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ON JR-RINGS

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ABSTRACT. We introduce the notion of a JR-ring. The class of JR-rings contains properly regular rings, local rings, J-clean rings, and P-clean rings. In support, we provide some examples and counterexamples. We establish some extensions of JR-rings and show that R is a JR-ring if and only if $K_0(R)$ is a JR-ring. A ring with the only idempotents 0 and 1 is a JR-ring if and only if it is a local ring. If R has no nonzero idempotents, then R is a J-clean ring if and only if R is a JR-ring. If R is J-semisimple and left or right quasi-duo ring, then R is a JR-ring if and only if R is an NR-clean ring.

1. INTRODUCTION AND PRELIMINARIES

Throughout, all rings are associative with unity unless otherwise stated. We denote the set of all regular elements, Jacobson radical, prime radical, set of all nilpotent elements, set of all idempotent elements, and set of units of a ring R by $reg(R)$, $J(R)$, $P(R)$, $N(R)$, $E(R)$, and $U(R)$, respectively. Moreover, $T_n(R)$ denotes the ring of all upper triangular matrices over a ring R. We refer readers to [7] for all undefined terms and notions.

Nicholson et al. [12] introduced the idea of a clean ring. According to them, an element in a ring R is said to be *clean* if it is the sum of an idempotent and a unit. A ring R is called *clean* if all its elements are clean. This class of rings and other generalizations have been widely studied by Anderson, Camillo, Nicholson, Khurana, Tong, Zhou, and others.

Chen [3] called a ring R, *P-clean* if every $x \in R$ has the form $x = p + u$, where $p \in P(R)$ and $u \in U(R)$. According to Chen [4], an element of a ring is called *J-clean* provided that it can be written as the sum of an idempotent and

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an element from its Jacobson radical. A ring is *J-clean* in the case where each of its element is J-clean.

From the motivation, we introduce the concept of a JR-ring. We call an element $x \in R$, JR if it is the sum of a regular element and an element from the Jacobson radical. A ring R is said to be JR if its all elements are JR. It generalizes notions of J-clean rings and P-clean rings, but the converse need not be true; in support, see Remark 2.3.

Let R be a ring and let $a \in R$. Then recall from [6] that R is said to be a *local ring* if either a or $1 - a$ is a unit. Also, a ring R is called *regular* if for each $r \in R$, there is some element $a \in R$ such that $r = rar$. All local and regular rings are JR-rings (see Proposition 2.2), but the converse need not be true. We give a sufficient condition for a JR-ring to be local in Proposition 3.1(1).

In Section 2, we give some examples of JR-rings. We also show that homomorphic images and the direct product of JR-rings are JR-rings. Finally, we discuss many extensions of JR-rings. In Section 3, we discuss some properties of JR-rings. We investigate many relations between JR-rings and other known rings.

2. JR-RINGS AND THEIR EXTENSIONS

Definition 2.1. An element $x \in R$ is said to be JR if it is the sum of a regular element and an element from the Jacobson radical, that is, $x = a + b$, $a \in reg(R)$ and $b \in J(R)$. A ring R is said to be JR if its all elements are JR.

Proposition 2.2. *Consider the following conditions:*

(1) *R is a regular ring.*
(2) *R is a J-clean ring.*
(3) *R is a P-clean ring.*
(4) *R is a local ring.*

If any one of the above statement holds, then R is a JR-ring.

Proof. (1). Clear.

(2). Let R be a J-clean ring. Therefore, for any $a \in R$, $a = e + j$, where $e \in E(R)$ and $j \in J(R)$. Since $e = eee = e$, so e is regular. Then $a \in reg(R) + J(R)$. Hence R is a JR-ring.

(3). Let R be a P-clean ring and let $a \in R$. Then $a = p + u$, $p \in P(R)$, and $u \in U(R)$. Since $P(R)$ is nil, $P(R) \subseteq J(R)$. Also, every unit is regular. It follows that $a \in reg(R) + J(R)$. Hence R is a JR-ring.

(4). Let R be a local ring. It follows from [12, Lemma 14] that R is a clean ring and that 0 and 1 are the only idempotents in R. Let $a \in R$ such that $a = e + u$, where $e \in E(R)$ and $u \in U(R)$. It follows that $a = u$ or $a = 1 + u$. In both cases, $a \in reg(R) + J(R)$, which implies that a is JR. Hence R is a JR-ring. □

Recall from [6] that a ring R is said to be *strongly regular* if for every $r \in R$, there is $x \in R$ such that $r^2x = r$.

Remark 2.3. (1) Since from [7] Boolean rings, strongly regular rings, division rings, and semisimple rings are regular rings, as a consequence of the above proposition, they are JR-rings.

(2) A JR-ring need not be regular. For example, $\mathbb{Z}_4$ is a JR-ring but not regular as 2 is not regular in $\mathbb{Z}_4$.

