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FINITE RANK LITTLE HANKEL OPERATORS ON $L^2_a(\mathbb{U}_+)$

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Abstract. Let $\psi \in L^\infty(\mathbb{U}_+)$, where $\mathbb{U}_+$ is the upper half plane in $\mathbb{C}$ and let S_ψ be the little Hankel operator with symbol ψ defined on the Bergman space $L^2_a(\mathbb{U}_+)$. In this article, we show that if S_ψ is of finite rank, then $\psi = \varphi + \chi$, where $\chi \in \left(\overline{L^2_a(\mathbb{U}_+)}\right)^\perp \bigcap L^\infty(\mathbb{U}_+)$ and $\overline{\varphi}$ is a linear combination of $d_{\overline{w}}, w \in \mathbb{U}_+$ and some of their derivatives.

1. Introduction

Let $\mathbb{U}_+ = \{z = x + iy \in \mathbb{C} : y > 0\}$ be the upper half plane in $\mathbb{C}$, and let $d\widetilde{A} = dxdy$ be the area measure on $\mathbb{U}_+$. Let $L^2(\mathbb{U}_+, d\widetilde{A})$ denote the Hilbert space of complex valued, absolutely square integrable, Lebesgue measurable functions on $\mathbb{U}_+$ with the inner product $\langle f, g\rangle = \int_{\mathbb{U}_+} f(s)\overline{g(s)}d\widetilde{A}(s)$, and the corresponding norm is defined by $||f||_2 = \langle f, f\rangle^{\frac{1}{2}} = \left[\int_{\mathbb{U}_+} |f(s)|^2 d\widetilde{A}(s)\right]^{\frac{1}{2}} < \infty$.

Let $L^2_a(\mathbb{U}_+)$ be the closed subspace of $L^2(\mathbb{U}_+, d\widetilde{A})$ consisting of all analytic functions in $L^2(\mathbb{U}_+, d\widetilde{A})$. The space $L^2_a(\mathbb{U}_+)$ is called the Bergman space on $\mathbb{U}_+$. It is a reproducing kernel Hilbert space and $K_w(s) = -\frac{1}{\pi(\overline{w}-s)^2}, w, s \in \mathbb{U}_+$, is the reproducing kernel for the Bergman space $L^2_a(\mathbb{U}_+)$. The Bergman (orthogonal) projection P_+ from $L^2(\mathbb{U}_+, d\widetilde{A})$ onto $L^2_a(\mathbb{U}_+)$ is given by $(P_+f)(w) = \langle f, K_w\rangle$. Let $L^\infty(\mathbb{U}_+)$ be the space of all complex valued, essentially bounded, Lebesgue

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measurable functions on $\mathbb{U}_+$. Define for $\varphi \in L^\infty(\mathbb{U}_+)$,

