



## ON GRADED STRONGLY 1-ABSORBING PRIMARY IDEALS

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**ABSTRACT.** Let  $G$  be a group with identity  $e$  and let  $R$  be a  $G$ -graded commutative ring with nonzero unity  $1$ . In this article, we introduce the concept of graded strongly 1-absorbing primary ideals. A proper graded ideal  $P$  of  $R$  is said to be a graded strongly 1-absorbing primary ideal of  $R$  if whenever nonunit homogeneous elements  $x, y, z \in R$  exist such that  $xyz \in P$ , then either  $xy \in P$  or  $z \in \text{Grad}(\{0\})$  (the graded radical of  $\{0\}$ ). Several properties of graded strongly 1-absorbing primary ideals are investigated. Many results are given to disclose the relations between this new concept and others that already exist, namely, the graded prime ideals, the graded primary ideals, and the graded 1-absorbing primary ideals.

### 1. INTRODUCTION AND PRELIMINARIES

Throughout this article, all rings are commutative with a nonzero unity  $1$ . Let  $G$  be a group with identity  $e$ . Then a ring  $R$  is said to be  $G$ -graded if  $R = \bigoplus_{g \in G} R_g$  with  $R_g R_h \subseteq R_{gh}$  for all  $g, h \in G$ , where  $R_g$  is an additive subgroup of  $R$  for all  $g \in G$ . The elements of  $R_g$  are called *homogeneous of degree  $g$* . If  $x \in R$ , then  $x$  can be written uniquely as  $\sum_{g \in G} x_g$ , where  $x_g$  is the component of  $x$  in  $R_g$ . Also, we set  $h(R) = \bigcup_{g \in G} R_g$ . Moreover, it has been proved in [10] that  $R_e$  is a subring of  $R$  and that  $1 \in R_e$ . Let  $I$  be an ideal of a graded ring  $R$ . Then  $I$  is said to be a *graded ideal* if  $I = \bigoplus_{g \in G} (I \cap R_g)$ , that is,  $x = \sum_{g \in G} x_g$ , for  $x \in I$ , where  $x_g \in I$  for all  $g \in G$ . An ideal of a graded ring does not need to be graded (see [10]). For more details and terminology, one can look in [1, 11].

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Let  $I$  be a proper graded ideal of  $R$ . Then the graded radical of  $I$  is

$$\text{Grad}(I) = \left\{ x = \sum_{g \in G} x_g \in R : \text{for all } g \in G, \text{ there exists } n_g \in \mathbb{N} \text{ such that } x_g^{n_g} \in I \right\}.$$

Note that  $\text{Grad}(I)$  is always a graded ideal of  $R$  (see [12]).

**Proposition 1.1** (see [6]). *Let  $R$  be a  $G$ -graded ring.*

- (1) *If  $I$  and  $J$  are graded ideals of  $R$ , then  $I + J$ ,  $IJ$ , and  $I \cap J$  are graded ideals of  $R$ .*
- (2) *If  $a \in h(R)$ , then  $Ra$  is a graded ideal of  $R$ .*

Let  $R$  be a  $G$ -graded ring and let  $K$  be a graded ideal of  $R$ . Then  $R/K$  is  $G$ -graded by  $(R/K)_g = (R_g + K)/K$  for all  $g \in G$ .

**Lemma 1.2.** *Let  $R$  be a graded ring and let  $K$  be a graded ideal of  $R$ . If  $P$  is a graded ideal of  $R$  with  $K \subseteq P$ , then  $P/K$  is a graded ideal of  $R/K$ .*

*Proof.* Clearly,  $P/K$  is an ideal of  $R/K$ . Let  $x + K \in P/K$ . Then  $x \in P$ , and  $x_g \in P$  for all  $g \in G$  as  $P$  is graded, which implies that  $x_g + K \in P/K$  for all  $g \in G$ . Hence,  $P/K$  is a graded ideal of  $R/K$ .  $\square$

Recall that a proper graded ideal  $Q$  of  $R$  is said to be *graded primary* if whenever  $x, y \in h(R)$  with  $xy \in Q$ , then  $x \in Q$  or  $y \in \text{Grad}(Q)$ . In this case  $P = \text{Grad}(Q)$  is a graded prime ideal of  $R$ , and  $Q$  is said to be *graded  $P$ -primary*. Al-Zoubi and Sharafat [4] introduced a generalization of graded primary ideals called *graded 2-absorbing primary ideals*. A proper graded ideal  $I$  of  $R$  is called a *graded 2-absorbing primary ideal of  $R$*  if whenever  $x, y, z \in h(R)$  and  $xyz \in I$ , then  $xy \in I$  or  $xz \in \text{Grad}(I)$  or  $yz \in \text{Grad}(I)$ . The concept of graded 2-absorbing primary ideals is generalized in many ways (see, for example, [3, 14]). Recently in [5], we considered a new class of graded ideals called the class of *graded 1-absorbing primary ideals*. A proper graded ideal  $P$  of  $R$  is said to be a *graded 1-absorbing primary ideal of  $R$*  if whenever nonunit elements  $x, y, z \in h(R)$  exist such that  $xyz \in P$ , then  $xy \in P$  or  $z \in \text{Grad}(P)$ . Clearly, every graded primary ideal is a graded 1-absorbing primary ideal. The next example shows that the converse is not true in general.