(3) A JR-ring need not be a J-clean ring. For example, $\mathbb{Z}_6$ is a JR-ring but not a J-clean ring.

(4) A JR-ring need not be local. For example, let $R = \begin{pmatrix} \mathbb{Z}_4 & \mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{pmatrix}$. By Proposition 2.9, R is a JR-ring. We have $J(R) = \begin{pmatrix} 2\mathbb{Z}_4 & \mathbb{Z}_4 \\ 0 & 2\mathbb{Z}_4 \end{pmatrix}$. Since $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in R/J(R)$ is not a unit, R is not a local ring.

In [6], authors called an element in a ring *NR-clean* if it is the sum of a (von-Neumann) regular and a nilpotent elements. A ring is called *NR-clean* if every element of the ring is NR-clean. In general, JR-rings need not be NR-clean rings (see Remark 3.6). Recall from [14] that an element in a ring R is called *clear* if it is the sum of unit-regular element and unit. From [13], an element a in a unital ring R is called *quasi-regular* if $1 - a$ is being invertible in $(R, \cdot)$. A JR element need not be a quasi-regular element. For example, in $\mathbb{Z}_4$, $\bar{3}$ is a JR element but not a quasi-regular element as $\bar{1} - \bar{3}$ is not a unit in $\mathbb{Z}_4$. In [10], a ring is called *UR-ring* if every element is the sum of unit and regular. This ring is a proper generalization of clean rings and regular rings.

In the following proposition, we give some examples of JR elements.

Proposition 2.4. *Let R be a ring and let $a \in R$.*

(1) *If $a - 1$ is NR-clean, then $1 + a$ is JR.*
(2) *If a is clear, then $1 + a$ is JR.*
(3) *If $a \in J(R)$, then $1 - a$ is JR.*
(4) *a is $UR(JR)$ if and only if $(1 - a)$ is $JR(UR)$.*
(5) *If a is quasi-regular, then a is JR. Indeed, if 2 is JR and $a \in reg(R)$, then a is quasi-regular.*

Proof. (1). Let $a - 1$ be NR-clean. Then $a - 1 = r + n$, where $r \in reg(R)$ and $n \in N(R)$. Thus $a = r + (1 + n)$. Since n is nilpotent, $(1 + n)$ is unit. Hence $a \in reg(R) + U(R)$. Now $1 + a = r + (1 + u)$, where $r \in reg(R)$ and $u \in U(R)$. Then $1 + a \in reg(R) + J(R)$. Hence $1 + a$ is JR.

(2). Let $a \in R$ be a clear element such that $a = r + u$, where r is unit-regular and $u \in U(R)$. Then $1 + a = r + (1 + u) \in reg(R) + J(R)$. Hence $1 + a$ is JR.

(3). Let $a \in J(R)$. Then $(1 - a) \in U(R)$, so $(1 - a)$ is regular. Hence $(1 - a)$ is JR.

(4). Clear.

(5). Let $a \in R$ such that a is quasi-regular. Then $1 - a$ is a unit, so $1 - a = u$, where $u \in U(R)$. Thus $a = 1 - u \in J(R)$, so $a \in reg(R) + J(R)$. Hence a is JR. Now, let 2 be JR. Then $2 = a + j$, $a \in reg(R)$ and $j \in J(R)$. This implies that $1 - a = j - 1 \in U(R)$. Then $1 - a$ is a unit. Hence a is quasi-regular.

□

Lemma 2.5. *Homomorphic image of a JR-ring is a JR-ring.*

Proof. Let $f : R \to S$ be a ring epimorphism. Let $s \in S$. Then there exists $a \in R$ such that $f(a) = s$. Since R is JR, so $a = r + j$, where $r \in reg(R)$ and $j \in J(R)$. Then $s = f(a) = f(r + j) = f(r) + f(j) \in reg(S) + J(S)$. Hence S is JR. □

Proposition 2.6. *Let I be an ideal of a ring R contained in $J(R)$ and idempotents lift modulo I. Then R is a JR-ring if and only if R/I is JR.*

Proof. Let $a \in R$, and write $a + I = (r + I) + (j + I)$, where $(r + I) \in reg(R/I)$ and $(j + I) \in J(R/I)$. By [10, Lemma 2.4], regular elements lift the modulo I. Then there exists a regular element $r^{'} \in R$ such that $r - r^{'} \in I \subsetneq J(R)$. Therefore, $r = r^{'} + j^{'}$, where $j^{'} \in J(R)$. We have $a + I = (r^{'} + j^{'} + I) + (j + I) = (r^{'} + j^{''}) + I$, where $j + j^{'} = j^{''}$ as $j \in J(R)$ since $I \subsetneq J(R)$. This implies that $a - (r^{'} + j^{''}) \in I \subsetneq J(R)$, so, $a = r^{'} + j^{'''} + j^{''} = r^{'} + j^{iv}$, where $j^{'''} + j^{''} = j^{iv}$. Then $a \in reg(R) + J(R)$. Hence R is a JR-ring. The converse is clear by Lemma 2.5. □

Recall from [6] that the prime radical $P(R)$ of a ring R is defined as the intersection of all prime ideals of R. Since $P(R) \subsetneq J(R)$, as a consequence of Proposition 2.6, we have the following result.

