



ON THE DEGREE OF APPROXIMATION OF FUNCTIONS BELONGING TO THE LIPSCHITZ CLASS BY $(E, q)(C, \alpha, \beta)$ MEANS

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Communicated by T. Riedel

ABSTRACT. In this paper two generalized theorems on the degree of approximation of conjugate functions belonging to the Lipschitz classes of the type $\text{Lip}\alpha$, $0 < \alpha \leq 1$, and $W(L_p, \xi(t))$ are proved. The first one gives the degree of approximation with respect to the L_∞ -norm, and the second one with respect to L_p -norm, $p \geq 1$. In addition, a correct condition in proving of the second mentioned theorem is employed.

1. INTRODUCTION AND PRELIMINARIES

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with its partial sums s_n . We denote by $C_n^{(\theta, \beta)}$ the n -th Cesàro means of order (θ, β) , with $\theta + \beta > -1$ of the sequence (s_n) , i.e. (see [2])

$$C_n^{(\theta, \beta)} = \frac{1}{A_n^{\theta+\beta}} \sum_{v=0}^n A_{n-v}^{\theta-1} A_v^\beta s_v,$$

where $A_n^{\theta+\beta} = O(n^{\theta+\beta})$, $\theta + \beta > -1$ and $A_0^{\theta+\beta} = 1$.

The series $\sum_{n=0}^{\infty} u_n$ is said to be (C, θ, β) summable to the definite number s if

$$C_n^{(\theta, \beta)} = \frac{1}{A_n^{\theta+\beta}} \sum_{v=0}^n A_{n-v}^{\theta-1} A_v^\beta s_v \rightarrow s, \quad \text{as } n \rightarrow \infty.$$

Date: Received: 4 June 2015; Accepted: 22 December 2015.

2010 Mathematics Subject Classification. Primary 42B05; Secondary 42B08.

Key words and phrases. Lipschitz classes, Fourier series, summability, degree of approximation.

Then, for $q > 0$ a real number the Euler means (E, q) of the sequence (s_n) are defined to be (see for example [2])

$$E_n^q = \frac{1}{(1+q)^n} \sum_{v=0}^n \binom{n}{v} q^{n-v} s_v.$$

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$$E_n^q = \frac{1}{(1+q)^n} \sum_{v=0}^n \binom{n}{v} q^{n-v} s_v \rightarrow s, \quad \text{as } n \rightarrow \infty.$$

The (E, q) transform of the (C, θ, β) transform, defines $(E, q)(C, \theta, \beta)$ transform and we shall denote it by $(EC)_n^{q, \theta, \beta}$.

Moreover, if

$$(EC)_n^{q, \theta, \beta} = \frac{1}{(1+q)^n} \sum_{k=0}^n \binom{n}{k} q^{n-k} C_k^{(\theta, \beta)} \rightarrow s, \quad \text{as } n \rightarrow \infty,$$

then we shall say that the infinite series $\sum_{n=0}^{\infty} u_n$ is $(E, q)(C, \theta, \beta)$ summable to the definite number s .

We note that for $q = 1$, $\theta = 1$ and $\beta = 0$ the concept of $(E, q)(C, \theta, \beta)$ summability reduces to the $(E, 1)(C, 1)$ summability introduced in [10].

Let $f(x)$ be a 2π periodic function and integrable in the sense of Lebesgue. Then, let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be its Fourier series with n -th partial sum $s_n(f; x)$.

The conjugate series of the above Fourier series is given by

$$\sum_{n=1}^{\infty} (a_n \cos nx - b_n \sin nx). \quad (1.1)$$

For a function $f : R \rightarrow R$ the equalities

$$\|f\|_{\infty} = \sup\{|f(x)| : x \in R\}$$

and

$$\|f\|_p = \left(\int_0^{2\pi} |f(x)|^p dx \right)^{1/p}, \quad p \geq 1$$

denote the L_{∞} -norm and L_p -norm, respectively.