**Example 1.3** ([5]). Let  $K$  be a field, and assume that  $R = K[X, Y]$  is  $\mathbb{Z}$ -graded with  $\deg X = 1 = \deg Y$ . Consider the graded ideal  $P = (X^2, XY)$  of  $R$ . Then  $\text{Grad}(P) = (X)$ . It is easily seen that  $P$  is a graded 1-absorbing primary ideal of  $R$ . On the other hand,  $P$  is not graded primary ideal of  $R$  by [13, Example 2.11].

Moreover, it is clear that every graded 1-absorbing primary ideal is graded 2-absorbing primary ideal. The next example shows that the converse is not true in general.

**Example 1.4** ([5]). Let  $R = \mathbb{Z}[i]$  and let  $G = \mathbb{Z}_2$ . Then  $R$  is  $G$ -graded by  $R_0 = \mathbb{Z}$  and  $R_1 = i\mathbb{Z}$ . Consider  $P = 12R$ . Then as  $12 \in h(R)$ ,  $P$  is a graded ideal of  $R$ . It follows from [4, Example 2.2(ii)] that  $P$  is a graded 2-absorbing primary ideal of  $R$ . On the other hand,  $2, 3 \in h(R)$  such that  $2 \cdot 2 \cdot 3 \in P$ , but neither  $2 \cdot 2 \in P$  nor  $3 \in \text{Grad}(P)$ . So,  $P$  is not a graded 1-absorbing primary ideal of  $R$ .

In this article, we follow [2] to introduce and study a subclass of the class of graded 1-absorbing primary ideals that does not necessarily contain all graded primary ideals. A proper graded ideal  $P$  of  $R$  is called *graded strongly 1-absorbing primary* if whenever nonunit elements  $x, y, z \in h(R)$  and  $xyz \in P$ , then  $xy \in P$  or  $z \in \text{Grad}(\{0\})$ . Several properties of graded strongly 1-absorbing primary ideals are investigated. Many results are given to disclose the relations between this new concept and others that already exist, namely, the graded prime ideals, the graded primary ideals, and the graded 1-absorbing primary ideals.

To see recent studies on generalizations of graded prime ideals (prime ideals) in graded commutative rings (commutative rings), one can look at [7, 8, 15].

## 2. GRADED STRONGLY 1-ABSORBING PRIMARY IDEALS

In this section, we introduce and study the concept of graded strongly 1-absorbing primary ideals.

**Definition 2.1.** Let  $R$  be a graded ring and let  $P$  be a proper graded ideal of  $R$ . Then  $P$  is said to be a *graded strongly 1-absorbing primary ideal* of  $R$  if whenever nonunits  $x, y, z \in h(R)$  exist such that  $xyz \in P$ , then either  $xy \in P$  or  $z \in \text{Grad}(\{0\})$ .

Clearly, every graded strongly 1-absorbing primary ideal is a graded 1-absorbing primary ideal. However, the next example shows that the converse is not true in general. The same example shows that a graded prime ideal (in particular, a graded primary ideal) is not necessarily a graded strongly 1-absorbing primary ideal.

**Example 2.2.** Let  $R = \mathbb{Z}[i]$  and let  $G = \mathbb{Z}_2$ . Then  $R$  is  $G$ -graded by  $R_0 = \mathbb{Z}$  and  $R_1 = i\mathbb{Z}$ . Consider  $P = 3R$ . Then as  $3 \in h(R)$ ,  $P$  is a graded ideal of  $R$ . It is clear that  $P$  is graded prime, and then graded 1-absorbing primary. However,  $P$  is not graded strongly 1-absorbing primary. Indeed,  $2, 3 \in h(R)$  with  $2 \cdot 2 \cdot 3 \in P$ , but neither  $2 \cdot 2 \in P$  nor  $3 \in \text{Grad}(\{0\}) = \{0\}$ .

A graded ring is said to be *graded local* if it has one graded maximal ideal only. The next theorem gives a characterization of graded strongly 1-absorbing primary ideals.

**Theorem 2.3.** *Let  $P$  be a proper graded ideal of  $R$ . Then  $P$  is graded strongly 1-absorbing primary if and only if*

- (1)  $P$  is graded 1-absorbing primary and  $\text{Grad}(P) = \text{Grad}(\{0\})$ , or
- (2)  $R$  is graded local with graded maximal ideal  $X = \text{Grad}(P)$  and  $X^2 \subseteq P$ .

*Proof.* Suppose that  $P$  is graded strongly 1-absorbing primary. Then it is clear that  $P$  is graded 1-absorbing primary. Assume that  $\text{Grad}(P) \neq \text{Grad}(\{0\})$ . Assume that there exist nonunit elements  $a, b \in h(R)$  such that  $ab \notin P$ . Let  $x \in P$ . Then since  $P$  is a graded ideal,  $x_g \in P$  for all  $g \in G$ , and then  $abx_g \in P$  for all  $g \in G$ . Since  $P$  is graded strongly 1-absorbing primary,  $x_g \in \text{Grad}(\{0\})$  for all  $g \in G$ , which implies that  $x \in \text{Grad}(\{0\})$ . So,  $P \subseteq \text{Grad}(\{0\})$ , and hence  $\text{Grad}(P) = \text{Grad}(\{0\})$ , which is a contradiction. Thus,  $ab \in P$  for each