$$||\varphi||_\infty = ess \sup_{s\in\mathbb{U}_+} |\varphi(s)| < \infty.$$

The space $L^\infty(\mathbb{U}_+)$ is a Banach space with respect to the essential supremum norm. Let $H^\infty(\mathbb{U}_+)$ be the space of all bounded analytic functions on $\mathbb{U}_+$. For $\varphi \in L^\infty(\mathbb{U}_+)$, we define the Toeplitz operator T_φ on $L^2_a(\mathbb{U}_+)$ by $T_\varphi f = P_+(\varphi f)$. The Toeplitz operator T_φ is bounded and $||T_\varphi|| \leq ||\varphi||_\infty$. For more details, see [3]. The big Hankel operator H_φ from $L^2_a(\mathbb{U}_+)$ into $(L^2_a(\mathbb{U}_+))^\perp$ is defined by $H_\varphi f = (I - P_+)(\varphi f), f \in L^2_a(\mathbb{U}_+)$. The little Hankel operator h_φ from $L^2_a(\mathbb{U}_+)$ into $\overline{(L^2_a(\mathbb{U}_+))} = \{\overline{f} : f \in L^2_a(\mathbb{U}_+)\}$ is defined by $h_\varphi f = \overline{P}_+(\varphi f)$, where $\overline{P}_+$ is the orthogonal projection operator from $L^2(\mathbb{U}_+, d\widetilde{A})$ onto $\overline{L^2_a(\mathbb{U}_+)}$. For $\psi \in L^\infty(\mathbb{U}_+)$, define the operator $S_\psi : L^2_a(\mathbb{U}_+) \to L^2_a(\mathbb{U}_+)$ as $S_\psi f = P_+ J(\psi f)$, where $J : L^2(\mathbb{U}_+, d\widetilde{A}) \to L^2(\mathbb{U}_+, d\widetilde{A})$ is defined by $Jf(s) = f(-\overline{s})$. The operator S_ψ is unitarily equivalent to h_φ for some $\varphi \in L^\infty(\mathbb{U}_+)$. Hence both operators h_φ and S_ψ are referred to as little Hankel operator on $L^2_a(\mathbb{U}_+)$. For $g \in L^\infty(\mathbb{D})$, the little Hankel operator $\widetilde{\Gamma}_g : L^2_a(\mathbb{D}) \to L^2_a(\mathbb{D})$ with symbol g is defined by $\widetilde{\Gamma}_g f = PJ(gf), f \in L^2_a(\mathbb{D})$, where P is the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $L^2_a(\mathbb{D})$ and $J : L^2(\mathbb{D}, dA) \to L^2(\mathbb{D}, dA)$ is defined by $Jf(z) = f(\overline{z})$. For details, see [7].

Define $M : \mathbb{U}_+ \to \mathbb{D}$ by $M(s) = \frac{i-s}{i+s} = z$. Then M is one-to-one and onto, and $M^{-1} : \mathbb{D} \to \mathbb{U}_+$ is given by $M^{-1}(z) = i\frac{1-z}{1+z}$. Thus M is its self inverse. Furthermore, $M'(s) = \frac{-2i}{(i+s)^2}$ and $(M^{-1})'(z) = \frac{-2i}{(1+z)^2}$. Let $W : L^2_a(\mathbb{D}) \to L^2_a(\mathbb{U}_+)$ be defined by $(Wg)(s) = g(Ms)\frac{(2i)}{\sqrt{\pi}(i+s)^2}$. The map W is one-to-one and onto. Hence W^{-1} exists and $W^{-1} : L^2_a(\mathbb{U}_+) \to L^2_a(\mathbb{D})$ is given by $(W^{-1}G)(z) = (2i)\sqrt{\pi}G(M^{-1}(z))\frac{1}{(1+z)^2}$.

In 1881, Kronecker [4,5] showed that the matrix $L = (a_{i+j})_{i,j=0}^\infty$ is of finite rank n if and only if $r(z) = a_0 z^{-1} + a_1 z^{-2} + \cdots$, is a rational function of z, and in this case, n is the number of poles of $r(z)$. That is, in the Hardy space $H^2(\mathbb{T})$, a Hankel operator, H_φ, is of finite rank if and only if $\varphi = z\overline{u}h$, where u is a finite Blaschke product and $h \in H^\infty(\mathbb{T})$. In this case, the rank of S is no greater than the number of zeros of u counted with multiplicity. Das [2] showed that if $\psi \in L^\infty(\mathbb{D})$ and the little Hankel operator S_ψ is of finite rank, then $\psi = \varphi + \chi$, where $\chi \in \left(\overline{L^2_a(\mathbb{D})}\right)^\perp \bigcap L^\infty(\mathbb{D})$ and $\overline{\varphi}$ is a linear combination of the Bergman kernels and some of its derivatives. In this article, we have extended the result of [2] to characterize finite rank little Hankel operators defined on $L^2_a(\mathbb{U}_+)$.

The organization of the article is as follows. In section 2, we introduce the elementary functions $d_{\overline{w}}(s)$ and $D_{\overline{w}}(s)$ and discuss some properties of these functions. We show that $D_{\overline{w}} \in L^\infty(\mathbb{U}_+)$ and that $S_{\overline{D_w}}$ is a rank-one operator. We also relate little Hankel operators defined on $L^2_a(\mathbb{D})$ and $L^2_a(\mathbb{U}_+)$ and prove that they are unitarily equivalent, and the symbol correspondence is obtained. In section

3, we show that if $S_{\overline{G}}$ is of finite rank, then $G = \displaystyle\sum_{i=1}^{n}\sum_{\nu=0}^{r_i-1} C_{i\nu}\frac{\partial^\nu}{\partial \overline{w_i}^\nu} D_{\overline{w_i}}$, for some constants $C_{i\nu}, i = 1, 2, \ldots, n$ and $\nu = 0, \ldots, r_i - 1$. That is, if $S_{\overline{G}}$ is a finite rank little Hankel operator, then G is a linear combination of $d_{\overline{w}}, w \in \mathbb{U}_+$ and some of their derivatives.