The degree of approximation of a function f by a trigonometric polynomial t_n of order n under the norm $\|\cdot\|_{\infty}$ is defined by Zygmund [16] with

$$\|f - t_n\|_{\infty} = \sup\{|f(x) - t_n(x)| : x \in R\}$$

and the best approximation $E_n(f)$ of a function $f \in L_p$ is defined by the equality

$$E_n(f) = \min_{t_n} \|f - t_n\|_p.$$

A function $f \in \text{Lip}\alpha$ or $f \in \text{Lip}(\alpha, p)$ if

$$|f(x+t) - f(x)| = O(|t|^{\alpha}) \quad \text{for } 0 < \alpha \leq 1$$

or

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^p dx \right)^{1/p} = O(|t|^\alpha) \quad \text{for } 0 < \alpha \leq 1 \quad \text{and } p \geq 1,$$

respectively.

For a given positive increasing function $\xi(t)$ and an integer $p \geq 1$, $f \in \text{Lip}(\xi(t), p)$ (see [14]) if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^p dx \right)^{1/p} = O(\xi(t))$$

and $f \in W(L_p, \xi(t))$ if

$$\left(\int_0^{2\pi} |[f(x+t) - f(x)] \sin^\gamma x|^p dx \right)^{1/p} = O(\xi(t)), \quad \gamma \geq 0, \quad p \geq 1.$$

We note here in these definitions that for $\beta = 0$ the class $W(L_p, \xi(t))$ reduces to the class $\text{Lip}(\xi(t), p)$ and if $\xi(t) = t^\alpha$ then the class $W(L_p, \xi(t))$ reduces to the class $\text{Lip}(\alpha, p)$, and if $p \rightarrow \infty$ then the class $\text{Lip}(\alpha, p)$ reduces to the class $\text{Lip}\alpha$.

A lot of authors have determined the degree of approximation of functions from above mentioned classes, using Cesàro and generalized Nörlund means (we refer the reader for details to the papers [1], and [3]–[15]. Very recently H. K. Nigam and K. Sharma [10] have established two theorems on determining the degree of approximation of conjugate functions using $(E, 1)(C, 1)$ means. The condition

$$\left\{ \int_0^{\frac{1}{n+1}} \left(\frac{t|\phi_x(t)|}{\xi(t)} \right)^p \sin^{\gamma p} t dt \right\}^{1/p} = O\left(\frac{1}{n+1}\right)$$

assumed in Theorem 2, of their result, is not sufficient for the validity of it. This condition leads to the divergent integral of type (see for details [6], page 14)

$$\left(\int_0^{\frac{\pi}{n}} t^{-(2+\gamma)p} dt \right)^{1/p}.$$

Here in this paper we shall generalize their theorems using $(E, q)(C, \theta, \beta)$ means instead of $(E, 1)(C, 1)$ means that are obviously particular cases of them. Moreover, we employ a correct condition in our result. To verify the main results we need first to prove some helpful statements given in the next section. Everywhere in this paper, we write $u = O(v)$ if there exists a positive constant C such that $u \leq Cv$.

2. AUXILIARY LEMMA

Throughout this paper we shall use notations

$$\phi_x(t) := f(x+t) + f(x-t),$$

$$\tilde{D}_n^{q;\theta,\beta}(t) := \frac{1}{\pi(1+q)^n} \sum_{k=0}^n \binom{n}{k} q^{n-k} \frac{1}{A_k^{\theta+\beta}} \sum_{v=0}^k \frac{A_{k-v}^{\theta-1} A_v^\beta \cos\left(v + \frac{1}{2}\right)t}{2 \sin \frac{t}{2}},$$

and we prove a lemma which plays a key role in the proof of the main results.

Lemma 2.1. *The estimate $|\widetilde{D}_n^{q;\theta,\beta}(t)| = \mathcal{O}\left(\frac{1}{t}\right)$ holds true for $0 \leq t \leq \pi$.*