nonunit elements  $a, b \in h(R)$ . Let  $X$  be a graded maximal ideal of  $R$ . We have that  $X^2 \subseteq P$ , and then  $X = \text{Grad}(X^2) \subseteq \text{Grad}(P)$ . Thus,  $X = \text{Grad}(P)$  for each graded maximal ideal  $X$  of  $R$ . Hence, we conclude that  $R$  is graded local with graded maximal ideal  $X = \text{Grad}(P)$  and that  $X^2 \subseteq P$ . Conversely, if  $P$  is graded 1-absorbing primary and  $\text{Grad}(P) = \text{Grad}(\{0\})$ , then  $P$  is clearly a graded strongly 1-absorbing primary ideal of  $R$ . Suppose that  $R$  is graded local with graded maximal ideal  $X = \text{Grad}(P)$  and  $X^2 \subseteq P$ . Then for each nonunit elements  $a, b \in h(R)$ ,  $ab \in X^2 \subseteq P$ , and then  $P$  is trivially a graded strongly 1-absorbing primary ideal of  $R$ .  $\square$

A graded commutative ring  $R$  with unity is said to be a *graded domain* if  $R$  has no homogeneous zero divisors. A graded commutative ring  $R$  with unity is said to be a *graded field* if every nonzero homogeneous element of  $R$  is unit. The next example shows that a graded field does not need to be a field.

**Example 2.4.** Let  $R$  be a field, and suppose that  $F = \{x + uy : x, y \in R, u^2 = 1\}$ . If  $G = \mathbb{Z}_2$ , then  $F$  is  $G$ -graded by  $F_0 = R$  and  $F_1 = uR$ . Let  $a \in h(F)$  such that  $a \neq 0$ . If  $a \in F_0$ , then  $a \in R$  and since  $R$  is a field, we have  $a$  is a unit element. Suppose that  $a \in F_1$ . Then  $a = uy$  for some  $y \in R$ . Since  $a \neq 0$ , we have  $y \neq 0$ , and since  $R$  is a field, we have  $y$  is a unit element, that is,  $zy = 1$  for some  $z \in R$ . Thus,  $uz \in F_1$  such that  $(uz)a = uz(uy) = u^2(zy) = 1 \cdot 1 = 1$ , which implies that  $a$  is a unit element. Hence,  $F$  is a graded field. On the other hand,  $F$  is not a field since  $1 + u \in F - \{0\}$  is not a unit element since  $(1 + u)(1 - u) = 0$ .

*Remark 2.5.* Let  $R$  be a graded local ring. If  $P$  is a graded strongly 1-absorbing primary ideal of  $R$ , then  $\text{Grad}(P)$  does not need to be graded maximal. To see that, it suffices to consider any graded local domain that is not graded field. Then  $\{0\} = \text{Grad}(\{0\})$  is graded strongly 1-absorbing primary, which is not graded maximal.

**Corollary 2.6.** *Let  $P$  be a graded prime ideal of  $R$ . Then  $P$  is graded strongly 1-absorbing primary if and only if*

- (1)  $P = \text{Grad}(\{0\})$ , or
- (2)  $R$  is graded local with graded maximal ideal  $P$ .

**Lemma 2.7** ([5]). *Let  $R$  be a graded ring and let  $P$  be a graded ideal of  $R$ . If  $P$  is a graded 1-absorbing primary ideal of  $R$ , then  $\text{Grad}(P)$  is a graded prime ideal of  $R$ .*

*Proof.* Let  $a, b \in h(R)$  such that  $ab \in \text{Grad}(P)$ . We may assume that  $a$  and  $b$  are nonunit elements of  $R$ . Let  $k \geq 2$  be an even positive integer such that  $(ab)^k \in P$ . Then  $k = 2s$  for some positive integer  $s \geq 1$ . Since  $(ab)^k = a^k b^k = a^s a^s b^k \in P$  and  $P$  is a graded 1-absorbing primary ideal of  $R$ , we conclude that  $a^s a^s = a^k \in P$  or that  $b^k \in \text{Grad}(P)$ . Hence,  $a \in \text{Grad}(P)$  or  $b \in \text{Grad}(P)$ . Thus  $\text{Grad}(P)$  is a graded prime ideal of  $R$ .  $\square$

**Theorem 2.8.** *Let  $R$  be a graded ring. Then there exists a graded strongly 1-absorbing primary ideal of  $R$  if and only if*

- (1)  $\text{Grad}(\{0\})$  is graded prime, or

(2)  $R$  is graded local.

*Proof.* Suppose that  $R$  contains a graded strongly 1-absorbing primary ideal  $P$ . If  $R$  is not graded local, then by Theorem 2.3,  $\text{Grad}(P) = \text{Grad}(\{0\})$ . Now, by Lemma 2.7,  $\text{Grad}(P)$  is graded prime since  $P$  is graded 1-absorbing primary. Thus,  $\text{Grad}(\{0\})$  is graded prime. Conversely, by Corollary 2.6, if  $(R, X)$  is graded local, then  $X$  is a graded strongly 1-absorbing primary ideal, and if  $P = \text{Grad}(\{0\})$  is graded prime, then it is a graded strongly 1-absorbing primary ideal.  $\square$

A graded ring  $R$  is said to be *graded Artinian* if every descending chain of graded ideals of  $R$  terminates.