2. Preliminaries

In this section, we introduce the elementary functions $d_{\overline{w}}(s)$ and $D_{\overline{w}}(s)$ and discuss some properties of these functions. We show that $D_{\overline{w}} \in L^\infty(\mathbb{U}_+)$ and that $S_{\overline{D_w}}$ is a rank-one operator. We also relate little Hankel operators defined on $L_a^2(\mathbb{D})$ and $L_a^2(\mathbb{U}_+)$ and prove that they are unitarily equivalent and the symbol correspondence is obtained.

For $s, w \in \mathbb{U}_+$, define $d_{\overline{w}}(s) = \dfrac{1}{\sqrt{\pi}}\dfrac{w+i}{\overline{w}-i}\dfrac{(-2i)Im\ w}{(s+w)^2}$. If $w = i\dfrac{1-\overline{a}}{1+\overline{a}} \in \mathbb{U}_+$, then $\overline{a} \in \mathbb{D}$ and $\overline{a} = \dfrac{i-w}{i+w} = Mw$. That is, $M^{-1}\overline{a} = w$. Then

$$\begin{aligned}
d_{\overline{w}}(-\overline{w}) &= \frac{1}{\sqrt{\pi}}\frac{w+i}{\overline{w}-i}\frac{(-2i)(Im\ w)}{(-\overline{w}+w)^2} \\
&= \frac{(-2i)}{\sqrt{\pi}}\frac{M^{-1}\overline{a}+i}{\overline{M^{-1}\overline{a}}-i}\frac{Im\ w}{(w-\overline{w})^2} \\
&= \frac{(-2i)}{\sqrt{\pi}}\frac{i\frac{1-\overline{a}}{1+\overline{a}}+i}{\overline{\left(i\frac{1-\overline{a}}{1+\overline{a}}\right)}-i}\frac{w-\overline{w}}{(2i)(w-\overline{w})^2} \\
&= -\frac{1}{\sqrt{\pi}}\frac{i\left[\frac{1-\overline{a}}{1+\overline{a}}+1\right]}{\left[-i\frac{1-a}{1+a}-i\right]}\frac{1}{w-\overline{w}} \\
&= \frac{1}{\sqrt{\pi}}\frac{2}{1+\overline{a}}\frac{1+a}{2}\frac{1}{i\frac{1-\overline{a}}{1+\overline{a}}+i\frac{1-a}{1+a}} \\
&= \frac{1}{\sqrt{\pi}}\frac{1+a}{(1+\overline{a})}\frac{(1+\overline{a})(1+a)}{i[(1-\overline{a})(1+a)+(1-a)(1+\overline{a})]} \\
&= \frac{1}{i\sqrt{\pi}}\frac{(1+a)^2}{[1+a-\overline{a}-|a|^2+1+\overline{a}-a-|a|^2]} \\
&= \frac{1}{i\sqrt{\pi}}\frac{(1+a)^2}{2(1-|a|^2)} \\
&= \frac{1}{(2i)\sqrt{\pi}}\frac{(1+a)^2}{(1-|a|^2)}.
\end{aligned}$$

