TOEPLITZ AND HANKEL OPERATORS ON A VECTOR-VALUED BERGMAN SPACE

NAMITA DAS

Communicated by A.R. Mirmostafaei

ABSTRACT. In this paper we derive certain algebraic properties of Toeplitz and Hankel operators defined on the vector-valued Bergman spaces $L_a^{2,c^n}(D)$ where $D$ is the open unit disk in $\mathbb{C}$ and $n \geq 1$. We show that the set of all Toeplitz operators $T_{\Phi}, \Phi \in L_{\infty}^M(D)$ is strongly dense in the set of all bounded linear operators $L(L_a^{2,c^n}(D))$ and characterize all finite rank little Hankel operators.

1. INTRODUCTION

Let $D = \{ z \in \mathbb{C} : |z| < 1 \}$ be the open unit disc in the complex plane $\mathbb{C}$ and let $dA(z) = \frac{1}{\pi} dx dy = \frac{1}{\pi} r dr d\theta$ be the area measure on $D$ normalised so that the area of $D$ is $1$. For $1 \leq p < \infty$, the Bergman space $L_a^p(D)$ is the space of all holomorphic functions $f$ in $D$ for which

$$
\| f \|_{L_a^p(D)} = \left( \int_D |f(z)|^p dA(z) \right)^{\frac{1}{p}} < \infty.
$$

The quantity $\| \cdot \|_{L_a^p(D)}$ is a norm if $p \geq 1$. Thus $L_a^p(D)$ is the subspace of holomorphic functions that are in the space $L^p(D, dA)$. The Bergman spaces are Banach spaces, which is a consequence of the estimate:

$$
\sup_{z \in K} |f(z)| \leq C_K \| f \|_{L_a^p(D)}
$$

valid on compact subsets $K$ of $D$. If $p = 2$, then $L_a^p(D)$ is a Hilbert space. Since point evaluation at $z \in D$ is a bounded linear functional [12] on the Hilbert
functions \( L^2(\mathbb{D}) \) form the standard orthonormal basis for \( L^2(\mathbb{D}) \) and is called the Bergman kernel. The sequence \( \{e_n(z)\}_{n \geq 0} = \{\sqrt{n + 1}z^n\}_{n \geq 0} \) of functions [12] form the standard orthonormal basis for \( L^2(\mathbb{D}) \) and

\[
K(z, w) = \sum_{n=1}^{\infty} e_n(z)\overline{e_n(w)}.
\]

The Bergman kernel is independent of the choice of orthonormal basis and \( K(z, w) = \frac{1}{(1 - z\bar{w})^2} \). Let \( k_\alpha(z) = \frac{K(z, \alpha)}{\sqrt{K(\alpha, \alpha)}} = \frac{1 - |\alpha|^2}{(1 - z\bar{\alpha})^2} \). These functions \( k_\alpha \) are called the normalized reproducing kernels of \( L^2(\mathbb{D}) \); it is clear that they are unit vectors in \( L^2(\mathbb{D}) \). Let \( L^\infty(\mathbb{D}, dA) \) denote the Banach space of Lebesgue measurable functions \( f \) on \( \mathbb{D} \) with

\[
\|f\|_\infty = \operatorname{esssup}\{ |f(z)| : z \in \mathbb{D} \} < \infty
\]

and \( H^\infty(\mathbb{D}) \) be the space of bounded analytic functions on \( \mathbb{D} \).

Let \( L^2,\mathbb{C}^n(\mathbb{D}) = L^2(\mathbb{D}) \otimes \mathbb{C}^n \) and \( L^\infty,\mathbb{M}_n(\mathbb{D}) = L^\infty(\mathbb{D}) \otimes \mathbb{M}_n \) where \( \mathbb{M}_n(\mathbb{C}) = M_n, n \geq 1 \) is the set of all \( n \times n \) matrices with entries in \( \mathbb{C} \). The space \( L^2,\mathbb{C}^n(\mathbb{D}), n \geq 1 \) is called the vector-valued Bergman space. The inner product on \( L^2,\mathbb{C}^n(\mathbb{D}) \) is defined as

\[
\langle f, g \rangle_{L^2,\mathbb{C}^n(\mathbb{D})} = \int_\mathbb{D} \langle f(z), g(z) \rangle_{\mathbb{C}^n} dA(z).
\]

With this inner product \( L^2,\mathbb{C}^n(\mathbb{D}) \) is a Hilbert space. The norm defined on \( L^2,\mathbb{C}^n(\mathbb{D}) \) is given by

\[
\|f\|^2_{L^2,\mathbb{C}^n(\mathbb{D}, dA)} = \int_\mathbb{D} \|f(z)\|^2_{\mathbb{C}^n} dA(z).
\]

It is a closed subspace of \( L^2,\mathbb{C}^n(\mathbb{D}, dA) = L^2(\mathbb{D}, dA) \otimes \mathbb{C}^n \). Let \( P \) denote the orthogonal projection from \( L^2,\mathbb{C}^n(\mathbb{D}, dA) \) onto \( L^2,\mathbb{C}^n(\mathbb{D}) \). For \( \Phi \in L^\infty,\mathbb{M}_n(\mathbb{D}) \), we define the Toeplitz operator \( T_\Phi \) from \( L^2,\mathbb{C}^n(\mathbb{D}) \) into itself as \( T_\Phi f = P(\Phi f) \) and the Hankel operator \( H_\Phi \) from \( L^2,\mathbb{C}^n(\mathbb{D}) \) into \( (L^2,\mathbb{C}^n(\mathbb{D}))^\perp = L^2,\mathbb{C}^n(\mathbb{D}, dA) \otimes L^2,\mathbb{C}^n(\mathbb{D}) \) as \( H_\Phi f = (I - P)(\Phi f) \). For \( \Phi \in L^\infty,\mathbb{M}_n(\mathbb{D}) \), define \( \|\Phi\|_\infty = \operatorname{esssup}_{z \in \mathbb{D}} \|\Phi(z)\| \). If \( \Phi \in L^\infty,\mathbb{M}_n(\mathbb{D}) \), then it is not difficult to see that \( \|T_\Phi\| \leq \|\Phi\|_\infty \) and \( \|H_\Phi\| \leq \|\Phi\|_\infty \).

This is so as \( \|P\| \leq 1 \) and \( \|I - P\| \leq 1 \).

For \( \Phi \in L^\infty,\mathbb{M}_n(\mathbb{D}) \), we define the little Hankel operator \( S_\Phi \) from \( L^2,\mathbb{C}^n(\mathbb{D}) \) into itself as \( S_\Phi f = PJ(\Phi f) \) where \( J : L^2,\mathbb{C}^n(\mathbb{D}, dA) \to L^2,\mathbb{C}^n(\mathbb{D}, dA) \) is defined as \( Jf(z) = f(\bar{z}) \). The map \( J \) is unitary. There are also many equivalent ways of defining little Hankel operators. Let \( L^2,\mathbb{C}^n(\mathbb{D}) = L^2,\mathbb{C}^n(\mathbb{D}) \otimes \mathbb{C}^n \). For \( \Phi \in L^\infty,\mathbb{M}_n(\mathbb{D}) \), define \( h_\Phi \) from \( L^2,\mathbb{C}^n(\mathbb{D}) \) into \( L^2,\mathbb{C}^n(\mathbb{D}) \) as \( h_\Phi f = \overline{\Phi f} \) where \( \overline{\Phi} \) is the orthogonal
projection from $L^2_{\mathcal{C}_n}(\mathbb{D}, dA)$ onto $\overline{L^2_{\mathcal{C}_n}(\mathbb{D})}$. It is not difficult to verify that $h_\Phi = J S \Phi$.