**Corollary 2.9.** *Let  $n \geq 2$  be an integer and let the ring  $\mathbb{Z}/n\mathbb{Z}$  be  $G$ -graded by a group  $G$ . Then  $\mathbb{Z}/n\mathbb{Z}$  has a graded strongly 1-absorbing primary ideal if and only if  $n = p^m$  for some prime  $p$  and positive integer  $m$ .*

*Proof.* By Theorem 2.8,  $\mathbb{Z}/n\mathbb{Z}$  has a graded strongly 1-absorbing primary ideal if and only if it is graded local or  $\text{Grad}(\{0\})$  is graded prime. On the other hand,  $\mathbb{Z}/n\mathbb{Z}$  is Artinian and so it is graded Artinian, and then  $\mathbb{Z}/n\mathbb{Z}$  is graded local. Now, if  $n = p_1^{k_1} \dots p_r^{k_r}$  (the primary decomposition of  $n$ ), then  $\text{Grad}(\{0\}) = p_1 \dots p_r \mathbb{Z}/n\mathbb{Z}$ . Hence,  $\text{Grad}(\{0\})$  is graded prime if and only if  $r = 1$ .  $\square$

Let  $R$  and  $S$  be two  $G$ -graded rings. Then  $R \times S$  is a  $G$ -graded ring by  $(R \times S)_g = R_g \times S_g$  for all  $g \in G$ .

**Corollary 2.10.** *Let  $R$  and  $S$  be two  $G$ -graded rings. Then  $R \times S$  has no graded strongly 1-absorbing primary ideal.*

*Proof.* The result holds from the facts that  $R \times S$  is not graded local and that  $\text{Grad}(\{0_{R \times S}\}) = \text{Grad}(\{0_R\}) \times \text{Grad}(\{0_S\})$  is never a graded prime ideal in  $R \times S$ .  $\square$

**Proposition 2.11.** *Let  $P$  be a proper graded ideal of  $R$ . Then  $P$  is a graded strongly 1-absorbing primary ideal of  $R$  if and only if whenever  $IJK \subseteq P$  for some proper graded ideals  $I, J$ , and  $K$  of  $R$ , then  $IJ \subseteq P$  or  $K \subseteq \text{Grad}(\{0\})$ .*

*Proof.* Suppose that  $P$  is a graded strongly 1-absorbing primary ideal of  $R$ . Assume that  $I, J$ , and  $K$  are proper graded ideals of  $R$  such that  $IJK \subseteq P$  and  $IJ \not\subseteq P$ . Then there exist  $a \in I$  and  $b \in J$  such that  $ab \notin P$ . Since  $a \in I$  and  $I$  is a graded ideal,  $a_g \in I$  for all  $g \in G$ . Similarly,  $b_g \in J$  for all  $g \in G$ . Since  $ab \notin P$ , there exist  $g, h \in G$  such that  $a_g b_h \notin P$ . Let  $x \in K$ . Then  $x_r \in K$  for all  $r \in G$ , and so  $a_g b_h x_r \in P$  for all  $r \in G$ . Since  $P$  is graded strongly 1-absorbing primary,  $x_r \in \text{Grad}(\{0\})$  for all  $r \in G$ , and then  $x \in \text{Grad}(\{0\})$ . Hence,  $K \subseteq \text{Grad}(\{0\})$ . Conversely, let  $x, y, z \in h(R)$  be nonunit elements such that  $xyz \in P$ . Then  $I = Rx$ ,  $J = Ry$ , and  $K = Rz$  are proper graded ideals of  $R$  with  $IJK \subseteq P$ , and then  $IJ \subseteq P$  or  $K \subseteq \text{Grad}(\{0\})$ , which implies that  $xy \in P$  or  $z \in \text{Grad}(\{0\})$ . Hence,  $P$  is a graded strongly 1-absorbing primary ideal of  $R$ .  $\square$

**Proposition 2.12.** *Let  $R$  be a graded ring and let  $P$  and  $K$  be two proper graded ideals of  $R$ . If  $P$  and  $K$  are graded strongly 1-absorbing primary, then so is  $P \cap K$ .*

*Proof.* By Proposition 1.1,  $P \cap K$  is a graded ideal of  $R$ . Let  $x, y, z \in h(R)$  be nonunit elements such that  $xyz \in P \cap K$ . Then  $xyz \in P$  and  $xyz \in K$ . If  $z \in \text{Grad}(\{0\})$ , then it is done. Suppose that  $z \notin \text{Grad}(\{0\})$ . Since  $xyz \in P$  and  $P$  is graded strongly 1-absorbing primary,  $xy \in P$ . Similarly,  $xy \in K$ . Hence,  $xy \in P \cap K$ , and thus  $P \cap K$  is a graded strongly 1-absorbing primary ideal of  $R$ .  $\square$

**Proposition 2.13.** *Let  $R$  be a graded ring such that every element of  $h(R)$  is either nilpotent or unit. Then  $Ra$  is a graded strongly 1-absorbing primary ideal of  $R$  for all nonunit  $a \in h(R)$ .*