Proof. First proof: Since for $0 \leq t \leq \pi$, $\sin(t/2) \geq t/\pi$, then

$$\begin{aligned} |\widetilde{D}_n^{q;\theta,\beta}(t)| &= \frac{1}{\pi(1+q)^n} \left| \sum_{k=0}^n \binom{n}{k} \frac{q^{n-k}}{A_k^{\theta+\beta}} \sum_{v=0}^k \frac{A_{k-v}^{\theta-1} A_v^\beta \cos\left(v + \frac{1}{2}\right)t}{2 \sin \frac{t}{2}} \right| \\ &\leq \frac{1}{\pi(1+q)^n} \sum_{k=0}^n \binom{n}{k} \frac{q^{n-k}}{A_k^{\theta+\beta}} \sum_{v=0}^k A_{k-v}^{\theta-1} A_v^\beta \left| \frac{\cos\left(v + \frac{1}{2}\right)t}{2 \sin \frac{t}{2}} \right| \\ &\leq \frac{1}{2t(1+q)^n} \sum_{k=0}^n \binom{n}{k} \frac{q^{n-k}}{A_k^{\theta+\beta}} \sum_{v=0}^k A_{k-v}^{\theta-1} A_v^\beta \\ &= \frac{1}{2t(1+q)^n} \sum_{k=0}^n \binom{n}{k} 1^k q^{n-k} \\ &= \mathcal{O}\left(\frac{1}{t}\right), \end{aligned}$$

because of

$$\sum_{v=0}^k A_{k-v}^{\theta-1} A_v^\beta = A_k^{\theta+\beta} \quad \text{and} \quad \sum_{k=0}^n \binom{n}{k} q^{n-k} = (1+q)^n.$$

The first proof of this lemma is completed.

Second proof: Applying the well-known inequality $\sin(t/2) \geq t/\pi$ for $0 \leq t \leq \pi$, we obtain

$$\begin{aligned} |\widetilde{D}_n^{q;\theta,\beta}(t)| &= \frac{1}{\pi(1+q)^n} \left| \sum_{k=0}^n \binom{n}{k} \frac{q^{n-k}}{A_k^{\theta+\beta}} \sum_{v=0}^k \frac{A_{k-v}^{\theta-1} A_v^\beta \cos\left(v + \frac{1}{2}\right)t}{2 \sin \frac{t}{2}} \right| \\ &\leq \frac{1}{2t(1+q)^n} \left| \sum_{k=0}^n \binom{n}{k} \frac{q^{n-k}}{A_k^{\theta+\beta}} \Re \sum_{v=0}^k A_{k-v}^{\theta-1} A_v^\beta e^{i\left(v + \frac{1}{2}\right)t} \right| \\ &\leq \frac{1}{2t(1+q)^n} \left| \sum_{k=0}^n \binom{n}{k} \frac{q^{n-k}}{A_k^{\theta+\beta}} \Re \sum_{v=0}^k A_{k-v}^{\theta-1} A_v^\beta e^{ivt} \right| |e^{\frac{it}{2}}| \\ &\leq \frac{1}{2t(1+q)^n} \left| \sum_{k=0}^n \binom{n}{k} \frac{q^{n-k}}{A_k^{\theta+\beta}} \Re \sum_{v=0}^k A_{k-v}^{\theta-1} A_v^\beta e^{ivt} \right| \\ &\leq \frac{1}{2t(1+q)^n} \left| \sum_{k=0}^{\tau-1} \binom{n}{k} \frac{q^{n-k}}{A_k^{\theta+\beta}} \Re \sum_{v=0}^k A_{k-v}^{\theta-1} A_v^\beta e^{ivt} \right| \\ &\quad + \frac{1}{2t(1+q)^n} \left| \sum_{k=\tau}^n \binom{n}{k} \frac{q^{n-k}}{A_k^{\theta+\beta}} \Re \sum_{v=0}^k A_{k-v}^{\theta-1} A_v^\beta e^{ivt} \right| \\ &=: J_1 + J_2. \end{aligned}$$

For the quantity J_1 , we have

$$\begin{aligned}
 J_1 &\leq \frac{1}{2t(1+q)^n} \left| \sum_{k=0}^{\tau-1} \binom{n}{k} \frac{q^{n-k}}{A_k^{\theta+\beta}} \Re \sum_{v=0}^k A_{k-v}^{\theta-1} A_v^\beta e^{ivt} \right| \\
 &\leq \frac{1}{2t(1+q)^n} \sum_{k=0}^{\tau-1} \binom{n}{k} \frac{q^{n-k}}{A_k^{\theta+\beta}} \sum_{v=0}^k A_{k-v}^{\theta-1} A_v^\beta |e^{ivt}| \\
 &\leq \frac{1}{2t(1+q)^n} \sum_{k=0}^{\tau-1} \binom{n}{k} q^{n-k}.
 \end{aligned} \tag{2.1}$$