Let $\mathcal{L}(H)$ be the set of all bounded linear operators from the Hilbert space $H$ into itself and $\mathcal{L}C(H)$ be the set of all compact operators in $\mathcal{L}(H)$.

Consider the direct sum $\sum_{k=1}^{n} \oplus L_k$, with each $L_k$ the same Hilbert space $L^2_a(\mathbb{D})$. Define the bounded linear operators

$$U_i : L^2_a(\mathbb{D}) \rightarrow \sum_{k=1}^{n} \oplus L_k, \quad V_i : \sum_{k=1}^{n} \oplus L_k \rightarrow L^2_a(\mathbb{D}),$$

for each $i \in \{1, 2, \cdots, n\}$ as follows. When $f \in L^2_a(\mathbb{D})$ and $g = \{g_k\} \in \sum_{k=1}^{n} \oplus L_k$, $V_i g = g_i$ and $U_i f$ is the family $\{h_k\}$ in which $h_i = f$ and all other $h_k$ are 0. Let $L_i'$ be the range of $U_i$. It consists of all elements $\{h_k\}$ of $\sum_{k=1}^{n} \oplus L_k$ in which $h_k = 0$ when $k \neq i$. The space $L_i'$ is a closed subspace of $\sum_{k=1}^{n} \oplus L_k$ and observe that $V_i U_i$ is the identity operator on $L^2_a(\mathbb{D})$ and $U_i V_i$ is the projection $E_i$ from $\sum_{k=1}^{n} \oplus L_k$ onto $L_i'$. Since the subspace $L_i', i \in \{1, 2, \cdots, n\}$ are pairwise orthogonal, and $\sum_{i=1}^{n} L_i' = \sum_{k=1}^{n} \oplus L_k$, it follows that the sum $\sum_{i=1}^{n} E_i = I$. Note that $U_i = V_i^*$, since

$$\langle U_i f, \{f_k\} \rangle = \langle f, f_i \rangle = \langle f, V_i \{f_k\} \rangle$$

whenever $f \in L^2_a(\mathbb{D})$ and $\{f_k\} \in \sum_{k=1}^{n} \oplus L_k$. With each bounded linear operator $T$ acting on $\sum_{k=1}^{n} \oplus L_k$, we associate a matrix $(T_{ij})_{1 \leq i, j \leq n}$, with entries $T_{ij}$ in $\mathcal{L}(L^2_a(\mathbb{D}))$ defined by

$$T_{ij} = V_i T U_j. \quad (1.1)$$

If $g = \{g_k\} \in \sum_{k=1}^{n} \oplus L_k$, then $T g$ is an element $\{p_k\}$ of $\sum_{k=1}^{n} \oplus L_k$ and

$$p_i = V_i T g = V_i T \left( \sum_{k=1}^{n} E_k g \right) = \sum_{k=1}^{n} V_i T U_j V_j g = \sum_{j=1}^{n} T_{ij} g_j.$$

Thus

$$T \left( \sum_{k=1}^{n} \oplus g_k \right) = \sum_{k=1}^{n} \oplus p_k \text{ where } p_i = \sum_{j=1}^{n} T_{ij} g_j, \quad i \in \{1, 2, \cdots, n\}. \quad (1.2)$$

The usual rules of matrix algebra have natural analogues in this situation. From (1.1), the matrix elements $T_{ij}$ depend linearly on $T$. Since

$$V_i T^* U_j = U_i^* T^* V_j^* = (V_j T U_i)^* = (T_{ji})^*,$$

the matrix of $T^*$ has $(T_{ji})^*$ in the $(i, j)$ position. If $S$ and $T$ are bounded linear operators acting on $\sum_{k=1}^{n} \oplus L_k$, and $R = ST$, then

$$R_{ij} = V_i R U_j = V_i S T U_j = \sum_{k=1}^{n} V_i S E_k T U_j$$

$$= \sum_{k=1}^{n} V_i S U_k V_k T U_j = \sum_{k=1}^{n} S_{ik} T_{kj}.$$
Thus we establish a one-to-one correspondence between elements of \( \mathcal{L}(\sum_{k=1}^{n} \oplus L_k) \) and certain matrices \((T_{ij})_{i,j=1}^{n}\) with entries \( T_{ij} \in \mathcal{L}(L_a^2(\mathbb{D})) \). Each such matrix corresponds to some bounded operator \( T \) acting on \( \sum_{k=1}^{n} \oplus L_k \); indeed, \( T \) is defined by (1.2), and its boundedness follows at once from the relations

\[
\|\{p_i\}\|^2 = \sum_{i=1}^{n} \|p_i\|^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} T_{ij} g_j \| g_j \|^2 \leq \sum_{i=1}^{n} \left( \sum_{j=1}^{n} \|T_{ij}\| \|g_j\| \right)^2 \\
\leq \sum_{i=1}^{n} \left( \sum_{j=1}^{n} \|T_{ij}\|^2 \right) \left( \sum_{j=1}^{n} \|g_j\|^2 \right) = \left( \sum_{i=1}^{n} \sum_{j=1}^{n} \|T_{ij}\|^2 \right) \|\{g_k\}\|^2.
\]

In this paper we derive certain algebraic properties of Toeplitz and Hankel operators defined on the vector-valued Bergman spaces \( L_a^{2,\mathbb{C}^n}(\mathbb{D}) \), \( n \geq 1 \). We have shown that if there exists \( A, B \in \mathcal{L}(L_a^{2,\mathbb{C}^n}(\mathbb{D})) \) such that \( AT_{\Phi} B = T_{\Phi} \) for all \( \Phi \in L_{M_n}(\mathbb{D}) \), then \( A = \alpha I_{\mathcal{L}(L_a^{2,\mathbb{C}^n}(\mathbb{D}))} \), \( B = \beta I_{\mathcal{L}(L_a^{2,\mathbb{C}^n}(\mathbb{D}))} \), \( \alpha, \beta \in \mathbb{C} \) and \( \alpha \beta = 1 \) and that the set of all Toeplitz operators \( T_{\Phi}, \Phi \in L_{M_n}(\mathbb{D}) \) is strongly dense in the set of all bounded linear operators \( \mathcal{L}(L_a^{2,\mathbb{C}^n}(\mathbb{D})) \) and characterize all finite rank little Hankel operators defined on the vector-valued Bergman space. The layout of this paper is as follows. In section 2, we establish that if \( AT_{\Phi} B = T_{\Phi} \) for all \( \Phi \in L_{M_n}(\mathbb{D}) \), then \( A = \alpha I_{\mathcal{L}(L_a^{2,\mathbb{C}^n}(\mathbb{D}))} \), \( B = \beta I_{\mathcal{L}(L_a^{2,\mathbb{C}^n}(\mathbb{D}))} \), \( \alpha, \beta \in \mathbb{C} \) and \( \alpha \beta = 1 \). Furthermore, it is shown that the set of all Toeplitz operators \( T_{\Phi}, \Phi \in L_{M_n}(\mathbb{D}) \) from \( L_a^{2,\mathbb{C}^n}(\mathbb{D}) \) into itself is strongly dense in the Banach space \( \mathcal{L}(L_a^{2,\mathbb{C}^n}(\mathbb{D})) \). In section 3, we prove that there exists no finite rank Hankel operator \( H_{\Phi} \) with nonconstant matrix-valued symbol \( \Phi \) that is diagonal. We further establish certain elementary properties of little Hankel operators and characterize all finite rank little Hankel operators with diagonal matrix-valued symbols.