*Proof.* Let  $a \in h(R)$  be nonunit element. By Proposition 1.1,  $Ra$  is a graded ideal of  $R$ , and as  $a$  is nonunit,  $Ra$  is proper. Assume that  $x, y, z \in h(R)$  are nonunit elements such that  $xyz \in Ra$ . Then  $z \in \text{Grad}(\{0\})$ , and hence  $Ra$  is a graded strongly 1-absorbing primary ideal of  $R$ .  $\square$

**Corollary 2.14.** *Let  $R$  be a graded ring such that every element of  $h(R)$  is either nilpotent or unit. Then every proper graded ideal of  $R$  is graded strongly 1-absorbing primary.*

*Proof.* Let  $P$  be a proper graded ideal of  $R$ . Assume that  $x, y, z \in h(R)$  are nonunit elements such that  $xyz \in P$  and  $z \notin \text{Grad}(\{0\})$ . Then  $xyz \in h(R)$  is a nonunit element with  $xyz \in R(xyz)$ . By Proposition 2.13,  $R(xyz)$  is graded strongly 1-absorbing primary, and then  $xy \in R(xyz) \subseteq P$ . Hence,  $P$  is a graded strongly 1-absorbing primary ideal of  $R$ .  $\square$

**Example 2.15.** Every element of  $R = \mathbb{Z}/9\mathbb{Z}$  is either unit or nilpotent. So, if  $R$  is  $G$ -graded by any group  $G$ , then every homogeneous element of  $R$  is either unit or nilpotent, and then every proper graded ideal of  $R$  is graded strongly 1-absorbing primary by Corollary 2.14.

**Proposition 2.16.** *Let  $R$  be a graded ring. Then every graded prime ideal of  $R$  is graded strongly 1-absorbing primary if and only if  $R$  is graded local and has at most one graded prime ideal that is not graded maximal.*

*Proof.* Suppose that every graded prime ideal of  $R$  is graded strongly 1-absorbing primary. Assume that  $R$  is not graded local, and let  $P$  be a graded prime ideal of  $R$ . Then since  $P$  is graded strongly 1-absorbing primary, we get  $\text{Grad}(\{0\}) = P$  by Corollary 2.6. Hence,  $\text{Grad}(\{0\})$  is the only graded prime ideal of  $R$ . Thus,  $R$  is graded local, which is a contradiction. Hence,  $R$  is graded local with graded maximal ideal  $X$ . Now, let  $P$  be a graded prime ideal that is not graded maximal. Then  $\text{Grad}(P) = \text{Grad}(\{0\})$ . Hence,  $R$  has at most two graded prime ideals  $\text{Grad}(\{0\})$  and  $X$ . Conversely, if every homogeneous element of  $R$  is either unit or nilpotent, then it is done by Corollary 2.14. Otherwise,  $R$  has exactly two graded prime ideals  $P \subsetneq X$ . Since  $\text{Grad}(\{0\})$  is the intersection of graded prime ideals, we get that  $\text{Grad}(\{0\}) = P$ . It is clear that both of  $\text{Grad}(\{0\})$  and  $X$  are graded strongly 1-absorbing primary ideals, as desired.  $\square$

**Proposition 2.17.** *Let  $R$  be a graded ring. Then every graded primary ideal of  $R$  is graded strongly 1-absorbing primary if and only if*

- (1) every element of  $h(R)$  is either nilpotent or unit, or
- (2)  $R$  is graded local with graded maximal ideal  $X$ , one graded prime ideal that is not graded maximal (which is  $\text{Grad}(\{0\})$ ), and every graded  $X$ -primary ideal contains  $X^2$ .

*Proof.* Suppose that every graded primary ideal of  $R$  is graded strongly 1-absorbing primary. If (1) does not hold, then by Proposition 2.16,  $R$  is graded local with exactly two graded prime ideals, which are  $\text{Grad}(\{0\})$  and  $X$  (the graded maximal ideal). Now, let  $P$  be a graded  $X$ -primary ideal of  $R$ . Then since  $\text{Grad}(\{0\}) \neq \text{Grad}(P) = X$ , we have necessarily  $X^2 \subseteq P$  (by Theorem 2.3). Conversely, if (1) holds, then the result follows trivially. Hence, suppose that (2) holds. Let  $P$  be a graded primary ideal of  $R$ . Then  $\text{Grad}(P) = \text{Grad}(\{0\})$  or  $\text{Grad}(P) = X$ . If  $\text{Grad}(P) = \text{Grad}(\{0\})$ , then by Theorem 2.3,  $P$  is graded strongly 1-absorbing primary since every graded primary is graded 1-absorbing primary. Now, if  $\text{Grad}(P) = X$ , then also  $P$  is graded strongly 1-absorbing primary since  $X^2 \subseteq P$ .  $\square$

**Proposition 2.18.** *Let  $R$  be a graded ring. Then  $\{0\}$  is the only graded strongly 1-absorbing primary ideal of  $R$  if and only if*

- (1)  $R$  is a graded field, or
- (2)  $R$  is a graded domain that is not graded local.