The use of Abel’s lemma leads to

$$\begin{aligned}
 J_2 &\leq \frac{1}{2t(1+q)^n} \left| \sum_{k=\tau}^n \binom{n}{k} \frac{q^{n-k}}{A_k^{\theta+\beta}} \Re \sum_{v=0}^k A_{k-v}^{\theta-1} A_v^\beta e^{ivt} \right| \\
 &\leq \frac{1}{2t(1+q)^n} \sum_{k=\tau}^n \binom{n}{k} \frac{q^{n-k}}{A_k^{\theta+\beta}} \max_{0 \leq j \leq k} \left| \sum_{v=0}^j A_{k-v}^{\theta-1} A_v^\beta e^{ivt} \right| \\
 &\leq \frac{1}{2t(1+q)^n} \sum_{k=\tau}^n \binom{n}{k} 1^k q^{n-k}.
 \end{aligned} \tag{2.2}$$

Thus, the estimations (2.1) and (2.2) give

$$|\tilde{D}_n^{q;\theta,\beta}(t)| \leq \frac{1}{2t(1+q)^n} \sum_{k=0}^{\tau-1} \binom{n}{k} q^{n-k} + \frac{1}{2t(1+q)^n} \sum_{k=\tau}^n \binom{n}{k} 1^k q^{n-k} = \mathcal{O}\left(\frac{1}{t}\right),$$

which as well verifies the statement of the lemma. □

3. MAIN RESULTS

At first, we prove the following theorem.

Theorem 3.1. *If a function \bar{f} , conjugate to a 2π periodic function f , belongs to Lip α class, then its degree of approximation by $(E, q)(C, \theta, \beta)$ means of conjugate Fourier series is given by*

$$\sup_{0 < x < 2\pi} \left| \overline{(EC)_n^{q;\theta,\beta}}(\bar{f}(x)) - \bar{f}(x) \right| = \left\| \overline{(EC)_n^{q;\theta,\beta}}(\bar{f}) - \bar{f} \right\|_\infty = \mathcal{O}\left(\frac{1}{(n+1)^\alpha}\right), \quad 0 < \alpha < 1,$$

where $\overline{(EC)_n^{q;\theta,\beta}}(\bar{f}(x))$ denotes the $(E, q)(C, \theta, \beta)$ transform of partial sums of the series (1.1).

Proof. Let $\bar{s}_v(x)$ be the partial sums of the series (1.1). Then, in [7] it is verified that

$$\bar{s}_v(x) - \bar{f}(x) = \frac{1}{2\pi} \int_0^\pi \phi_x(t) \frac{\cos\left(v + \frac{1}{2}\right)t}{\sin \frac{t}{2}} dt.$$

Thus, the (C, θ, β) transform $\overline{C_k^{\theta,\beta}}(x)$ of $\bar{s}_v(x)$ is

$$\overline{C_k^{\theta,\beta}}(x) - \bar{f}(x) = \frac{1}{2\pi A_k^{\theta+\beta}} \int_0^\pi \phi_x(t) \sum_{v=0}^k \frac{A_{k-v}^{\theta-1} A_v^\beta \cos\left(v + \frac{1}{2}\right)t}{\sin \frac{t}{2}} dt. \tag{3.1}$$

Further, denoting the $(E, q)(C, \theta, \beta)$ transform of $\bar{s}_v(x)$ by $\overline{(EC)_n^{q;\theta,\beta}}$, we have

$$\begin{aligned} \overline{(EC)_n^{q;\theta,\beta}}(\bar{f}(x)) - \bar{f}(x) &= \\ &= \frac{1}{(1+q)^n} \sum_{k=0}^n \binom{n}{k} q^{n-k} \int_0^\pi \frac{\phi_x(t)}{\pi A_k^{\theta+\beta}} \sum_{v=0}^k \frac{A_{k-v}^{\theta-1} A_v^\beta \cos\left(v + \frac{1}{2}\right)t}{2 \sin \frac{t}{2}} dt \\ &= \int_0^\pi \phi_x(t) \tilde{D}_n^{q;\theta,\beta}(t) dt \\ &= \left(\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^\pi \right) \phi_x(t) \tilde{D}_n^{q;\theta,\beta}(t) dt \\ &=: I_1 + I_2. \end{aligned} \tag{3.2}$$