2. Toeplitz operators with symbols in \( L_{M_n}(\mathbb{D}) \)

In this section we have shown that if there exists \( A, B \in \mathcal{L}(L_a^{2,\mathbb{C}^n}(\mathbb{D})) \) such that \( AT_{\Phi} B = T_{\Phi} \) for all \( \Phi \in L_{M_n}(\mathbb{D}) \), then \( A = \alpha I_{\mathcal{L}(L_a^{2,\mathbb{C}^n}(\mathbb{D}))} \), \( B = \beta I_{\mathcal{L}(L_a^{2,\mathbb{C}^n}(\mathbb{D}))} \), \( \alpha, \beta \in \mathbb{C} \) and \( \alpha \beta = 1 \) Here \( I_{\mathcal{L}(L_a^{2,\mathbb{C}^n}(\mathbb{D}))} \) is the identity operator from the space \( L_a^{2,\mathbb{C}^n}(\mathbb{D}) \) into itself. Further, we show that the set of all Toeplitz operators \( T_{\Phi}, \Phi \in L_{M_n}(\mathbb{D}) \) from \( L_a^{2,\mathbb{C}^n}(\mathbb{D}) \) into itself is strongly dense in the Banach space \( \mathcal{L}(L_a^{2,\mathbb{C}^n}(\mathbb{D})) \).

**Theorem 2.1.** If \( A, B \in \mathcal{L}(L_a^{2,\mathbb{C}^n}(\mathbb{D})), n \geq 1 \) and \( AT_{\Phi} B = T_{\Phi} \) for all \( \Phi \in L_{M_n}(\mathbb{D}) \), then \( A = \alpha I_{\mathcal{L}(L_a^{2,\mathbb{C}^n}(\mathbb{D}))} \), \( B = \beta I_{\mathcal{L}(L_a^{2,\mathbb{C}^n}(\mathbb{D}))} \), \( \alpha, \beta \in \mathbb{C} \) and \( \alpha \beta = 1 \).

**Proof.** Suppose \( A, B \in \mathcal{L}(L_a^{2,\mathbb{C}^n}(\mathbb{D})), n \geq 1 \) and \( AT_{\Phi} B = T_{\Phi} \) for all \( \Phi \in L_{M_n}(\mathbb{D}) \). Since \( L_a^{2,\mathbb{C}^n}(\mathbb{D}) = L_2(\mathbb{D}) \otimes \mathbb{C}^n \), we obtain

\[
A = \begin{pmatrix}
A_{11} & A_{12} & \cdots & A_{1n} \\
A_{21} & A_{22} & \cdots & A_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
A_{n1} & A_{n2} & \cdots & A_{nn}
\end{pmatrix}
\quad \text{and} \quad
B = \begin{pmatrix}
B_{11} & B_{12} & \cdots & B_{1n} \\
B_{21} & B_{22} & \cdots & B_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
B_{n1} & B_{n2} & \cdots & B_{nn}
\end{pmatrix},
\]

where \( A_{ij}, B_{ij} \in \mathcal{L}(L_a^2(\mathbb{D})) \) for all \( i, j \in \{1, 2, \ldots, n\} \). Here \( A_{ij} = V_i A U_j \) and \( B_{ij} = \cdots \)
$V_i B U_j$ for all $i, j \in \{1, 2, \cdots, n\}$. Further, as $\Phi \in L^\infty_{M_n}(\mathbb{D}) = L^\infty(\mathbb{D}) \otimes M_n$, we have

$$\Phi = \begin{pmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{pmatrix},$$

where $\phi_{ij} \in L^\infty(\mathbb{D})$ for all $i, j \in \{1, 2, \cdots, n\}$.

Hence

$$T \Phi = \begin{pmatrix} T_{\phi_{11}} & T_{\phi_{12}} & \cdots & T_{\phi_{1n}} \\ T_{\phi_{21}} & T_{\phi_{22}} & \cdots & T_{\phi_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ T_{\phi_{n1}} & T_{\phi_{n2}} & \cdots & T_{\phi_{nn}} \end{pmatrix}. $$

By considering elementary matrices of the type

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & T_{\phi_{ij}} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \end{pmatrix},$$

with just one nonzero $(i, j)$th entry $T_{\phi_{ij}}$, $\phi_{ij} \in L^\infty(\mathbb{D})$, $i, j \in \{1, 2, \cdots, n\}$ and using the operator equations

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix} \begin{pmatrix} 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & T_{\phi_{ij}} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ B_{21} & B_{22} & \cdots & B_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ B_{n1} & B_{n2} & \cdots & B_{nn} \end{pmatrix},$$

it follows from [5] that $V_i A U_j = V_i B U_j = 0$ if $i \neq j$, $i, j = 1, 2, \cdots, n$ and $V_i A U_i = \alpha I_{\mathcal{L}(L_2^2(\mathbb{D}))}$, $V_i B U_i = \beta I_{\mathcal{L}(L_2^2(\mathbb{D}))}$ for all $i = 1, 2, \cdots, n$ and for some $\alpha, \beta \in \mathbb{C}$ such that $\alpha \beta = 1$. This implies $A = \alpha I_{\mathcal{L}(L_2^2, C^n(\mathbb{D}))}$ and $B = \beta I_{\mathcal{L}(L_2^2, C^n(\mathbb{D}))}$. The theorem follows. □
Theorem 2.2. Let $T \in \mathcal{L}(L^2_a, \mathbb{C}^n(\mathbb{D})), n \geq 1, F_i = \begin{pmatrix} F_{i1} \\ \vdots \\ F_{in} \end{pmatrix} \in L^2_a, \mathbb{C}^n(\mathbb{D}), G_i = \begin{pmatrix} G_{i1} \\ \vdots \\ G_{in} \end{pmatrix} \in L^2_a, \mathbb{C}^n(\mathbb{D}), i = 1, \cdots, N$. Then there exists $\Phi \in L^\infty_M(\mathbb{D})$ such that 

$$\langle T\Phi F_i, G_i \rangle = \langle TF_i, G_i \rangle, i = 1, \cdots, N.$$ 

Proof. Let $f_1, f_2, \cdots, f_k$ and $g_1, g_2, \cdots, g_m$ respectively be bases of the finite-dimensional subspaces of $L^2_a, \mathbb{C}^n(\mathbb{D})$ generated by $F_1, \cdots, F_N$ and $G_1, \cdots, G_N$. We shall find $\Phi \in L^\infty_M(\mathbb{D})$ such that 

$$\langle T\Phi f_i, g_j \rangle = \langle Tf_i, g_j \rangle$$ 

for all $i = 1, \cdots, k$ and $j = 1, \cdots, m$.