Now

$$\begin{aligned}
d_{\overline{w}}(s)d_{\overline{w}}(-\overline{w}) &= \frac{(-2i)}{\sqrt{\pi}}\frac{w+i}{\overline{w}-i}\frac{Im\ w}{(s+w)^2}\frac{1}{(2i)\sqrt{\pi}}\frac{(1+a)^2}{1-|a|^2} \\
&= \frac{(-2i)}{\sqrt{\pi}}\left(\frac{i\frac{1-\overline{a}}{1+\overline{a}}+i}{-i\frac{1-a}{1+a}-i}\right)\frac{\left(\frac{w-\overline{w}}{2i}\right)}{(s+i\frac{1-\overline{a}}{1+\overline{a}})^2}\frac{1}{(2i)\sqrt{\pi}}\frac{(1+a)^2}{1-|a|^2}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(-2i)}{\sqrt{\pi}} \frac{\left(\frac{1-\overline{a}}{1+\overline{a}}+1\right)}{-\left(\frac{1-a}{1+a}+1\right)} \frac{\left[\left(i\frac{1-\overline{a}}{1+\overline{a}}\right)-\left(-i\frac{1-a}{1+a}\right)\right](1+\overline{a})^2}{(2i)[s(1+\overline{a})+i(1-\overline{a})]^2} \frac{1}{(2i)\sqrt{\pi}} \frac{(1+a)^2}{1-|a|^2} \\
&= \frac{1}{(2i)\pi} \left(\frac{\frac{1-\overline{a}+1+\overline{a}}{1+\overline{a}}}{\frac{1-a+1+a}{1+a}} \right) \frac{i\left[\frac{1-\overline{a}}{1+\overline{a}}+\frac{1-a}{1+a}\right]}{[s(1+\overline{a})+i(1-\overline{a})]^2} \frac{(1+a)^2}{1-|a|^2}(1+\overline{a})^2 \\
&= \frac{1}{2\pi} \frac{1+a}{1+\overline{a}} \frac{(1+a)^2}{(1-|a|^2)} \frac{2(1-|a|^2)}{(1+a)(1+\overline{a})} \frac{(1+\overline{a})^2}{[s(1+\overline{a})+i(1-\overline{a})]^2} \\
&= \frac{1}{\pi}\left(\frac{1+a}{1+\overline{a}}\right)^2 \frac{(1+\overline{a})^2}{[i+s+\overline{a}(s-i)]^2} \\
&= \frac{1}{\pi}\left(\frac{1+a}{1+\overline{a}}\right)^2 \frac{(1+\overline{a})^2}{[i+s-\overline{a}(i-s)]^2} \\
&= \frac{1}{\pi}\left(\frac{1+a}{1+\overline{a}}\right)^2 \frac{(1+\overline{a})^2}{(i+s)^2[1-\overline{a}\left(\frac{i-s}{i+s}\right)]^2} \\
&= \frac{1}{\pi} \frac{(1+a)^2}{(i+s)^2} \frac{1}{(1-\overline{a}Ms)^2} \\
&= D(s,w) \\
&= D_{\overline{w}}(s).
\end{aligned}
$$

Hence, $d_{\overline{w}}(s) = \frac{D(s,w)}{d_{\overline{w}}(-\overline{w})}$ and $(d_{\overline{w}}(-\overline{w}))^2 = D(\overline{w}, w)$. Now

$$
\begin{aligned}
||D_{\overline{w}}||^2 &= \langle D_{\overline{w}}, D_{\overline{w}} \rangle \\
&= \int_{\mathbb{U}_+} |D_{\overline{w}}(s)|^2 d\widetilde{A}(s) \\
&= \int_{\mathbb{U}_+} |D(s,w)|^2 d\widetilde{A}(s) \\
&= \int_{\mathbb{U}_+} |d_{\overline{w}}(-\overline{w})|^2 |d_{\overline{w}}(s)|^2 d\widetilde{A}(s) \\
&= |d_{\overline{w}}(-\overline{w})|^2 \int_{\mathbb{U}_+} |d_{\overline{w}}(s)|^2 d\widetilde{A}(s) \\
&= |d_{\overline{w}}(-\overline{w})|^2 ||d_{\overline{w}}||_2^2 \\
&= |d_{\overline{w}}(-\overline{w})|^2 \quad \text{since } ||d_{\overline{w}}||_2 = 1.
\end{aligned}
$$

Thus

$$||D_{\overline{w}}|| = |d_{\overline{w}}(-\overline{w})| \text{ and } |d_{\overline{w}}(s)|\ ||D_{\overline{w}}|| = |D_{\overline{w}}(s)|. \text{ Furthermore, } D_{\overline{w}} \in L^\infty(\mathbb{U}_+).$$