*Proof.* Suppose that  $\{0\}$  is the only graded strongly 1-absorbing primary ideal of  $R$ . If  $R$  is graded local with graded maximal ideal  $X$ , then  $X = \{0\}$  since  $X$  is a graded strongly 1-absorbing primary ideal of  $R$ . Hence,  $R$  is a graded field. Now, if  $R$  is not graded local, then  $\text{Grad}(\{0\}) = \{0\}$  is graded prime since  $\text{Grad}(\{0\})$  is a graded strongly 1-absorbing primary ideal. Thus,  $R$  is a graded domain. The converse is obvious.  $\square$

**Lemma 2.19.** *Let  $R$  be a graded ring. Suppose that  $P$  and  $K$  are graded ideals of  $R$ . Then  $(P : K) = \{r \in R : rK \subseteq P\}$  is a graded ideal of  $R$ .*

*Proof.* Clearly,  $(P : K)$  is an ideal of  $R$ . Let  $r \in (P : K)$ . Then  $rK \subseteq P$  and  $r = \sum_{g \in G} r_g$ . Let  $x \in K$ . Then since  $K$  is graded,  $x = \sum_{h \in G} x_h$ , where  $x_h \in K$  for all  $h \in G$ . Now,  $r_g x_h \in h(R)$  for all  $g, h \in G$  with  $\sum_{g \in G} r_g x_h = \left(\sum_{g \in G} r_g\right) x_h = r x_h \in rK \subseteq P$  for all  $h \in G$ . Since  $P$  is graded,  $r_g x_h \in P$  for all  $g, h \in G$ , and then  $r_g x \in P$  for all  $g \in G$ , which implies that  $r_g K \subseteq P$  for all  $g \in G$ , so  $r_g \in (P : K)$  for all  $g \in G$ . Hence,  $(P : K)$  is a graded ideal of  $R$ .  $\square$

**Lemma 2.20.** *Let  $P$  be a graded 1-absorbing primary ideal of  $R$  and let  $K \not\subseteq P$  be a proper graded ideal of  $R$ . Then  $(P : K)$  is a graded primary ideal of  $R$ .*

*Proof.* By Lemma 2.19,  $(P : K)$  is a graded ideal of  $R$  and it is proper as  $K \not\subseteq P$ . Let  $x, y \in h(R)$  with  $xy \in (P : K)$  and  $x \notin (P : K)$ . Clearly,  $y$  is a nonunit element. If  $x$  is unit, then  $y \in (P : K) \subseteq \text{Grad}((P : K))$ . Hence, we may assume that  $x$  and  $y$  are nonunit elements of  $R$ . Since  $x \notin (P : K)$ , there is  $z \in K$  such that  $xz \notin P$ , and then  $xz_g \notin P$  for some  $g \in G$ . Note that  $z_g \in K$  as  $K$  is graded. Indeed  $xyz_g \in P$  and  $P$  is graded 1-absorbing primary. Then

$y \in \text{Grad}(P) \subseteq \text{Grad}((P : K))$ . Thus,  $(P : K)$  is a graded primary ideal of  $R$ .  $\square$

**Proposition 2.21.** *Let  $P$  be a graded strongly 1-absorbing primary ideal of  $R$  and let  $K \not\subseteq \text{Grad}(P)$  be a proper graded ideal of  $R$ . Then  $(P : K)$  is a graded strongly 1-absorbing primary ideal of  $R$ .*

*Proof.* If  $\text{Grad}(P) \neq \text{Grad}(\{0\})$ , then  $R$  is graded local with graded maximal ideal  $\text{Grad}(P)$ . In this situation, our assumption  $K \not\subseteq \text{Grad}(P)$  is not satisfied. Then we must have that  $\text{Grad}(P) = \text{Grad}(\{0\})$  is graded prime. Let  $a \in (P : K)$ . Then  $a_g \in (P : K)$  for all  $g \in G$  as  $(P : K)$  is a graded ideal by Lemma 2.19, and then  $a_g K \subseteq P \subseteq \text{Grad}(P)$  for all  $g \in G$ . Since  $K \not\subseteq \text{Grad}(P)$ , we get  $a_g \in \text{Grad}(P)$  for all  $g \in G$ , which implies that  $a \in \text{Grad}(P)$ . Hence,  $P \subseteq (P : K) \subseteq \text{Grad}(P)$ . Thus,  $\text{Grad}((P : K)) = \text{Grad}(P) = \text{Grad}(\{0\})$ . Now, the result follows from Theorem 2.3 and Lemma 2.20.  $\square$

Let  $R$  and  $S$  be two  $G$ -graded rings. Then a ring homomorphism  $f : R \rightarrow S$  is said to be graded homomorphism if  $f(R_g) \subseteq S_g$  for all  $g \in G$ .

**Lemma 2.22.** *Suppose that  $f : R \rightarrow S$  is a graded homomorphism.*

- (1) *If  $K$  is a graded ideal of  $S$ , then  $f^{-1}(K)$  is a graded ideal of  $R$ .*
- (2) *If  $P$  is a graded ideal of  $R$  with  $\ker(f) \subseteq P$ , then  $f(P)$  is a graded ideal of  $f(R)$ .*

*Proof.* (1) Clearly,  $f^{-1}(K)$  is an ideal of  $R$ . Let  $x \in f^{-1}(K)$ . Then  $x \in R$  with  $f(x) \in K$ , and  $x = \sum_{g \in G} x_g$ , where  $x_g \in R_g$  for all  $g \in G$ . Hence, for every  $g \in G$ ,  $f(x_g) \in f(R_g) \subseteq S_g$  such that  $\sum_{g \in G} f(x_g) = f\left(\sum_{g \in G} x_g\right) = f(x) \in K$ . Since  $K$  is graded,  $f(x_g) \in K$  for all  $g \in G$ , that is,  $x_g \in f^{-1}(K)$  for all  $g \in G$ . Hence,  $f^{-1}(K)$  is a graded ideal of  $R$ .