Consider the operator $R : L^\infty_M(\mathbb{D}) \to \mathbb{C}^{k \times m}$, defined by $(R\Phi)_{ij} = \langle T\Phi f_i, g_j \rangle$, $i = 1, \cdots, k$ and $j = 1, \cdots, m$. Suppose $u \in \mathbb{C}^{k \times m}$ is orthogonal to the range of $R$. That is, let

$$\sum_{i=1}^k \sum_{j=1}^m (R\Phi)_{ij} u_{ij} = 0$$

for all $\Phi \in L^\infty_M(\mathbb{D})$. This implies (taking $\Phi = I_{n \times n}$, the identity matrix)

$$\sum_{i=1}^k \sum_{j=1}^m \langle f_i, g_j \rangle_{L^2_a, \mathbb{C}^n(\mathbb{D})} u_{ij} = 0.$$ 

Hence

$$\sum_{i=1}^k \sum_{j=1}^m \langle f_i(z), g_j(z) \rangle_{\mathbb{C}^n} u_{ij} = 0$$ 

almost everywhere on $\mathbb{D}$. Since the left hand side is obviously continuous on $\mathbb{D}$, this equality holds, in fact, on the whole of $\mathbb{D}$. Thus the function

$$\Omega(x, y) = \sum_{i=1}^k \sum_{j=1}^m \langle f_i(x), g_j(y) \rangle_{\mathbb{C}^n} u_{ij}$$ 

which is analytic in $\mathbb{D} \times \mathbb{D}$, equals zero when $x = y$. By the uniqueness theorem [11], this implies that $\Omega \equiv 0$ on $\mathbb{D} \times \mathbb{D}$. Because, functions $f_i, i = 1, 2, \cdots, k$, are linearly independent, we obtain

$$\sum_{j=1}^m u_{ij} g_j(y) = 0$$

for all $y \in \mathbb{D}, i = 1, 2, \cdots, k$; but $g_j, j = 1, 2, \cdots, m$, are also linearly independent, and so $u_{ij} = 0$ for all $i, j$; i.e., $u = 0$. This means that the range of $R$ is all of $\mathbb{C}^{k \times m}$ and the result follows. \hfill \Box

Theorem 2.3. The set of all Toeplitz operators $T_\Phi, \Phi \in L^\infty_M(\mathbb{D})$ is dense in $\mathcal{L}(L^2_a, \mathbb{C}^n(\mathbb{D}))$ in the strong operator topology.
Proof. From Theorem 2.2, it follows that the collection \( \mathcal{N} = \{ T_\Phi : \Phi \in L^\infty_{M_n}(\mathbb{D}) \} \) is dense in \( L(L^2_a, C^n)(\mathbb{D}) \) in the weak operator topology. As \( \mathcal{N} \) is a subspace, i.e., a convex set, its weak operator topology and strong operator topology closures coincide. Hence \( \mathcal{N} \) is dense in \( L(L^2_a, C^n)(\mathbb{D}) \) in the strong operator topology. Let \( T \in L(L^2_a, C^n)(\mathbb{D}) \). Then there exists \( \Phi_N \in L^\infty_{M_n}(\mathbb{D}) \) such that \( T_{\Phi_N} \to T \) in the strong operator topology. This can also be verified as follows: Let \( T = \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{pmatrix} \) where \( T_{ij} = V_i T U_j \in L(L^2_a(\mathbb{D})) \). From [6] and [7], it follows that \( \{ T_\Phi : \Phi \in L^\infty(\mathbb{D}) \} \) is dense in \( L(L^2_a(\mathbb{D})) \) in the strong operator topology. Thus there exists a sequence \( T_{\phi^{ij}_m} \) that converges to \( T_{ij} \) strongly for all \( i, j \in \{1, 2, \cdots , n\} \). Let \( \Phi_m = (\phi^{ij}_m)_{i,j=1}^n \). Then for \( F = (f_1, f_2, \cdots , f_n)^T \in L(L^2_a(\mathbb{D})) \), we obtain

\[
\| T_{\Phi_m} F - TF \| = \left\| \begin{pmatrix} T_{\phi^{11}_m} - T_{11} & T_{\phi^{12}_m} - T_{12} & \cdots & T_{\phi^{1n}_m} - T_{1n} \\ T_{\phi^{21}_m} - T_{21} & T_{\phi^{22}_m} - T_{22} & \cdots & T_{\phi^{2n}_m} - T_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ T_{\phi^{n1}_m} - T_{n1} & T_{\phi^{n2}_m} - T_{n2} & \cdots & T_{\phi^{nn}_m} - T_{nn} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} \right\|^2 
\]

\[
= \left\| \begin{pmatrix} (T_{\phi^{11}_m} - T_{11}) f_1 + (T_{\phi^{12}_m} - T_{12}) f_2 + \cdots + (T_{\phi^{1n}_m} - T_{1n}) f_n \\ (T_{\phi^{21}_m} - T_{21}) f_1 + (T_{\phi^{22}_m} - T_{22}) f_2 + \cdots + (T_{\phi^{2n}_m} - T_{2n}) f_n \\ \vdots \\ (T_{\phi^{n1}_m} - T_{n1}) f_1 + (T_{\phi^{n2}_m} - T_{n2}) f_2 + \cdots + (T_{\phi^{nn}_m} - T_{nn}) f_n \end{pmatrix} \right\|^2 
\]

\[
\leq \sum_{i,j=1}^n \| T_{\phi^{ij}_m} f_j - T_{ij} f_j \|^2 \to 0 
\]

as \( m \to \infty \). Hence the set of all Toeplitz operators \( \{ T_\Phi, \Phi \in L^\infty_{M_n}(\mathbb{D}) \} \) is dense in \( L(L^2_a, C^n)(\mathbb{D}) \) in the strong operator topology.

3. Hankel operators with matrix-valued symbols

Suppose \( \Phi \in L^\infty_{M_n}(\mathbb{D}) \). In this section we show that \( H_\Phi \equiv 0 \) if and only if \( \Phi \in H^\infty_{M_n}(\mathbb{D}) \) and that there exists no finite rank Hankel operator \( H_\Phi \) with nonconstant matrix-valued symbol \( \Phi \) that is diagonal. We further establish certain elementary properties of little Hankel operators and characterize all finite rank little Hankel operators with diagonal matrix-valued symbols.

**Theorem 3.1.** Let \( \Phi \in L^\infty_{M_n}(\mathbb{D}) \) and \( \Phi = \begin{pmatrix} \phi_{11} & 0 & \cdots & 0 \\ 0 & \phi_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi_{nn} \end{pmatrix} \), where \( \phi_{ii} \in L^\infty(\mathbb{D}), 1 \leq i \leq n \). The following hold:

(i) The operator \( H_\Phi \equiv 0 \) if and only if \( \Phi \in H^\infty_{M_n}(\mathbb{D}) \).
(ii) The operator $H_{\phi jj} \neq 0$ for all $j \in \{1, 2, \ldots, n\}$ if and only if $\ker H_{\phi} = \{0\}$. Further $H_{\phi} \equiv 0$ if and only if $H_{\phi} = L_{a, C^\infty}(D)$.

(iii) If in addition, $\Phi \in H_{M_n}(D)$, then the operator $H_{\phi \phi}$ is a finite rank Hankel operator if and only if $\Phi$ is a diagonal matrix with entries in $\mathbb{C}$.

Proof. It is not difficult to see that $H_{\phi} = \begin{pmatrix} H_{\phi_{11}} & 0 & \cdots & 0 \\ 0 & H_{\phi_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & H_{\phi_{nn}} \end{pmatrix}$, where $H_{\phi_{ii}} \in L(L_a^2(D))$ is a Hankel operator with symbol $\phi_{ii} \in L^\infty(D)$.

Suppose $\phi \in L^\infty(D)$. Before we begin the proof of the theorem, the points to note are the following:

(a) If $\phi f \in L_a^2(D)$ for all $f \in L_a^2(D)$ then $\phi \in H^\infty(D)$.