Corollary 2.7. *If R is a ring and idempotents lift modulo $P(R)$, then R is JR if and only if $R/P(R)$ is JR.*

Now, we discuss some extensions of JR-rings.

Proposition 2.8. *A finite product of rings is a JR-ring if and only if each ring is a JR-ring.*

Proof. Let $R = \prod_{i=1}^{n} R_i$ be a JR-ring. Then each R_i is JR by Lemma 2.5. Conversely, suppose that each R_i is JR, and let $a = (a_1, a_2, \ldots, a_n) \in R$. Since each a_i is JR, so $a_i = r_i + j_i$, where $r_i \in reg(R_i)$, $j_i \in J(R_i)$. Then $r = (r_1, \ldots, r_n) \in reg(R)$ and $j = (j_1, \ldots, j_n) \in J(R)$. Thus $a = r + j$. Hence R is a JR-ring. □

Proposition 2.9. *Let R and T be any two rings. Let M be an R-T-bimodule and let P be the formal triangular ring $\begin{pmatrix} R & M \\ 0 & T \end{pmatrix}$. Then P is a JR-ring if and only if R and T are JR-rings.*

Proof. Let $P = \begin{pmatrix} R & M \\ 0 & T \end{pmatrix}$ be a JR-ring. It is well known that

$$J(P) = \begin{pmatrix} J(R) & M \\ 0 & J(T) \end{pmatrix}.$$

Let $\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \in P = \begin{pmatrix} R & M \\ 0 & T \end{pmatrix}$, where $a \in R$, $m \in M$, and $b \in T$. Then $\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix} + \begin{pmatrix} j_1 & m \\ 0 & j_3 \end{pmatrix}$, where $\begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix} \in reg(P)$ and $\begin{pmatrix} j_1 & m \\ 0 & j_3 \end{pmatrix} \in$

$J(P)$. Then $a = r_1 + j_1$ and $b = r_3 + j_3$. Since $\begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix} \in reg(P)$, there exists $\begin{pmatrix} y_1 & y_2 \\ 0 & y_3 \end{pmatrix} \in P$ such that $\begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix}\begin{pmatrix} y_1 & y_2 \\ 0 & y_3 \end{pmatrix}\begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix} = \begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix}$. This implies that $r_1y_1r_1 = r_1$ and $r_3y_3r_3 = r_3$. Thus $r_1 \in reg(R)$ and $r_3 \in reg(T)$. Also $j_1 \in J(R)$ and $j_3 \in J(T)$. Hence R and T are JR-rings.

Conversely, let R and T be JR-rings. Let $\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \in P$. Then $\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} r_1 + j_1 & m \\ 0 & r_3 + j_3 \end{pmatrix} = \begin{pmatrix} r_1 & 0 \\ 0 & r_3 \end{pmatrix} + \begin{pmatrix} j_1 & m \\ 0 & j_3 \end{pmatrix}$. Since r_1 and r_3 are regular in R and T, respectively, there exist x_1 and x_3 such that $r_1x_1r_1 = r_1$ and $r_3x_3r_3 = r_3$. Then $\begin{pmatrix} r_1 & 0 \\ 0 & r_3 \end{pmatrix}\begin{pmatrix} x_1 & 0 \\ 0 & x_3 \end{pmatrix}\begin{pmatrix} r_1 & 0 \\ 0 & r_3 \end{pmatrix} = \begin{pmatrix} r_1x_1r_1 & 0 \\ 0 & r_3x_3r_3 \end{pmatrix} = \begin{pmatrix} r_1 & 0 \\ 0 & r_3 \end{pmatrix}$. Thus $\begin{pmatrix} r_1 & 0 \\ 0 & r_3 \end{pmatrix} \in reg(P)$ and $\begin{pmatrix} j_1 & m \\ 0 & j_3 \end{pmatrix} \in J(P)$. Hence P is a JR-ring. □

Let R be a ring and let $s \in R$ be central. Following [5], the set

$$K_s(R) = \{[a_{ij}] \in M_2(R) | \text{ for all } a_{ij} \in R\} \textit{ with the following operations}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a + a' & b + b' \\ c + c' & d + d' \end{pmatrix} \textit{ and}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} aa' + sbc' & ab' + bd' \\ ca + dc' & scb' + dd' \end{pmatrix} \textit{ is a ring.}$$