Lemma 2.1. *If $\psi \in L^\infty(\mathbb{U}_+)$, then the little Hankel operator S_ψ defined on $L_a^2(\mathbb{U}_+)$ with symbol ψ is unitarily equivalent to the little Hankel operator $\widetilde{\Gamma}_g$ defined on $L_a^2(\mathbb{D})$ with symbol $g(z) = (\psi \circ M^{-1})(z)\left(\frac{1+z}{1+\overline{z}}\right)^2$.*

Proof. The operator W maps $z^n\sqrt{n+1}$ to the function $\frac{2i}{\sqrt{\pi}}(Ms)^n\sqrt{n+1}\frac{1}{(i+s)^2} = \frac{2i}{\sqrt{\pi}}\left(\frac{i-s}{i+s}\right)^n\sqrt{n+1}\frac{1}{(i+s)^2}$, which belongs to $L_a^2(\mathbb{U}_+)$.

Now

$$\begin{aligned}
&S_\psi\left(\frac{2i}{\sqrt{\pi}}\left(\frac{i-s}{i+s}\right)^n\sqrt{n+1}\frac{1}{(i+s)^2}\right)\\
&= P_+\left(J\left(\psi(s)\frac{2i}{\sqrt{\pi}}\left(\frac{i-s}{i+s}\right)^n\sqrt{n+1}\frac{1}{(i+s)^2}\right)\right)\\
&= P_+\left(\psi(-\overline{s})\frac{2i}{\sqrt{\pi}}\sqrt{n+1}\left(\frac{i+\overline{s}}{i-\overline{s}}\right)^n\frac{1}{(i-\overline{s})^2}\right)\\
&= WPW^{-1}\left(\psi(-\overline{s})\frac{2i}{\sqrt{\pi}}\sqrt{n+1}\left(\frac{i+\overline{s}}{i-\overline{s}}\right)^n\frac{1}{(i-\overline{s})^2}\right)\\
&= WP\left((2i)\sqrt{\pi}\frac{2i}{\sqrt{\pi}}\sqrt{n+1}\psi\left(-\overline{M^{-1}(z)}\right)\left(\frac{i+\overline{M^{-1}(z)}}{i-\overline{M^{-1}(z)}}\right)^n\frac{1}{(i-\overline{M^{-1}(z)})^2}\frac{1}{(1+z)^2}\right)\\
&= WP\left((-4)\sqrt{n+1}\psi\left(i\frac{1-\overline{z}}{1+\overline{z}}\right)\left(\frac{i-i\frac{1-\overline{z}}{1+\overline{z}}}{i+i\frac{1-\overline{z}}{1+\overline{z}}}\right)^n\frac{1}{(i+i\frac{1-\overline{z}}{1+\overline{z}})^2}\frac{1}{(1+z)^2}\right)\\
&= WP\left((-4)\sqrt{n+1}\psi\left(i\frac{1-\overline{z}}{1+\overline{z}}\right)\left(\frac{1-\frac{1-\overline{z}}{1+\overline{z}}}{1+\frac{1-\overline{z}}{1+\overline{z}}}\right)^n\frac{-1}{(1+\frac{1-\overline{z}}{1+\overline{z}})^2}\frac{1}{(1+z)^2}\right)\\
&= WP\left((-4)\sqrt{n+1}\psi\left(i\frac{1-\overline{z}}{1+\overline{z}}\right)\left(\frac{2\overline{z}}{2}\right)^n(-1)\frac{(1+\overline{z})^2}{4}\frac{1}{(1+z)^2}\right)\\
&= WP\left(\sqrt{n+1}\psi\left(i\frac{1-\overline{z}}{1+\overline{z}}\right)(\overline{z})^n\left(\frac{1+\overline{z}}{1+z}\right)^2\right)\\
&= WP\left(J\left(\sqrt{n+1}z^n\psi\left(i\frac{1-z}{1+z}\right)\left(\frac{1+z}{1+\overline{z}}\right)^2\right)\right)\\
&= W\widetilde{\Gamma}_{\psi(i\frac{1-z}{1+z})(\frac{1+z}{1+\overline{z}})^2}(z^n\sqrt{n+1})\\
&= W\widetilde{\Gamma}_{(\psi\circ M^{-1})(z)(\frac{1+z}{1+\overline{z}})^2}(z^n\sqrt{n+1}).
\end{aligned}$$