(b) $H_{\phi} \equiv 0$ if and only if $\phi \in H^\infty(D)$.

The statement (a) can be verified as follows: Suppose $\phi L_a^2(D) \subset L_a^2(D)$. Then $T_\phi f = \phi f$ and therefore $\phi(z) = T_\phi f(z) = \phi(z)$. Hence $\phi$ is analytic on $\mathbb{D} \setminus \{0\}$ of $f$. Each isolated singularity of $\phi$ in $\mathbb{D}$ is removable, since $\phi$ is assumed to be bounded. Thus $\phi$ is analytic on $\mathbb{D}$. Since $\phi \in L^\infty(D)$, we have $\phi \in H^\infty(D)$.

To establish (b), suppose $H_{\phi} \equiv 0$. Then $H_{\phi} f = 0$ for all $f \in L_a^2(D)$. That is, $T_\phi f = \phi f$. From (a) it follows that $\phi \in H^\infty(D)$. Conversely, if $\phi \in H^\infty(D)$, then $\phi f \in L_a^2(D)$ for all $f \in L_a^2(D)$. Hence $H_{\phi} f = 0$ for all $f \in L_a^2(D)$. Therefore $H_{\phi} \equiv 0$.

Now (i) follows from (a) and (b) since $H_{\phi} \equiv 0$ if and only if $H_{\phi_{jj}} \equiv 0$ for all $j \in \{1, 2, \ldots, n\}$. That is, if and only if $\phi_{jj} \in H^\infty(D)$ for all $j \in \{1, 2, \ldots, n\}$. Thus $H_{\phi} \equiv 0$ if and only if $\Phi \in H_{M_n}(D)$.

To prove (ii), suppose $\phi \in L^\infty(D)$. Then

$$\ker H_{\phi} = \{ f \in L_a^2(D) : (I - P)(\phi f) = 0 \} = \{ f \in L_a^2(D) : \phi f \in L_a^2(D) \}.$$

Now if $\ker H_{\phi} \neq \{0\}$, then $\phi \in H^\infty(D)$ (proceed as in (a)). This implies $H_{\phi}$ is equivalent to zero and $\ker H_{\phi} = L_a^2(D)$. Thus if $H_{\phi} \neq 0$, then $\ker H_{\phi} = \{0\}$.

Further, if $\ker H_{\phi} = \{0\}$ then it follows that $\phi \not\in H^\infty(D)$ and $H_{\phi} \neq 0$. To prove (ii), let $\Phi \in L^\infty(M_n(D))$. Then $\ker H_{\phi}$ is equal to

$$\left\{(f_1, f_2, \ldots, f_n) \in L_a^2(C^\infty(D)) : \begin{pmatrix} H_{\phi_{11}} & 0 & \cdots & 0 \\ 0 & H_{\phi_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & H_{\phi_{nn}} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\} = \{(f_1, f_2, \ldots, f_n) \in L_a^2(C^\infty(D)) : (H_{\phi_{jj}} f_j = 0 \text{ for all } j \in \{1, 2, \ldots, n\}) \}.$$

Thus it follows that $\ker H_{\phi} = \{0\}$ if and only if $\ker H_{\phi_{jj}} = \{0\}$ for all $j \in \{1, 2, \ldots, n\}$. But $\ker H_{\phi_{jj}} = \{0\}$ for all $j \in \{1, 2, \ldots, n\}$ if and only if $H_{\phi_{jj}} \neq 0$ for all $j \in \{1, 2, \ldots, n\}$.

To prove (iii), we shall first show that if $\phi \in H^\infty(D)$, then $H_{\phi \phi}$ is a finite rank Hankel operator if and only if $\phi$ is a constant. This can be verified as follows:
Sufficiency is obvious. For the necessity, suppose that $H_{\bar{\phi}}$ is a finite rank operator, where $\phi$ is analytic on $\mathbb{D}$. Then

$$\ker H_{\bar{\phi}} = \{ f \in L_a^2(\mathbb{D}) : (I - P)\bar{\phi}f = 0 \} = \{ f \in L_a^2(\mathbb{D}) : \bar{\phi}f \in L_a^2(\mathbb{D}) \}$$

has finite codimension and is invariant under multiplication by $z$. By the result of Axler and Bourdon [1], there exists a polynomial $q$ whose roots lie in $\mathbb{D}$ such that $\ker H_{\bar{\phi}} = qL_a^2(\mathbb{D})$. Let $\phi(z) = \sum c_kz^k$; then $\bar{\phi}(z)q(z) \in L_a^2(\mathbb{D})$ implies that either $\phi$ is a constant or $q = 0$. If $q = 0$ then $\ker H_{\bar{\phi}} = \{ 0 \}$. This implies $(\text{Range}H_{\bar{\phi}})^\perp = \{ 0 \}$. Hence $(\text{Range}H_{\bar{\phi}}) = L_a^2(\mathbb{D})$. This implies $H_{\bar{\phi}}$ is not of finite rank. Hence $q \neq 0$ since $H_{\bar{\phi}}$ has finite rank, so the claim is verified.

Now if $\Phi \in H_{M_n}^\infty (\mathbb{D})$ then $H_{\Phi}$ is a finite rank Hankel operator if and only if 

$$H_{\Phi j} = \text{ker} \Phi_{j} = \{ f \in L_a^2(\mathbb{D}) : f = 0 \text{ for all } j \in \{ 1, 2, \cdots, n \} \}$$

is of finite rank for all $j \in \{ 1, 2, \cdots, n \}$. That is, if and only if $\Phi_{jj}$ is a constant for all $j \in \{ 1, 2, \cdots, n \}$. That is, if and only if $\Phi$ is a diagonal matrix with entries in $\mathbb{C}$.

**Definition-3.1** A function $G \in L_a^2(\mathbb{D})$ is called an inner function in $L_a^2(\mathbb{D})$ if $|G|^2 - 1$ is orthogonal to $H^\infty$.

This definition of inner function in a Bergman space was given by Korenblum and Steassin [10]. If $N$ is a subspace of $L_a^2(\mathbb{D})$, let $Z(N) = \{ z \in \mathbb{D} : f(z) = 0 \text{ for all } f \in N \}$, which is called the common zero set of functions in $N$. Hence if $z_1$ is a zero of multiplicity at most $n$ of all functions in $N$, then $z_1$ appears $n$ times in the set $Z(N)$, and each $z_1$ is treated as a distinct element of $Z(N)$.

**Theorem 3.2.** Let $\Phi = (\phi_{ij})$ where $\phi_{ij} \in L^\infty(\mathbb{D})$, $1 \leq i, j \leq n$. Suppose $\phi_{ij} = 0$ if $i \neq j$ and let $S_{\Phi} \in \mathcal{L}(L_a^{2,\mathbb{C}^n}(\mathbb{D}))$ be the little Hankel operator with symbol $\Phi$. The following hold:

(i) The operator $S_{\Phi} = 0$ if and only if $\Phi \in (L_a^{2,\mathbb{C}^n}(\mathbb{D}))^\perp$.

(ii) The operator $S_{\Phi} \in \mathcal{L}(L_a^{2,\mathbb{C}^n}(\mathbb{D}))$ is a little Hankel operator if and only if $T_{i_{1}x_{1}n} = ST_{i_{1}x_{1}n}$, where $I_{1}x_{1}n$ is the identity matrix of order $n$.