Proposition 2.10. *A ring R is a JR-ring if and only if $K_0(R)$ is a JR-ring.*

Proof. Let R be a JR-ring. We have to show that $K_0(R)$ is a JR-ring. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(R)$. Then $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} r_1 + j_1 & b \\ c & r_4 + j_4 \end{pmatrix}$, as R is a JR-ring, where $r_1, r_4 \in reg(R)$ and $j_1, j_4 \in J(R)$. Then $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} r_1 & 0 \\ 0 & r_4 \end{pmatrix} + \begin{pmatrix} j_1 & b \\ c & j_4 \end{pmatrix}$. Since $r_1, r_4 \in reg(R)$, there exist x and y in R, respectively, such that $r_1xr_1 = r_1$ and $r_4yr_4 = r_4$. Then $\begin{pmatrix} r_1 & 0 \\ 0 & r_4 \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\begin{pmatrix} r_1 & 0 \\ 0 & r_4 \end{pmatrix} = \begin{pmatrix} r_1xr_1 & 0 \\ 0 & r_4yr_4 \end{pmatrix} = \begin{pmatrix} r_1 & 0 \\ 0 & r_4 \end{pmatrix}$. Then $\begin{pmatrix} r_1 & 0 \\ 0 & r_4 \end{pmatrix} \in reg(K_0(R))$ and $\begin{pmatrix} j_1 & b \\ c & j_4 \end{pmatrix} \in J(K_0(R))$. Hence $K_0(R)$ is a JR-ring.

Conversely, let $K_0(R)$ be a JR-ring. Then R is so, since clearly the mapping $f\colon K_0(R) \to R$ defined by $f(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = a$ is a ring epimorphism. □

In view of [5], B is called a unital subring of a ring A if $\phi \neq B \subseteq A$ and for any $x, y \in B$, $x - y, xy \in B$ and $1_A \in B$. Let A be a ring, let B be a unital subring of A, and let $R[A, B]$ denote the set $\{(a_1, a_2, \ldots, a_n, b, b, \ldots) : a_i \in A, b \in B, 1 \leq i \leq n\}$. Then $R[A, B]$ is a ring under the componentwise addition and multiplication. Also, $J(R[A, B]) = R[J(A), J(A) \cap J(B)]$ (see [5]).

Proposition 2.11. *Let A be a ring and let B be a unital subring of A such that $J(B) \subseteq J(A)$. Then $R[A, B]$ is a JR-ring if and only if A and B are JR-rings.*

Proof. Let $R[A, B]$ be a JR-ring. Let $a \in A$ such that $(a, 0, 0, \ldots) = (r, b_1, b_1, \ldots) + (j, b_2, b_2, \ldots)$, where $(r, b_1, b_1, \ldots) \in reg(R[A, B])$, $(j, b_2, b_2, \ldots) \in J(R[A, B])$, $j \in J(A)$, and $b_2 \in J(A) \cap J(B)$. Since $(r, b_1, b_1, \ldots)$ is regular in $R[A, B]$, there exists $(x, b, b, \ldots)$ such that $(r, b_1, b_1, \ldots)(x, b, b, \ldots)(r, b_1, b_1, \ldots) = (r, b_1, b_1, \ldots)$. Hence $rxr = r$, and then r is regular in A and $j \in J(A)$. Thus A is JR. Now, let $b \in B$ such that $(0, b, b, \ldots) = (r, b_3, b_3, \ldots) + (j, b_4, b_4, \ldots)$, where $(r, b_3, b_3, \ldots) \in reg(R[A, B])$, $(j, b_4, b_4, \ldots) \in J(R[A, B])$, $j \in J(A)$, and $b_4 \in J(A) \cap J(B)$. Since $(r, b_3, b_3, \ldots) \in reg(R[A, B])$, there exists $(y, b', b', \ldots)$ such that $(r, b_3, b_3, \ldots)(y, b', b', \ldots)(r, b_3, b_3, \ldots) = (r, b_3, b_3, \ldots)$. Then $b_3 b' b_3 = b_3$. Thus b_3 is regular in B. Since $J(B) \subseteq J(A)$, so $b_4 \in J(A) \cap J(B) = J(B)$. Therefore, $b_3 \in reg(B)$ and $b_4 \in J(B)$. Thus B is JR. Hence A and B are JR-rings.