We apply Lemma 2.1 in order to estimate I_1 :

$$\begin{aligned} |I_1| &\leq \int_0^{\frac{1}{n+1}} |\phi_x(t)| |\tilde{D}_n^{q;\theta,\beta}(t)| dt \\ &= \mathcal{O} \left(\int_0^{\frac{1}{n+1}} t^{\alpha-1} dt \right) = \mathcal{O} \left(\frac{1}{(n+1)^\alpha} \right). \end{aligned} \tag{3.3}$$

Also, applying again Lemma 2.1, we have

$$\begin{aligned} |I_2| &\leq \int_{\frac{1}{n+1}}^\pi |\phi_x(t)| |\tilde{D}_n^{q;\theta,\beta}(t)| dt \\ &= \mathcal{O} \left(\int_{\frac{1}{n+1}}^\pi t^{\alpha-1} dt \right) = \mathcal{O} \left(\frac{1}{(n+1)^\alpha} \right). \end{aligned} \tag{3.4}$$

Based on (3.3), (3.4), and (3.2), the required estimation is an immediate result. The proof of the theorem is completed. \square

The following result gives the degree of approximation of conjugate functions with respect to L_p -norm, $1 \leq p < \infty$.

Theorem 3.2. *If \bar{f} , conjugate to a 2π periodic function f , belongs to $W(L_p, \xi(t))$ class, then its degree of approximation by $(E, q)(C, \theta, \beta)$ means of conjugate Fourier series is given by*

$$\| \overline{(EC)_n^{q;\theta,\beta}}(\bar{f}) - \bar{f} \|_p = \mathcal{O} \left((n+1)^{\gamma+\frac{1}{p}} \xi \left(\frac{1}{n+1} \right) \right)$$

provided that $\xi(t)$ satisfies the following conditions:

$$\left\{ \frac{\xi(t)}{t} \right\} \text{ is a decreasing sequence,} \tag{3.5}$$

$$\left\{ \int_0^{\frac{1}{n+1}} \left(\frac{|\phi_x(t)|}{\xi(t)} \right)^p \sin^{\gamma p} \frac{t}{2} dt \right\}^{1/p} = \mathcal{O} \left(\frac{1}{(n+1)^{1/p}} \right), \tag{3.6}$$

and

$$\left\{ \int_{\frac{1}{n+1}}^{\pi} \left(\frac{t^{-\delta} |\phi_x(t)|}{\xi(t)} \right)^p dt \right\}^{1/p} = \mathcal{O}((n+1)^\delta) \tag{3.7}$$

where δ is an arbitrary number such that $s(1 - \delta) - 1 > 0$, $1/p + 1/s = 1$, $1 \leq p < \infty$, conditions (3.6) and (3.7) hold uniformly in x and $(EC)_n^{q;\theta,\beta}$ are $(E, q)(C_n^{\theta,\beta})$ means of the series (1.1), and

$$\bar{f}(x) = -\frac{1}{2\pi} \int_0^{2\pi} \phi_x(t) \cot\left(\frac{t}{2}\right) dt.$$

Proof. We shall use the equality

$$\overline{(EC)_n^{q;\theta,\beta}(\bar{f}(x))} - \bar{f}(x) = \left(\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi} \right) \phi_x(t) \tilde{D}_n^{q;\theta,\beta}(t) dt =: J_1 + J_2, \tag{3.8}$$

obtained earlier in the proof of theorem 3.1.