Thus $S_\psi W = W\widetilde{\Gamma}_g$, where $g(z) = (\psi\circ M^{-1})(z)\left(\frac{1+z}{1+\overline{z}}\right)^2$. Since the sequence of vectors $\{\sqrt{n+1}z^n\}_{n=0}^\infty$ forms an orthonormal basis for $L^2_a(\mathbb{D})$, this proves our claim. Thus the little Hankel operator S_ψ defined on $L^2_a(\mathbb{U}_+)$ with symbol ψ is unitarily equivalent to the little Hankel operator $\widetilde{\Gamma}_g$ defined on $L^2_a(\mathbb{D})$. □

Lemma 2.2. *Let $S_{\overline{D_w}}$ be the little Hankel operator defined on $L^2_a(\mathbb{U}_+)$ with symbol $\overline{D_w}$. Then $S_{\overline{D_w}} = D_{\overline{w}}\otimes D_w$.*

Proof. Let $f, g\in L^2_a(\mathbb{U}_+)$ and let $\overline{a} = Mw\in\mathbb{D}$. Also let $f = Wf_1$ and $g = Wg_1$, $f_1, g_1\in L^2_a(\mathbb{D})$. Then

$$\begin{aligned}
\langle(D_{\overline{w}}\otimes D_w)f, g\rangle &= \langle\langle f, D_w\rangle D_{\overline{w}}, g\rangle\\
&= \langle f, D_w\rangle\langle D_{\overline{w}}, g\rangle\\
&= \langle Wf_1, WK_{\overline{a}}\rangle\langle WK_a, Wg_1\rangle\\
&= \langle Wf_1, WK_{\overline{a}}\rangle\overline{\langle Wg_1, WK_a\rangle}\\
&= f_1(\overline{a})\overline{g_1(a)}.
\end{aligned}$$

Now let $g^+(z) = \overline{g(\overline{z})}$. Then for polynomials f and g in $L^2_a(\mathbb{U}_+)$ on w, we have

$$\begin{aligned}
\langle S_{\overline{D_w}} f, g\rangle &= \langle P_+ J(\overline{D_w} f), g\rangle \\
&= \langle \overline{D_w} f, Jg\rangle \\
&= \langle f g^+, D_w\rangle \\
&= \langle W(f_1 g_1^+), W K_{\overline{a}}\rangle \\
&= \langle f_1 g_1^+, K_{\overline{a}}\rangle \\
&= f_1(\overline{a})\overline{g_1(a)}.
\end{aligned}$$

□

Lemma 2.3 (see [6]). *Let f be a linear functional defined on a vector space V, and let $f_1, f_2, \ldots, f_n$ be linear functionals on V. If* $\ker f \supseteq \bigcap_{i=1}^{n} \ker f_i$*, then* $f = \sum_{i=1}^{n} \lambda_i f_i$*, where λ_i's are complex numbers.*

3. Characterization of finite rank little Hankel operators

In this section, we show that if $S_{\overline{G}}$ is of finite rank, then

$$G = \sum_{i=1}^{n} \sum_{\nu=0}^{r_i-1} C_{i\nu} \frac{\partial^\nu}{\partial \overline{w_i}^\nu} D_{\overline{w_i}}$$

for some constants $C_{i\nu}, i = 1, 2, \ldots, n$ and $\nu = 0, \ldots, r_i - 1$. That is, if $S_{\overline{G}}$ is a finite rank little Hankel operator, then G is a linear combination of $d_{\overline{w}}, w \in \mathbb{U}_+$ and some of their derivatives.

Theorem 3.1. *Let $\overline{G} \in L^\infty(\mathbb{U}_+)$, where G is analytic on $\mathbb{U}_+$ and let $S_{\overline{G}}$ be the little Hankel operator defined on $L^2_a(\mathbb{U}_+)$ with symbol $\overline{G}$. If $S_{\overline{G}}$ is of finite rank, then* $G = \sum_{i=1}^{n} \sum_{\nu=0}^{r_i-1} C_{i\nu} \frac{\partial^\nu}{\partial \overline{w_i}^\nu} D_{\overline{w_i}}$.