Conversely, let A and B be JR-rings. Let $z = (a_1, a_2, \ldots, a_n, b, b, \ldots) \in R[A, B]$, where, for each i, $a_i = r_i + j_i$, $r_i \in reg(A)$ and $j_i \in J(A)$, and $b = r + j$, $r \in reg(B)$ and $j \in J(B)$. Then $z = (r_1, r_2, \ldots, r_n, r, r, \ldots) + (j_1, j_2, \ldots, j_n, j, j, \ldots)$. Since each r_i is regular in A and r is regular in B, there exist $z_i \in A$ and $z_{11} \in B$ such that $r_i z_i r_i = r_i$ and $r z_{11} r = r$. Hence, $(r_1, r_2, \ldots, r_n, r, r, \ldots)\,(z_1, z_2, \ldots, z_n, z_{11}, z_{11}, \ldots)\,(r_1, r_2, \ldots, r_n, r, r, \ldots) = (r_1, r_2, \ldots, r_n, r, r, \ldots)$ is in $reg(R[A, B])$. Also, for each i, $j_i \in J(A)$ and $j \in J(B) = J(A) \cap J(B)$. Hence $R[A, B]$ is a JR-ring. □

According to [5], let R be a ring and let M be an (R, R)-bimodule, which is a general ring (with or without identity) such that for all $m, l \in M$ and $a \in R$, we have $(ml)a = m(la)$ and $(am)l = a(ml)$. The ideal-extension $I(R, M)$ of R by M is defined as the additive Abelian group $I(R, M) = R \oplus M$ with multiplication $(a, m)(c, l) = (ac, al + mc + ml)$.

Proposition 2.12. *Let R and M be defined as above. If $mxs + mxm + sxm = m$ for all $m \in M$ and $x, s \in R$, then $I(R, M)$ is a JR-ring if and only if R is a JR-ring.*

Proof. Let the ideal-extension $I(R, M)$ be a JR-ring. Define $f \colon I(R, M) \to R$ by $f(r, m) = r$. Then clearly it is a ring epimorphism. Thus, by Lemma 2.5, R is a JR-ring. Conversely, let R be a JR-ring. We have to show that $I(R, M)$ is a JR-ring. Let $(r, m) \in I(R, M)$. Then $r = s + j$, where $s \in reg(R)$ and $j \in J(R)$. Since $s \in reg(R)$, there exists $x \in R$ such that $sxs = s$. Thus $(r, m) = (s, m) + (j, 0)$, where $(j, 0) \in J(I(R, M))$. Also $(s, m)(x, 0)(s, m) = (sx, 0 + mx + 0)(s, m) = (sxs, sxm + mxs + mxm) = (s, m)$ as $mxs + mxm + sxm = m$ for all $m \in M$ and $x, s \in R$. Then $(s, m) \in reg(I(R, M))$. Thus $(r, m) \in reg(I(R, M)) + J(I(R, M))$. Hence $I(R, M)$ is a JR-ring. □

Remark 2.13. Recall from [8] that a ring R is said to be *Abelian* if every idempotent in R is central. A JR-ring and an Abelian ring do not imply each other. Consider $R = \mathbb{Z}_,$. Then it is an Abelian ring but not a JR-ring. Consider $R = \begin{pmatrix} \mathbb{Z}_4 & \mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{pmatrix}$.

Then by Proposition 2.9, R is a JR-ring. All idempotents of R are $\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $\begin{pmatrix}1&0\\0&0\end{pmatrix}$, $\begin{pmatrix}0&0\\0&1\end{pmatrix}$, and $\begin{pmatrix}1&1\\0&0\end{pmatrix}$. Indeed for $\begin{pmatrix}1&1\\0&0\end{pmatrix}$, $\begin{pmatrix}1&1\\0&0\end{pmatrix}\begin{pmatrix}0&0\\0&1\end{pmatrix}=\begin{pmatrix}0&1\\0&0\end{pmatrix}$ and $\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}1&1\\0&0\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$. Thus $\begin{pmatrix}1&1\\0&0\end{pmatrix}$ is not central in R. Hence R is not an Abelian ring.

It is clear from the above discussion that all idempotent elements of a JR-ring need not be central.

In the following proposition, we have extensions of a JR-ring with the help of its central idempotent elements.

Proposition 2.14. *Let R be a ring and let e be a central idempotent of R.*

(1) *If R is a JR-ring, then eRe is a JR-ring.*
(2) *If eRe and $(1-e)R(1-e)$ are JR-rings, then R is a JR-ring.*

Proof. (1). Let R be a JR-ring. Since e is a central idempotent of R, so eRe is a homomorphic image of R. It follows from Lemma 2.5 that eRe is a JR-ring.