(iii) If $\Psi \in L_{M_n}^\infty (\mathbb{D})$, then the subspace ker $S_{\Psi}$ is an invariant subspace of $T_{i_{1}x_{1}n}$.

(iv) Let $\Psi = (\psi_{ij})$, $\psi_{ij} \in L^\infty(\mathbb{D})$ and $\psi_{ij}^\dagger(z) = \psi_{ij}(\bar{z})$, $1 \leq i, j \leq n$. Then $S_{\Psi} = S_{\Psi^*}$ where $\Psi^* = (\psi_{ij}^\dagger)^{1\leq i,j\leq n}$.

(v) If for $j \in \{ 1, 2, \cdots, n \}$, ker $S_{\phi_{jj}} = \{ f \in L_a^2(\mathbb{D}) : f = 0 \text{ on } b_{jj} \}$ where $b_{jj} = \{ b_{jj}^k \}_{k=1}^{\infty}$ is an infinite sequence of points in $\mathbb{D}$, then there exists an inner function $G \in L_a^2(\mathbb{D})$ such that ker $S_{\Phi} = GL_a^{2,\mathbb{C}^n}(\mathbb{D}) \cap L_a^{2,\mathbb{C}^n}(\mathbb{D})$.

(vi) If $S_{\Phi}$ is a finite rank little Hankel operator on $L_a^{2,\mathbb{C}^n}(\mathbb{D})$ then ker $S_{\Phi} = GL_a^{2,\mathbb{C}^n}(\mathbb{D})$ for some inner function $G \in L_a^2(\mathbb{D})$ and the following hold: (1) $G$ vanishes on $a_{j} = \{ a_{j} \}_{j=1}^{N}$, a finite sequence of points in $\mathbb{D}$, (2) $||G||_{L^2} = 1$. (3) $G$ is equal to a constant plus a linear combination of the Bergman kernel functions $K(z, a_{j})$, $K(z, a_{2})$, $\cdots$, $K(z, a_{n})$ and certain of their derivatives. (4) $|G|^2 - 1$ is orthogonal to $L_a^{1,\mathbb{C}}$, the class of harmonic functions in $L^1$ of the disc.

**Proof.** To prove (i), assume $\Phi \in L^\infty(\mathbb{D})$. We shall first verify that $S_{\Phi} = 0$ if and only if $\phi \in (L_a^2(\mathbb{D}))^\perp$. Suppose $S_{\Phi} = 0$. Then $S_{\Phi}f = 0$ for all $f \in L_a^2(\mathbb{D})$. Thus $PJ(\phi) = 0$ and hence $\phi f \in (L_a^2(\mathbb{D}))^\perp$, for all $f \in L_a^2(\mathbb{D})$. Since $1 \in L_a^2(\mathbb{D})$,
$\phi \in (L^2_\alpha(\mathbb{D}))^\perp$. Now suppose $\phi \in (L^2_\alpha(\mathbb{D}))^\perp$. This implies $\langle \phi, g \rangle = 0$ for all $g \in L^2_\alpha(\mathbb{D})$. Hence $\langle \phi f, g \rangle = \langle \phi, \overline{f}g \rangle = 0$ for all $g \in L^2_\alpha(\mathbb{D})$ and $f \in H^\infty(\mathbb{D})$. Thus $\langle h_\phi f, g \rangle = \langle \overline{f}(\phi f), g \rangle = 0$ for all $g \in L^2_\alpha(\mathbb{D})$ and $f \in H^\infty(\mathbb{D})$. Thus $h_\phi f = 0$ for all $f \in H^\infty(\mathbb{D})$. Since $H^\infty(\mathbb{D})$ is dense in $L^2_\alpha(\mathbb{D})$, we obtain $h_\phi \equiv 0$. That is, $S_\phi = \overline{h_\phi} \equiv 0$.

Now to prove (i), notice that $S_\Psi \equiv 0$ if and only if $S_{\phi_j} \equiv 0$ for all $j \in \{1, 2, \ldots, n\}$. This is true if and only if $\phi_j \in (L^2_\alpha(\mathbb{D}))^\perp$. That is, if $\Phi \in (L^2_\alpha(\mathbb{D}))^\perp$.

Now we prove (ii). Let $S \in \mathcal{L}(L^2_\alpha(\mathbb{D}))$. Since $L^2_\alpha(\mathbb{D}) = L^2_\alpha(\mathbb{D}) \oplus L^2_\alpha(\mathbb{D}) \oplus \cdots \oplus L^2_\alpha(\mathbb{D})$, the operator $S = \left( \begin{array}{ccc} S_{11} & S_{12} & \cdots & S_{1n} \\ S_{21} & S_{22} & \cdots & S_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ S_{n1} & S_{n2} & \cdots & S_{nn} \end{array} \right)$ for some $S_{ij} \in \mathcal{L}(L^2_\alpha(\mathbb{D}))$, $1 \leq i, j \leq n$. Suppose $T^*_zn_{1 \times n}S = STz_{1 \times n}$. This implies $T^*_zn_{1 \times n}S_{ij} = S_{ij}T_z$. From [8], it follows that $S_{ij} = S_{\psi_{ij}}$ for $\psi_{ij} \in L^\infty(\mathbb{D})$, $1 \leq i, j \leq n$. Thus

$$S = \left( \begin{array}{ccc} S_{\psi_{11}} & S_{\psi_{12}} & \cdots & S_{\psi_{1n}} \\ S_{\psi_{21}} & S_{\psi_{22}} & \cdots & S_{\psi_{2n}} \\ \vdots & \vdots & \cdots & \vdots \\ S_{\psi_{n1}} & S_{\psi_{n2}} & \cdots & S_{\psi_{nn}} \end{array} \right).$$

That is, $S = S_{\Psi}$ where $\Psi = \left( \begin{array}{ccccc} \psi_{11} & \psi_{12} & \cdots & \psi_{1n} \\ \psi_{21} & \psi_{22} & \cdots & \psi_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \psi_{n1} & \psi_{n2} & \cdots & \psi_{nn} \end{array} \right)$. Conversely, suppose $S \in \mathcal{L}(L^2_\alpha(\mathbb{D}))$ is a little Hankel operator. That is, $S = S_{\Psi}$ where $\Psi \in L^\infty_M(\mathbb{D})$.

Let $\Psi = (\psi_{ij})_{1 \leq i, j \leq n}$. Then $S_{\Psi} = (S_{\psi_{ij}})_{1 \leq i, j \leq n}$. From [8], it follows that $T^*_zn_{1 \times n}S_{\psi_{ij}} = S_{\psi_{ij}}T_z$. This implies $T^*_zn_{1 \times n}S_{\Psi} = S_{\Psi}T_zn_{1 \times n}$.

To prove (iii), let $f \in \ker S_{\Psi}$. Then $S_{\Psi}T_zn_{1 \times n}f = T^*_zn_{1 \times n}S_{\Psi}f = 0$. That is, $T^*_zn_{1 \times n}f \in \ker S_{\Psi}$.