Proof. The little Hankel operator $S_{\overline{G}}$ on $L^2_a(\mathbb{U}_+)$ is unitarily equivalent to the little Hankel operator $\widetilde{\Gamma}_{\overline{g}}$ defined on $L^2_a(\mathbb{D})$, where $\overline{g}(z) = (\overline{G} \circ M^{-1})(z) \left(\frac{1+z}{1+\overline{z}}\right)^2$. Now since $T_{\overline{z}} \widetilde{\Gamma}_{\overline{g}} = \widetilde{\Gamma}_{\overline{g}} T_z$, it follows that $\ker \widetilde{\Gamma}_{\overline{g}}$ is an invariant subspace of T_z. Since the rank of $\widetilde{\Gamma}_{\overline{g}}$ is finite, it follows from [1] that

$$\ker \widetilde{\Gamma}_g = \theta L^2_a(\mathbb{D}) = (z - a_1)^{r_1} (z - a_2)^{r_2} \cdots (z - a_n)^{r_n} L^2_a(\mathbb{D}) = q L^2_a(\mathbb{D}),$$

where θ is a finite Blaschke product and a_i are the zeros of θ counted according to their multiplicities $r_i (1 \leq i \leq n)$ and $\sum_{i=1}^{n} r_i$ is the rank of $\widetilde{\Gamma}_{\overline{g}}$ and $q(z) = (z - a_1)^{r_1} (z - a_2)^{r_2} \cdots (z - a_n)^{r_n}$. Define on the space $\mathcal{P}$ of polynomials, the linear functional $\widehat{\phi}$ by $\widehat{\phi}(p) = \langle \widetilde{\Gamma}_{\overline{g}} p, 1\rangle, p \in \mathcal{P}$. Note that $q\mathcal{P} \subseteq$

$\{p : p(a_i) = 0, i = 1, 2, \ldots, n, p \in \mathcal{P}\}$. For $r \in \mathbb{N}$ and $f \in L^2_a(\mathbb{D})$, we have $f^{(r)}(a) = \langle f, K_{a,r}\rangle$, where $K_{a,r} = \frac{\partial^r}{\partial a^r} K_a$, $a \in \mathbb{D}$. Using this and the fact that $\langle \widetilde{\Gamma}_{\overline{K_{a,r}}} p, 1\rangle = \langle p, K_{a,r}\rangle$, it follows that $(z-a)^r p(z) \in \ker \widetilde{\Gamma}_{\overline{K_{a,r-1}}}$. Thus $\ker \widehat{\phi} \supset \bigcap_{i=1}^{n} \ker \widetilde{\Gamma}_{\overline{K_{a,r-1}}}$. Using Lemma 2.3, we obtain

$$\widehat{\phi} = \sum_{i=1}^{n} \beta_{i\nu} \widetilde{\Gamma}_{\overline{K_{a_i,r_i-1}}} = \sum_{i=1}^{n} \sum_{\nu=0}^{r_i-1} \beta_{i\nu} \widetilde{\Gamma}_{\overline{\frac{\partial^\nu}{\partial a_i^\nu} K_{a_i}}}.$$

Since $\langle \widetilde{\Gamma}_{\overline{g}} f, h^+\rangle = \langle \widetilde{\Gamma}_{\overline{g}} fh, 1\rangle$, $f, h, fh \in L^2_a(\mathbb{D})$, and $\{z^n : n \geq 0\}$ is dense in $L^2_a(\mathbb{D})$, it follows $g = \sum_{i=1}^{n} \beta_i K_{a_i, r_i-1}$. Thus since $W^{-1} S_{\overline{G}} W = \widetilde{\Gamma}_{\overline{g}}$, hence

$$G = \sum_{i=1}^{n} \sum_{\nu=0}^{r_i-1} C_{i\nu} \frac{\partial^\nu}{\partial \overline{w_i}^\nu} D_{\overline{w_i}}.$$