(2). Let eRe and $(1-e)R(1-e)$ be JR-rings. By the Pierce decomposition for the ring R, $R = eRe \oplus eR(1-e) \oplus (1-e)Re \oplus (1-e)R(1-e)$. Since e is a central idempotent of R, so $R = eRe \oplus (1-e)R(1-e) \simeq \begin{pmatrix}eRe & 0\\ 0 & (1-e)R(1-e)\end{pmatrix}$. Now let $\begin{pmatrix}a&0\\0&b\end{pmatrix} \in R$, where $a \in eRe$ and $b \in (1-e)R(1-e)$. Since eRe and $(1-e)R(1-e)$ are JR-rings, $a = r_1 + j_1$ and $b = r_2 + j_2$. Thus $\begin{pmatrix}a&0\\0&b\end{pmatrix} = \begin{pmatrix}r_1+j_1&0\\0&r_2+j_2\end{pmatrix} = \begin{pmatrix}r_1&0\\0&r_2\end{pmatrix} + \begin{pmatrix}j_1&0\\0&j_2\end{pmatrix}$. Since r_1 and r_2 are regular in R, there exist x_1 and x_2 in R such that $r_1x_1r_1 = r_1$ and $r_2x_2r_2 = r_2$. Thus $\begin{pmatrix}r_1&0\\0&r_2\end{pmatrix}\begin{pmatrix}x_1&0\\0&x_2\end{pmatrix}\begin{pmatrix}r_1&0\\0&r_2\end{pmatrix} = \begin{pmatrix}r_1x_1r_1&0\\0&r_2x_2r_2\end{pmatrix} = \begin{pmatrix}r_1&0\\0&r_2\end{pmatrix}$. Then $\begin{pmatrix}r_1&0\\0&r_2\end{pmatrix} \in reg(R)$, and also $\begin{pmatrix}j_1&0\\0&j_2\end{pmatrix} \in J(R)$. Hence R is a JR-ring. □

Corollary 2.15. *Let R be an Abelian ring. If for some $n \in \mathbb{N}$, $T_n(R)$ is a JR-ring, then R is a JR-ring.*

Proof. Let $a = diag(1,0,\dots,0) \in T_n(R)$, which is an idempotent. Clearly, $R \simeq aT_n(R)a$. Then by Proposition 2.14(1), R is a JR-ring. □

Let R be a ring and let $a \in R$. Recall from [4] that $ann_l(a) = \{r \in R | ra = 0\}$ and $ann_r(a) = \{r \in R | ar = 0\}$.

Lemma 2.16. *Let R be a ring in which all regular elements are central. If $a = b + j$ is a JR-element in R, where $b \in reg(R)$ and $j \in J(R)$, then $ann_l(a) \subseteq ann_l(b)$ and $ann_r(a) \subseteq ann_r(b)$.*

Proof. Suppose that $a = b + j$ is a JR-element in R, where $b \in reg(R)$ and $j \in J(R)$. Let $x \in ann_l(a)$. Then $xa = 0$ and so $0 = xa = xb + xj$. This implies that $xb = -xj$. Since b is regular in R, there exists $y \in R$ such that $byb = b$. It follows that $xby = -xjy = -xj'$, where $jy = j'$. Then, $xbyb = xb = -xj'b$. Since all regular elements are central in R, so $xb = -xbj'$. Thus $xb(1 + j') = 0$. Then $xb = 0$ as $(1+j') \in U(R)$, which implies that $x \in ann_l(b)$. Hence $ann_l(a) \subseteq ann_l(b)$. By a similar argument, we have $ann_r(a) \subseteq ann_r(b)$. □

Proposition 2.17. *Let R be a ring and let e be an idempotent element of R. If all regular elements are central in R, then a is JR in eRe if and only if $a \in eRe$ is JR in R.*

Proof. Suppose that a is JR in eRe. Therefore $a = b + j$, where $b \in reg(eRe)$ and $j \in J(eRe)$. Then $b \in reg(R)$, as all regular elements are central in R, and $j \in eJ(R)e \subseteq J(R)$. Hence $a \in eRe$ is JR in R.

Conversely, let $a \in eRe$ be JR in R. Then $a = b + j$, where $b \in reg(R)$ and $j \in J(R)$. Now $(1 - e)a = a - ea = exe - e(exe) = exe - exe = 0$, then $(1-e) \in ann_l(a)$ as $a \in eRe$. Similarly, we can show that $(1-e) \in ann_r(a)$. Hence $(1 - e) \in ann_l(a) \cap ann_r(a)$. Then by Lemma 2.16, $(1 - e) \in ann_l(b) \cap ann_r(b)$. We have $be = b = eb$. Since $b \in reg(R)$, there exists $y \in R$ such that $byb = b$. Then we have $(ebe)(eye)(ebe) = ebeyebe = ebybe = ebe$, that is, $ebe \in reg(eRe)$ and also $eje \in eJ(R)e = J(eRe)$. Therefore, $a = ebe+eje$, where $ebe \in reg(eRe)$ and $eje \in J(eRe)$. Hence a is JR in eRe. □

3. Properties of JR-rings

Let R be a ring and let $a \in R$. Recall from [6] that R is said to be a *local ring* if either a or $1-a$ is a unit. We know that a local ring is always a JR-ring, but the converse need not be true (see Section 2). Also, a ring R is called an (S, n)*-ring* if every element is either a unit or the sum of no more than n units. A JR-ring need not be an $(S, 2)$-ring. For example, $\mathbb{Z}_6$ is a JR-ring but not an $(S, 2)$-ring as 3 is neither a unit nor the sum of two units.

In the following proposition, we provide sufficient conditions for a JR-ring to be a local ring and an $(S, 2)$-ring.