To prove (iv), we shall first verify that if $\psi \in L^\infty(\mathbb{D})$ then $S_{\psi}^* = S_{\psi^+}$ where $\psi^+(z) = \overline{\psi(\overline{z})}$. Let $f, g \in L^2_\alpha(\mathbb{D})$. Then

$$\langle S_{\psi}^*f, g \rangle = \langle f, S_{\psi}g \rangle = \langle f, PJ(\psi g) \rangle = \langle f, (J\psi)Jg \rangle = \langle J\psi f, Jg \rangle = \langle \psi^+ f, Jg \rangle = \langle J(\psi^+)g, f \rangle = \langle PJ(\psi^+)g, f \rangle = \langle S_{\psi^+}f, g \rangle.$$
\{ f \in L^2_a(\mathbb{D}) : f = 0 \text{ on } b_{jj} \}, \text{ then by [3],[4] and [9], ker } S_{phi_j} = G_{jj}L^2_a(\mathbb{D}) \cap L^2_a(\mathbb{D}) \text{ for some inner functions } G_{jj} \in L^2_a(\mathbb{D}) \text{ formed by the corresponding zeros } \{b_{jj}^{k}\}_{k=1}^{\infty}, j = 1, 2, \ldots, n. \text{ Let } G \text{ be the inner function formed by the union of zeros of the functions } G_{jj}, j = 1, 2, \ldots, n \text{ counting multiplicities. It is not difficult to see that ker } S_{\phi} = GL^2_a,_{\mathbb{C}}(\mathbb{D}) \cap L^2_a,_{\mathbb{C}}(\mathbb{D}) \text{ as ker } S_{\phi} \text{ is an invariant subspace of } T_z L_{\nu, \nu}. \\
To prove (vi), first we shall verify that if } \phi \in L^\infty(\mathbb{D}) \text{ and } S_{\phi} \text{ is a finite rank little Hankel operator on } L^2_a(\mathbb{D}), \text{ then ker } S_{\phi} = GL^2_a(\mathbb{D}) \text{ for some inner function } G \in L^2_a(\mathbb{D}). \\
Since } S_{\phi} \text{ is a little Hankel operator on } L^2_a(\mathbb{D}), \text{ hence } T_z S_{\phi} = S_{\phi} T_z. \text{ So ker } S_{\phi} \text{ is invariant under multiplication by } z \text{ and ker } S_{\phi} \text{ has finite codimension since } S_{\phi} \text{ is of finite rank. Let } a = \{a_j\}_{j=1}^{N} \text{ be the common zeroes (counting multiplicities) of functions in ker } S_{\phi} \text{ i.e., } \mathcal{Z}(\text{ker } S_{\phi}) = \{a_j\}_{j=1}^{N}. \text{ Let } G \text{ be the extremal function for the problem} \\
\sup \{Re f^{(k)}(0) : f \in L^2_a, \|f\|_{L^2} \leq 1, f = 0 \text{ on } a\}, \\
where } \kappa \text{ is the multiplicity of the number of times zero appears in } a = \{a_j\}_{j=1}^{N}(\kappa = 0 \text{ if } 0 \notin \{a_j\}_{j=1}^{N}). \text{ It is clear from [2],[3], [4] and [9] that } G \text{ satisfies the conditions (1)-(4) and } G \text{ vanishes precisely on } a \text{ in } \overline{D} \text{ counting multiplicities. Moreover, for every function } f \in L^2_a(\mathbb{D}) \text{ that vanishes on } a = \{a_j\}_{j=1}^{N} \text{ there exists } g \in L^2_a(\mathbb{D}) \text{ such that } f = G g. \text{ Hence ker } S_{\phi} = GL^2_a(\mathbb{D}). \\
Now suppose } \Phi \in L^\infty_{\mathbb{C}}(\mathbb{D}) \text{ and } \Phi = \begin{pmatrix} \phi_{11} & 0 & \cdots & 0 \\ 0 & \phi_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi_{nn} \end{pmatrix}, \phi_{jj} \in L^\infty(\mathbb{D}) \\
and } S_{\Phi} \text{ is a finite rank little Hankel operator on } L^2_a,_{\mathbb{C}}(\mathbb{D}). \text{ Then} \\
S_{\Phi} = \begin{pmatrix} S_{\phi_{11}} & 0 & \cdots & 0 \\ 0 & S_{\phi_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S_{\phi_{nn}} \end{pmatrix} \text{ and each } S_{\phi_{jj}}, 1 \leq j \leq n \text{ is a finite rank little} \\
Hankel operator on } L^2_a(\mathbb{D}). \text{ From the argument above, it follows that ker } S_{\phi_{jj}} = G_{jj}L^2_a(\mathbb{D}), 1 \leq j \leq n \text{ where } G_{jj} \in L^2_a(\mathbb{D}) \text{ is an inner function and each } G_{jj} \text{ vanishes on a finite set of points in } \overline{D}, \|G_{jj}\|_{L^2} = 1 \text{ and each } G_{jj} \text{ is a linear combination of the Bergman kernels and some of their derivatives and } |G_{jj}|^2 - 1 \text{ is orthogonal to } L^1_k. \text{ Let } \{\gamma_1, \gamma_2, \ldots, \gamma_l\} \text{ be the union of the zeros of the functions } G_{jj}, 1 \leq j \leq n \text{ counting multiplicities. Let } G \in L^2_a(\mathbb{D}) \text{ be the inner function formed by the zeros } \gamma_1, \gamma_2, \ldots, \gamma_l \text{ taking multiplicities into account. It is not difficult to verify that ker } S_{\phi} = GL^2_a,_{\mathbb{C}}(\mathbb{D}) \text{ and } G \text{ is formed by a linear combination of (see [2],[3], [4] and [9]) the Bergman kernels and some of their derivatives and } G \text{ satisfies the conditions (1)-(4).} \square \\
\textbf{Theorem 3.3.} If } \Psi = (\psi_{ij}) \in L^\infty_{\mathbb{C}}(\mathbb{D}) \text{ where } \psi_{ij} = 0, i \neq j \text{ and } S_{\Psi} \text{ is a finite rank little Hankel operator on } L^2_a,_{\mathbb{C}}(\mathbb{D}) \text{ then } \Psi = \Phi + \chi \text{ where } \Phi = (\phi_{ij}), \phi_{ij} \in L^\infty(\mathbb{D}), 1 \leq i, j \leq n, \phi_{ij} = 0, i \neq j \text{ and each } \phi_{jj} \text{ is a linear combination of}
the Bergman kernels and some of their derivatives and \( \chi = (\theta_{ij}) \) where \( \theta_{ij} \in (L^2_a)_{\perp} \cap L^\infty(\mathbb{D}) \) and \( \theta_{ij} = 0, i \neq j \).

**Proof.** Since \( \Psi = (\psi_{ij})_{1 \leq i, j \leq n} \in L^\infty_{M_n}(\mathbb{D}) \) and \( \psi_{ij} = 0, i \neq j \), we have

\[
S_\Psi = \begin{pmatrix}
S_{\psi_{11}} & 0 & \cdots & 0 \\
0 & S_{\psi_{22}} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & S_{\psi_{nn}}
\end{pmatrix}.
\]