□

Corollary 3.2. *If $\psi \in L^\infty(\mathbb{U}_+)$ and S_ψ is a finite rank little Hankel operator on $L^2_a(\mathbb{U}_+)$, then $\psi = \varphi + \chi$, where $\chi \in \left(\overline{L^2_a(\mathbb{U}_+)}\right)^\perp \bigcap L^\infty(\mathbb{U}_+)$ and $\overline{\varphi} = \sum_{i=1}^{n} \sum_{\nu=1}^{m_i-1} \beta_{i\nu} \frac{\partial^\nu}{\partial \overline{w_i}^\nu} d_{\overline{w_i}}$.*

Proof. The result follows from the fact that $S_\chi \equiv 0$ if and only if $\chi \in \left(\overline{L^2_a(\mathbb{U}_+)}\right)^\perp$. This can be verified as follows: If $S_\chi = 0$, then $S_\chi f = P_+ J(\chi f) = 0$ for all $f \in L^2_a(\mathbb{U}_+)$. Hence $J(\chi f) \in (L^2_a(\mathbb{U}_+))^\perp$ or $\chi f \in \left(\overline{L^2_a(\mathbb{U}_+)}\right)^\perp$ for all $f \in L^2_a(\mathbb{U}_+)$. Thus $\langle \chi f, \overline{g}\rangle = 0$ for all $g \in H^\infty(\mathbb{U}_+)$. Therefore $\langle \chi, \overline{f}\overline{g}\rangle = 0$ for all $g \in H^\infty(\mathbb{U}_+)$. Thus we get $\langle \chi, \overline{h}\rangle = 0$ for all $h \in L^2_a(\mathbb{U}_+)$, and hence $\chi \in \left(\overline{L^2_a(\mathbb{U}_+)}\right)^\perp$. Similarly one can show that if $\chi \in \left(\overline{L^2_a(\mathbb{U}_+)}\right)^\perp \bigcap L^\infty(\mathbb{U}_+)$, then $S_\chi \equiv 0$. □

Now let $\psi \in L^\infty(\mathbb{U}_+)$ and let $\psi = \varphi + \chi$, where

$$\varphi \in \left(\overline{L^2_a(\mathbb{U}_+)}\right), \chi \in \left(\overline{L^2_a(\mathbb{U}_+)}\right)^\perp \bigcap L^\infty(\mathbb{U}_+).$$

Then $S_\chi \equiv 0$. Thus $S_\psi = S_\varphi$, where $\overline{\varphi}$ is a linear combination of $d_{\overline{w_i}}$, and some of its derivatives. From the proof of Theorem 3.1, it follows that $W^{-1} S_\varphi W = \widetilde{\Gamma}_{\overline{\theta}}$, where θ is a linear combination of the Bergman kernels K_{α_i} and some of its derivatives K_{α_i, r_i}. Note that $\widetilde{\Gamma}_{\overline{\theta}}$ is a little Hankel operator on $L^2_a(\mathbb{D})$ with $\ker \widetilde{\Gamma}_\theta = G L^2_a(\mathbb{D})$ for some inner functions $G \in L^2_a(\mathbb{D})$ and the space $(G L^2_a)^\perp$ is finite-dimensional. For proof, see [2]. It thus follows that $\ker S_\varphi$ has also finite-codimensional and the operator $S_\psi = S_\varphi$ is of finite rank.

References

1. S. Axler and P. Bourdon, *Finite codimensional invariant subspaces of Bergman spaces*, Trans. Amer. Math. Soc. **31** (1988), no. 2, 805–817.
2. N. Das, *The kernel of a Hankel operator on the Bergman space*, Bull. Lond. Math. Soc. **31** (1999) 75–80.
3. S. Elliott and A. Wynn, *Composition operators on weighted Bergman spaces of a half plane*, Proc. Edinb. Math. Soc. **54** (2011) 373–379.
4. V.V. Peller, *Hankel Operators and Their Applications*, Springer-Verlag, New York, 2003.
5. S.C. Power, *Hankel Operators on Hilbert Space*, Pitman, Boston-London, 1982.
6. W. Rudin, *Functional Analysis*, McGraw-Hill, New York, 1991.
7. K. Zhu, *Operator Theory in Function Spaces*, Marcell Dekker, New York, 1990.

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