Proposition 3.1. *Let R be a ring with the only idempotents 0 and 1.*

1. *R is a JR-ring if and only if R is a local ring.*
2. *If R is a JR-ring, then R is an $(S, 2)$-ring.*

Proof. (1). Let R be a JR-ring and let $a \in R$ such that $a = r + j$, where $r \in reg(R)$ and $j \in J(R)$. Since $r \in reg(R)$, there exists $y \in R$ such that $ryr = r$. Then ry and yr are idempotents. If $ry = 0$ or $yr = 0$ and so $r = ryr = 0$, then $a \in J(R)$. This implies that $1 - a \in U(R)$. If $ry = yr = 1$, then $r \in U(R)$. Thus $a = r + j \in U(R) + J(R)$. It follows that $1 - a = 1 - r + j \in J(R) + J(R)$, so $1 - a \in J(R)$. This implies that $a \in U(R)$. Hence R is a local ring. The converse is clear from Proposition 2.2(4).

(2). Let $a \in R$, and write $a - 1 = r + j$, where $r \in reg(R)$ and $j \in J(R)$. Since $r \in reg(R)$, there exists $y \in R$ such that $ryr = r$. Then $(ry)^2 = ryry = ry$, and also $(yr)^2 = yryr = yr$. Thus ry and yr are idempotents. Since R has only 0 and 1 idempotents, either $ry = yr = 1$ or at least one of ry and yr is zero. Suppose

$ry = yr = 1$. Then $r \in U(R)$. Then from $a - 1 = r + j$, we have $a = r + (1 + j)$, so a is the sum of two units. Suppose that at least one of ry and yr is zero. Then $r = 0$. Thus from $a - 1 = r + j$, we have $a = (1 + j)$, so a is unit. Hence R is an $(S, 2)$-ring. □

Proposition 3.2. *Let R be a ring.*

(1) *The only JR-ring that has no nonzero idempotents is $R = 0$.*
(2) *If R is a JR-ring and aR has no nonzero idempotents for $a \in R$, then a is the sum of a right unit and an element from Jacobson radical.*

Proof. (1). Suppose that R is a JR-ring. Therefore, for any $a \in R$, $a = r + j$, where $r \in reg(R)$ and $j \in J(R)$. Since $r \in reg(R)$, there exists $y \in R$ such that $ryr = r$, so $(ry)^2 = ryry = ry$. This implies that ry is an idempotent in R. Similarly yr is an idempotent of R. Since R has no nonzero idempotents, $ry = yr = 0$. In both cases, when $r = 0$, we have $a = 0 + j \in J(R)$. Thus $R = J(R)$ and so $R = 0$.

(2). Let R be a JR-ring and $a \in R$. Choose a regular element and an element from Jacobson radical such that $a - 1 = r + j$, where $r \in reg(R)$ and $j \in J(R)$. Then $a = r + (1 + j)$. Since $r \in reg(R)$, there exists $y \in R$ such that $ryr = r$. Then $ayr = ryr + (1 + j)yr$, also $a(1 - yr) = (j + 1)(1 - yr)$. It follows that $(j + 1)(1 - yr)(j + 1)^{-1} = a(1 - yr)(j + 1)^{-1} \in aR$ as $(j + 1) \in U(R)$. Now $((j + 1)(1 - yr)(j + 1)^{-1})^2 = ((j + 1)(j + 1)^{-1} - (j + 1)yr(j + 1)^{-1})^2 = (1 - (j + 1)yr(j+1)^{-1})^2 = 1 + (j+1)yr(j+1)^{-1} - 2(j+1)yr(j+1)^{-1} = 1 - (j+1)yr(j+1)^{-1}$ as yr is an idempotent. Then $(j + 1)(1 - yr)(j + 1)^{-1}$ is an idempotent, so $(j + 1)(1 - yr)(j + 1)^{-1} = 0$ as aR has no nonzero idempotents. It follows that $(1 - yr) = 0$, so $yr = 1$. Hence a is the sum of a right unit and an element from the Jacobson radical.

□

Let c be a condition on an element in a ring R. Recall from [11] that c is called a *translation invariant* if whenever $a \in R$ satisfies the condition c, then ua and au both satisfy c for every unit $u \in R$.

Proposition 3.3. *JR-ness is a translation invariant.*

Proof. Let $a \in R$ be JR. Then $a = r + j$, where $r \in reg(R)$ and $j \in J(R)$. Thus $au = ru + ju$, $u \in U(R)$. It follows that there exists $x \in R$ such that $rxr = r$ and $ru(u^{-1}x)ru = rxru = ru$. Then, ru is regular and $au \in reg(R) + J(R)$. Similarly, $ua = ur + uj$ and ur is regular. Therefore, $ua \in reg(R) + J(R)$. Hence au and ua are JR. This implies that JR-ness is a translation invariant. □

Remark 3.4. A JR-ring and a reduced ring do not imply each other. The ring of integers $\mathbb{Z}$ is a reduced ring but not a JR-ring. Consider $R = \begin{pmatrix} \mathbb{Z}_4 & 0 \\ 0 & \mathbb{Z}_4 \end{pmatrix}$. By Proposition 2.9, R is a JR-ring. Indeed, for $a = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $a^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ implies that R is not a reduced ring. From Section 2, it is clear that every regular ring is a JR-ring, but the converse need not be true.