Moreover, using Hölder’s inequality and the fact that $\phi \in W(L_p, \xi(t))$, condition (3.6), $\sin t \geq \frac{2t}{\pi}$, Lemma 2.1, and second mean value theorem for integrals, we have

$$\begin{aligned} |J_1| &\leq \left\{ \int_0^{\frac{1}{n+1}} \left(\frac{|\phi(t)|}{\xi(t)} \right)^p \sin^{\gamma p} \frac{t}{2} dt \right\}^{1/p} \left\{ \int_0^{\frac{1}{n+1}} \left(\frac{\xi(t) |\tilde{D}_n^{q;\theta,\beta}(t)|}{\sin^\gamma \frac{t}{2}} \right)^s dt \right\}^{1/s} \\ &= \mathcal{O} \left(\frac{1}{(n+1)^{1/p}} \right) \left\{ \int_0^{\frac{1}{n+1}} \left(\frac{\xi(t)}{t^{1+\gamma}} \right)^s dt \right\}^{1/s} \\ &= \mathcal{O} \left(\frac{1}{(n+1)^{1/p}} \xi \left(\frac{1}{n+1} \right) \right) \left\{ \int_\varepsilon^{\frac{1}{n+1}} \frac{dt}{t^{(1+\gamma)s}} \right\}^{1/s}, \quad \left(0 < \varepsilon < \frac{1}{n+1} \right) \\ &= \mathcal{O} \left((n+1)^{-\frac{1}{p}} \xi \left(\frac{1}{n+1} \right) (n+1)^{1+\gamma} \right) \\ &= \mathcal{O} \left((n+1)^\gamma \xi \left(\frac{1}{n+1} \right) \right), \quad \text{because of } 1/p + 1/s = 1. \end{aligned} \tag{3.9}$$

Again, using Hölder's inequality, $|\sin t| \leq 1$, $\sin t \geq \frac{2t}{\pi}$, conditions (3.5) and (3.7), Lemma 2.1, and second mean value theorem for integrals, we obtain

$$\begin{aligned}
|J_2| &\leq \left\{ \int_{\frac{1}{n+1}}^{\pi} \left(\frac{t^{-\delta} |\phi(t)|}{\xi(t)} \right)^p \sin^{\gamma p} \frac{t}{2} dt \right\}^{1/p} \left\{ \int_{\frac{1}{n+1}}^{\pi} \left(\frac{\xi(t) |\tilde{D}_n^{q;\theta,\beta}(t)|}{t^{-\delta} \sin^{\gamma} \frac{t}{2}} \right)^s dt \right\}^{1/s} \\
&= \mathcal{O}((n+1)^\delta) \left\{ \int_{\frac{1}{n+1}}^{\pi} \left(\frac{\xi(t)}{t^{\gamma+1-\delta}} \right)^s dt \right\}^{1/s} \\
&= \mathcal{O}((n+1)^\delta) \left\{ \int_{\frac{1}{\pi}}^{n+1} \left(\frac{\xi(1/u)}{u^{\delta-1-\gamma}} \right)^s \frac{du}{u^2} \right\}^{1/s} \\
&= \mathcal{O} \left((n+1)^\delta \xi \left(\frac{1}{n+1} \right) \right) \left\{ \int_{\frac{1}{\pi}}^{n+1} \frac{du}{u^{s(\delta-1-\gamma)+2}} \right\}^{1/s} \\
&= \mathcal{O} \left((n+1)^\delta \xi \left(\frac{1}{n+1} \right) \right) \left\{ \frac{(n+1)^{s(\gamma+1-\delta)-1} - \pi^{s(\delta-1-\gamma)+1}}{s(\gamma+1-\delta)-1} \right\}^{1/s} \\
&= \mathcal{O} \left((n+1)^\delta \xi \left(\frac{1}{n+1} \right) \right) \{(n+1)^{\gamma+1-\delta-1/s}\} \\
&= \mathcal{O} \left((n+1)^{\gamma+\frac{1}{p}} \xi \left(\frac{1}{n+1} \right) \right), \quad \text{where } 1/p + 1/s = 1. \tag{3.10}
\end{aligned}$$

Inserting (3.9) and (3.10) into (3.8), we obtain

$$\left| \overline{(EC)_n^{q;\theta,\beta}}(\bar{f}(x)) - \bar{f}(x) \right| = \mathcal{O} \left((n+1)^{\gamma+\frac{1}{p}} \xi \left(\frac{1}{n+1} \right) \right),$$

and whence,

$$\| \overline{(EC)_n^{q;\theta,\beta}}(\bar{f}) - \bar{f} \|_p = \mathcal{O} \left((n+1)^{\gamma+\frac{1}{p}} \xi \left(\frac{1}{n+1} \right) \right).$$