(2) Clearly,  $f(P)$  is an ideal of  $f(R)$ . Let  $y \in f(P)$ . Then  $y \in f(R)$ , and so there exists  $x \in R$  such that  $y = f(x)$ . Thus  $f(x) \in f(P)$ , which implies that  $x \in P$  since  $\ker(f) \subseteq P$ , and hence  $x_g \in P$  for all  $g \in G$  since  $P$  is graded. Thus,  $y_g = (f(x))_g = f(x_g) \in f(P)$  for all  $g \in G$ . Therefore,  $f(P)$  is a graded ideal of  $f(R)$ .  $\square$

**Proposition 2.23.** *Suppose that  $f : R \rightarrow S$  is a graded homomorphism.*

- (1) *If  $f$  is a graded epimorphism and  $P$  is a graded strongly 1-absorbing primary ideal of  $R$  containing  $\ker(f)$ , then  $f(P)$  is a graded strongly 1-absorbing primary ideal of  $S$ .*
- (2) *If  $f$  is a graded monomorphism,  $K$  is a graded strongly 1-absorbing primary ideal of  $S$ , and  $f(x)$  is nonunit in  $S$  for all nonunits homogeneous elements  $x \in R$ , then  $f^{-1}(K)$  is a graded strongly 1-absorbing primary ideal of  $R$ .*

*Proof.* (1) By Lemma 2.22,  $f(P)$  is a graded ideal of  $S$ . Let  $a, b, c \in h(S)$  be nonunit elements such that  $abc \in f(P)$ . Since  $f$  is a graded epimorphism, there exist nonunit elements  $x, y, z \in h(R)$  such that  $a = f(x), b = f(y)$ , and  $c = f(z)$ . Suppose that  $ab \notin f(P)$ . Then  $xy \notin P$ . Hence, since  $P$  is a graded strongly



1-absorbing primary ideal of  $R$  and  $xyz \in P$  (because  $\ker(f) \subseteq P$ ), we get that  $z \in \text{Grad}(\{0_R\})$ . Thus,  $c = f(z) \in \text{Grad}(\{0_S\})$ . Consequently,  $f(P)$  is a graded strongly 1-absorbing primary ideal of  $S$ .

(2) By Lemma 2.22,  $f^{-1}(K)$  is a graded ideal of  $R$ . Let  $a, b, c \in h(R)$  be nonunit elements such that  $abc \in f^{-1}(K)$ . Suppose that  $ab \notin f^{-1}(K)$ . Then  $f(a)f(b) \notin K$ . Hence, since  $K$  is a graded strongly 1-absorbing primary ideal of  $S$  and  $f(a)f(b)f(c) \in K$ , we conclude that  $f(c) \in \text{Grad}(\{0_S\})$ . Thus, since  $f$  is a graded monomorphism,  $c \in \text{Grad}(\{0_R\})$ . Consequently,  $f^{-1}(K)$  is a graded strongly 1-absorbing primary ideal of  $R$ .  $\square$

*Remark 2.24.* The condition “ $f(x)$  is nonunit in  $S$  for all nonunits homogeneous elements  $x \in R$ ” in Proposition 2.23 (2) is necessary by [9, Example 7].

**Corollary 2.25.** *Let  $K \subseteq P$  be two proper graded ideals of a graded ring  $R$ . If  $P$  is a graded strongly 1-absorbing primary ideal of  $R$ , then  $P/K$  is a graded strongly 1-absorbing primary ideal of  $R/K$ .*

*Proof.* Define  $f : R \rightarrow R/K$  by  $f(x) = x + K$ . Then  $f$  is a graded epimorphism with  $\ker(f) = K \subseteq P$ . So, by Proposition 2.23,  $f(P) = P/K$  is a graded strongly 1-absorbing primary ideal of  $R/K$ .  $\square$

**Corollary 2.26.** *Let  $P$  be a graded strongly 1-absorbing primary ideal of  $R$ , let  $S$  be a graded subring of  $R$ , and let every homogeneous nonunit elements of  $S$  be nonunit in  $R$ . Then  $P \cap S$  is a graded strongly 1-absorbing primary ideal of  $S$ .*

*Proof.* Define  $f : S \rightarrow R$  by  $f(x) = x$ . Then  $f$  is a graded monomorphism. So, by Proposition 2.23,  $f^{-1}(P) = P \cap S$  is a graded strongly 1-absorbing primary ideal of  $S$ .  $\square$

**Corollary 2.27.** *Let  $P$  be a graded strongly 1-absorbing primary ideal of  $R$ . Then  $P_e$  is a strongly 1-absorbing primary ideal of  $R_e$ .*

*Proof.* Apply Corollary 2.26 with  $S = R_e$ , to prove.  $\square$

Let  $S$  be a multiplicatively closed subset of  $h(R)$ . Then  $S^{-1}R$  is a graded ring with  $(S^{-1}R)_g = \left\{ \frac{a}{s} : a \in R_h, s \in S \cap R_{hg^{-1}} \right\}$ .