The operator \( S_\Psi \) is a finite rank little Hankel operator if and only if each \( S_{\psi_{jj}} \) is a finite rank little Hankel operator on \( L^2_a(\mathbb{D}) \) for all \( j \in \{1, 2, \ldots, n\} \). Now let \( 1 \leq j \leq n \). Since for each \( j \), \( S_{\psi_{jj}} \) is a finite rank little Hankel operator on \( L^2_a(\mathbb{D}) \), there exist inner functions \( G_{jj} \in L^2_a(\mathbb{D}) \) such that \( \ker S_{\psi_{jj}} = G_{jj}L^2_a(\mathbb{D}) \). Thus \( \psi_{jj}G_{jj} \in (L^2_a)_{\perp} \). So \( \langle \psi_{jj}G_{jj}, \bar{h} \rangle = 0 \) for all \( h \in L^2_a(\mathbb{D}) \), that is, \( \langle G_{jj}h, \psi_{jj} \rangle = 0 \) for all \( h \in L^2_a(\mathbb{D}) \) and so \( \psi_{jj} = \bar{\phi}_{jj} + \bar{\theta}_{jj} \) where \( \bar{\theta}_{jj} \in (L^2_a)_{\perp} \), the orthogonal complement of \( L^2_a(\mathbb{D}) \) with respect to \( L^2(\mathbb{D}, dA) \) and \( \bar{\phi}_{jj} \in (G_{jj}L^2_a)_{\perp} \), the orthogonal complement of \( G_{jj}L^2_a(\mathbb{D}) \) with respect to \( L^2_a(\mathbb{D}) \). Suppose the function \( G_{jj} \) vanishes precisely at \( d^j = \{d^j_1, d^j_2, \ldots, d^j_{m_j}\} \), a finite number of points in \( \mathbb{D} \) counting multiplicities. Since \( K_{d^j_1}, K_{d^j_2}, \ldots, K_{d^j_{m_j}} \) and their derivatives (where if the point \( \alpha \in \mathbb{D} \) occurs \( k \) times in \( d^j \) then we include the functions \((1-\bar{\alpha}z)^{-k}, z(1-\bar{\alpha}z)^{-k-1}, \ldots, z^{k-1}(1-\bar{\alpha}z)^{-k-1}\)) form a basis for \((G_{jj}L^2_a(\mathbb{D}))_{\perp}, j \in \{1, 2, \ldots, n\}\), hence \( \bar{\phi}_{jj} \) is a linear combination of the Bergman kernels and some of their derivatives and \( \bar{\theta}_{jj} \in (L^2_a(\mathbb{D}))_{\perp} \cap L^\infty(\mathbb{D}) \) since \( \psi_{jj}, \bar{\phi}_{jj}, \bar{\theta}_{jj} \in L^\infty(\mathbb{D}) \). Thus \( \Psi = \Phi + \chi \) where \( \Phi = (\bar{\phi}_{jj}), \chi = (\bar{\theta}_{jj}) \) and \( \bar{\phi}_{jj} \) is a linear combination of the Bergman kernels and some of their derivatives and \( \bar{\theta}_{jj} \in (L^2_a(\mathbb{D}))_{\perp} \cap L^\infty(\mathbb{D}) \). \( \Box \)

Now let \( b = \{b_j\}_{j=1}^\infty \) be an infinite sequence of points in \( \mathbb{D} \). Let \( \mathcal{I} = I(b) = \{f \in L^2_a(\mathbb{D}) : f = 0 \text{ on } b\} \). Let \( G_b \) be the solution of the extremal problem

\[
\sup\{Re f^{(n)}(0) : f \in \mathcal{I}, \|f\|_{L^2} \leq 1\},
\]

where \( n \) is the number of times zero appears in the sequence \( b \) (i.e., the functions in \( \mathcal{I} \) have a common zero of order \( n \) at the origin). The natural question that arises at this point is to see if it is possible to construct a little Hankel operator \( S_\Phi, \Phi \in L^\infty_{M_n}(\mathbb{D}) \) whose kernel is \( G_bL^2_a, c^n(\mathbb{D}) \cap L^2_a, c^n(\mathbb{D}) \). In the case that \( b = \{b_j\}_{j=1}^N \) is a finite set of points in \( \mathbb{D} \), it is possible to construct a little Hankel operator \( S_\Phi, \Phi \in L^\infty_{M_N}(\mathbb{D}) \) such that \( \ker S_\Phi = G_bL^2_a, c^n(\mathbb{D}) \) as follows:

**Theorem 3.4.** Let \( b = \{b_j\}_{j=1}^N \) be a finite set of points in \( \mathbb{D} \) and \( \mathcal{I} = I(b) = \{f \in L^2_a(\mathbb{D}) : f = 0 \text{ on } b\} \) and let \( G_b \) be the solution of the extremal problem (3.1). Let

\[
\bar{\phi} = \sum_{j=1}^N \sum_{\nu=0}^{m_j-1} c_{j\nu} \frac{\partial^n}{\partial \overline{b_j}} K_{b_j}(z),
\]

where \( c_{j\nu} \neq 0 \) for all \( j, \nu \) and \( m_j \) is the number of times \( b_j \) appears in \( b \). Then \( \ker S_\Phi = G_bL^2_a, c^n(\mathbb{D}) \) where \( \Phi = (\phi_{rs})_{r,s=1}^N \) and \( \phi_{rs} = \phi \) if \( r = s \) and \( 0 \), if \( r \neq s \).
Proof. The set of vectors \( \{ K_{b_1}, \ldots, \frac{\partial^{m_1-1}}{\partial b_1^{m_1}}K_{b_1}, \ldots, K_{b_N}, \ldots, \frac{\partial^{m_N-1}}{\partial b_N^{m_N}}K_{b_N} \} \) forms a basis [9] for \((G_b L_a^2(\mathbb{D}))^\perp\). By the Gram-Schmidt orthogonalization process we can get an orthonormal basis \( \{ \psi_j \}_{j=1}^1 \) for \((G_b L_a^2(\mathbb{D}))^\perp\). If \( \phi \in (G_b L_a^2(\mathbb{D}))^\perp \) then \( \langle \phi, G_b t \rangle = 0 \) for all \( t \in L_a^2(\mathbb{D}) \), i.e., \( \langle \phi, G_b \rangle = 0 \) for all \( t \in L_a^2(\mathbb{D}) \) and so \( G_b \in \ker S_\phi \). Since \( \ker S_\phi \) is invariant under the operator of multiplication by \( z \) we have that

\[
G_b L_a^2(\mathbb{D}) \subset \ker S_\phi. \tag{3.2}
\]

Suppose \( f \in \ker S_\phi \); then \( \langle \phi f, h \rangle = 0 \) for all \( h \in L_a^2(\mathbb{D}) \), so in particular \( \langle \phi f, \overline{K_{b_j}} \rangle = 0 \) for all \( j = 1, 2, \ldots, N \). Therefore, \( \langle \phi f, K_{b_j} \rangle = 0 \) for all \( j = 1, 2, \ldots, N \). Thus \( \phi(b_j)f(b_j) = 0 \) for all \( j = 1, 2, \ldots, N \). Since \( \phi(b_j) \neq 0 \) for all \( j = 1, 2, \ldots, N \), hence \( f(b_j) = 0 \) for all \( j = 1, 2, \ldots, N \). Thus \( f \in \mathcal{I} \). Since \( G_b \) is the solution of the extremal problem (3.1) therefore, \( f \in G_b L_a^2 \). Hence

\[
\ker S_\phi \subset G_b L_a^2. \tag{3.3}
\]

From (3.2) and (3.3), \( \ker S_\phi = G_b L_a^2(\mathbb{D}) = \mathcal{I} \). Now let \( \Phi = (\phi_{rs})_{r,s=1}^n \) where \( \phi_{rs} = \phi \) if \( r = s \) and 0, if \( r \neq s \). It is not difficult now to verify that \( \ker S_\Phi = G_b L_a^2 C^n(\mathbb{D}) \).

\( \square \)

References


P.G. Department of Mathematics, Utkal University, Vanivihar, Bhubaneswar, 751004, Odisha, India
E-mail address: namitadas440@yahoo.co.in