclude that $\varepsilon(\sum_{j=1}^m g_j \boxtimes \phi_j) \leq \pi'(\sum_{j=1}^m g_j \boxtimes \phi_j)$ and this completes the proof. \square

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