**Proposition 2.28.** *Let  $R$  be a graded ring, let  $S$  be a multiplicatively closed subset of  $h(R)$ , and let  $P$  be a graded strongly 1-absorbing primary ideal of  $R$  such that  $P \cap S = \emptyset$ . Then  $S^{-1}P$  is a graded strongly 1-absorbing primary ideal of  $S^{-1}R$ .*

*Proof.* Since  $P \cap S = \emptyset$ ,  $S^{-1}P$  is a proper graded ideal of  $S^{-1}R$ . Let  $\frac{x}{s_1} \frac{y}{s_2} \frac{z}{s_3} \in S^{-1}P$  for some nonunit elements  $x, y, z \in h(R)$  and  $s_1, s_2, s_3 \in S$ . Then there is  $u \in S$  such that  $uxyz \in P$ . Suppose that  $\frac{x}{s_1} \frac{y}{s_2} \notin S^{-1}P$ . Then  $uxy \notin P$ . Hence,  $z \in \text{Grad}(\{0_R\})$  since  $P$  is a graded strongly 1-absorbing primary ideal of  $R$ . Thus,  $\frac{z}{s_3} \in S^{-1}\text{Grad}(\{0_R\}) = \text{Grad}(\{0_{S^{-1}R}\})$ .  $\square$

Let  $R$  be a  $G$ -graded ring. Then  $R[X]$  is  $G$ -graded by  $(R[X])_g = R_g[X]$  for all  $g \in G$ .

**Lemma 2.29.** *Let  $R$  be a  $G$ -graded ring and let  $P$  be a graded ideal of  $R$ . Then  $P[X]$  is a graded ideal of  $R[X]$ .*

*Proof.* Let  $f(X) \in P[X]$ . Then  $f(X) = a_0 + a_1X + \cdots + a_nX^n$  for some  $a_0, a_1, \dots, a_n \in P$ . Since  $P$  is a graded ideal, we have  $(a_k)_g \in P$  for all  $k = 0, 1, \dots, n$  and  $g \in G$ , and then  $(f(X))_g = (a_0)_g + (a_1)_gX + \cdots + (a_n)_gX^n \in P[X]$  for all  $g \in G$ . Hence,  $P[X]$  is a graded ideal of  $R[X]$ .  $\square$

**Proposition 2.30.** *Let  $R$  be a graded ring and let  $P$  be a proper graded ideal of  $R$ .*

- (1)  *$R[X]$  has a graded strongly 1-absorbing primary ideal if and only if  $\text{Grad}(\{0_R\})$  is a graded prime ideal of  $R$ .*
- (2) *If  $P[X]$  is a graded strongly 1-absorbing primary ideal of  $R[X]$ , then  $P$  is a graded strongly 1-absorbing primary ideal of  $R$ .*
- (3) *The graded ideal  $P + XR[X]$  is never a graded strongly 1-absorbing primary ideal of  $R[X]$ .*
- (4)  *$P[X]$  is a graded strongly 1-absorbing primary ideal of  $R[X]$  if and only if  $P[X]$  is graded primary and  $\text{Grad}(P) = \text{Grad}(\{0\})$ .*

*Proof.* (1) From Theorem 2.8,  $R[X]$  has a graded strongly 1-absorbing primary ideal if and only if  $\text{Grad}(\{0_{R[X]}\})$  is graded prime (since  $R[X]$  is never graded local). Moreover,  $\text{Grad}(\{0_{R[X]}\}) = (\text{Grad}(\{0_R\})) [X]$  and  $(\text{Grad}(\{0_R\})) [X]$  is a graded prime ideal of  $R[X]$  if and only if  $\text{Grad}(\{0_R\})$  is a graded prime ideal of  $R$ .

(2) If  $P[X]$  is a graded strongly 1-absorbing primary ideal of  $R[X]$ , then by Corollary 2.26,  $P = P[X] \cap R$  is a graded strongly 1-absorbing primary ideal of  $R$ .

(3) Note that  $X = 1 \cdot X \in R_e X \subseteq R_e[X] = (R[X])_e$ , and so  $X \in h(R[X])$ , and then by Lemma 1.1,  $XR[X]$  is a graded ideal of  $R[X]$ , and then also by Lemma 1.1,  $P + XR[X]$  is a graded ideal of  $R[X]$ . Since  $R[X]$  is not graded local and  $P + XR[X] \not\subseteq \text{Grad}(\{0_{R[X]}\})$  (because  $X \notin \text{Grad}(\{0_{R[X]}\})$ ),  $P + XR[X]$  is never a graded strongly 1-absorbing primary ideal of  $R[X]$ .

(4) Since  $R[X]$  is not graded local,  $P[X]$  is a graded strongly 1-absorbing primary ideal of  $R[X]$  if and only if  $P[X]$  is graded  $\text{Grad}(\{0_{R[X]}\})$ -primary if and only if  $P[X]$  is graded primary and

$$(\text{Grad}(\{0_R\})) [X] = \text{Grad}(\{0_{R[X]}\}) = \text{Grad}(P[X]) = (\text{Grad}(P)) [X]$$

if and only if  $P[X]$  is graded primary and  $\text{Grad}(\{0_R\}) = \text{Grad}(P)$ .  $\square$

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