The proof is completed. \square

4. COROLLARIES

In this section we give some direct consequences of the main results. First it is clear that $(E, q)(C, \theta, \beta)$ means can be reduced to the following means:

1. If $\beta = 0$ then we obtain $(E, q)(C, \theta, \beta) \equiv (E, q)(C, \theta, 0) \equiv (E, q)(C, \theta)$ means.
2. If $\theta = 1$ then we obtain $(E, q)(C, \theta, \beta) \equiv (E, q)(C, 1, \beta)$ means.
3. If $\beta = 0, q = 1$ then we obtain $(E, q)(C, \theta, \beta) \equiv (E, 1)(C, \theta, 0) \equiv (E, 1)(C, \theta)$ means.
4. If $\theta = 1, q = 1$ then we obtain $(E, q)(C, \theta, \beta) \equiv (E, 1)(C, 1, \beta)$ means.
5. If $\theta = q = 1, \beta = 0$ then we obtain $(E, q)(C, \theta, \beta) \equiv (E, 1)(C, 1)$ means.

Denoting $(E, q)(C, \theta)$, $(E, q)(C, 1, \beta)$, $(E, 1)(C, \theta)$, $(E, 1)(C, 1, \beta)$, $(E, 1)(C, 1)$ means of $\bar{s}_n(f; x)$, respectively, by $\overline{(EC)}_n^{(q; \theta, 0)}(f; x)$, $\overline{(EC)}_n^{(q; 1, \beta)}(f; x)$, $\overline{(EC)}_n^{(1; \theta, 0)}(f; x)$, $\overline{(EC)}_n^{(1; 1, \beta)}(f; x)$ and $\overline{(EC)}_n^{(1; 1, 0)}(f; x)$, then from theorems 3.1 and 3.2 lots of corollaries can be derived.

We shall formulate below only some of them.

Corollary 4.1 ([10]). *If $\theta = q = 1$, $\beta = 0$ and all conditions of Theorem 3.1 are satisfied, then*

$$\|\overline{(EC)}_n^{1; 1, 0}(\bar{f}) - \bar{f}\|_\infty = \mathcal{O}\left(\frac{1}{(n+1)^\alpha}\right), \quad 0 < \alpha < 1.$$

Corollary 4.2 ([10]). *If $\theta = q = 1$, $\beta = 0$ and all conditions of Theorem 3.2 are satisfied (with corrected condition (3.6)), then*

$$\|\overline{(EC)}_n^{1; 1, 0}(\bar{f}) - \bar{f}\|_p = \mathcal{O}\left((n+1)^{\gamma + \frac{1}{p}} \xi\left(\frac{1}{n+1}\right)\right).$$

Corollary 4.3 ([10]). *If $\gamma = \beta = 0$, $\theta = q = 1$, and $\xi(t) = t^\alpha$, then the degree of approximation of a function \bar{f} , conjugate to a 2π -periodic function $f \in \text{Lip}(\alpha, p)$, $1/p \leq \alpha \leq 1$, is given by*

$$\|\overline{(EC)}_n^{1; 1, 0}(\bar{f}) - \bar{f}\|_p = \mathcal{O}\left(\frac{1}{(n+1)^{\alpha - \frac{1}{p}}}\right).$$

Corollary 4.4 ([10]). *Let $\gamma = \beta = 0$, $\theta = q = 1$, and $\xi(t) = t^\alpha$. If $p \rightarrow \infty$ in Corollary 4.3, then $f \in \text{Lip}(\alpha, p)$ reduces to $\text{Lip}\alpha$ for $0 < \alpha < 1$, and we have*

$$\|\overline{(EC)}_n^{1; 1, 0}(\bar{f}) - \bar{f}\|_\infty = \mathcal{O}\left(\frac{1}{(n+1)^\alpha}\right).$$

Acknowledgement. The author wishes to thank the referee for her/his suggestions which definitely improved the final form of this paper.

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