In the following proposition, we give a sufficient condition for a JR-ring to be a regular ring.

Proposition 3.5. *A reduced ring R is a regular ring if and only if R is a J-semisimple JR-ring.*

Proof. From [6], a regular reduced ring is strongly regular. Therefore, R is strongly regular. It follows from [8, Exercise 12.5] that R is a division ring and so R is a JR-ring with $J(R) = 0$ as R is regular. Conversely, since R is a JR-ring, for any $a \in R$, $a = r + j$, where $r \in reg(R)$ and $j \in J(R)$. Also $J(R) = 0$ implies that $a = r$, that is, each element of R is regular. Hence R is a regular ring. □

Remark 3.6. (1) A JR-ring need not be an NR-clean ring. For example, let $R = \begin{pmatrix} \mathbb{Z}_4 & \mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{pmatrix}$. By Proposition 2.9, R is a JR-ring. We see that all the elements of $N(R)$ are $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}$, $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$, and $\begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}$ and that all elements of $reg(R)$ are $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$, $\begin{pmatrix} 1 & 3 \\ 0 & 3 \end{pmatrix}$, $\begin{pmatrix} 0 & 3 \\ 0 & 3 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$, and $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$. An element $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ cannot be written as the sum of an element from $N(R)$ and $reg(R)$. Hence R is not an NR-clean ring.

(2) A JR-ring and a J-semisimple ring do not imply each other. The ring of integers $\mathbb{Z}$ is J-semisimple but not a JR-ring, and the ring $\mathbb{Z}_4$ is JR but not J-semisimple.

Recall from [9] that a ring is called left (resp. right) *quasi-duo* if every maximal left (resp. right) ideal is two-sided. In the following proposition, we provide a sufficient condition for an NR-clean ring to be a JR-ring.

Proposition 3.7. *If R is J-semisimple and left or right quasi-duo ring, then R is a JR-ring if and only if R is an NR-clean ring.*

Proof. Let R be a JR-ring. Therefore, for any $a \in R$, $a = r + j$, where $r \in reg(R)$ and $j \in J(R)$. Since R is J-semisimple, $J(R) = 0$. Thus $a = r$. Hence R is an NR-clean ring.

Conversely, suppose R is an NR-clean ring. Therefore, for any $a \in R$, $a = r + n$, where $r \in reg(R)$ and $n \in N(R)$. Since R is a left or right quasi-duo ring, from [9], we have $N(R) \subseteq J(R)$. Thus $a \in reg(R) + J(R)$. Hence R is a JR-ring. □

Remark 3.8. (1) A JR-ring need not be clean. For example, let F be a field with $char(F) \neq 2$, let $A = F[[x]]$, and let K be the field of fractions of A. All the ideals of A are generated by the power of x, denoted by (x^n). Now, if A_F denotes the vector space A over F, then define

$R = \{r \in \text{End}(A_F) :$ there exist $q \in K$and a positive integer n, with $r(a) = qa,$ for all $a \in (x^n)\}$. Then R is regular but not clean (see [1, Definition 2.1]). By Proposition 2.2, every regular ring is a JR-ring. It follows that R is a JR-ring but not clean.

(2) Following [6], a ring R is called *unit regular* if for each $r \in R$, $r = rur$ for some unit u in R. A JR-ring need not be unit regular. For example, consider the above example of ring R. By [2, Theorem 5], a unit regular ring is clean. It follows that R is not unit regular but a JR-ring.

Proposition 3.9. *Let R be a J-semisimple ring with the only idempotents* 0 *and* 1.

(1) *R is a JR-ring if and only if R is a clean ring.*
(2) *R is a JR-ring if and only if R is unit regular.*

Proof. (1). Let R be a JR-ring and let $a \in R$. If $a = 0$, then $a = 1 + (-1) \in E(R) + U(R)$ is a clean element of R. Consider $0 \neq a \in R$. Then $a = r + j$, where $r \in reg(R)$ and $j \in J(R)$. Since R is J-semisimple, $J(R) = 0$. Thus $a = r$. Since r is regular in R, there exists $y \in R$ such that $ryr = r$. Then ry and yr are idempotents. Since R has only 0 and 1 idempotents, either $ry = yr = 1$ or at least one of ry and yr is zero. If at least one of ry and yr is zero, then $r = 0$. Thus $a = 0$, which is a contradiction as $a \neq 0$. If $ry = yr = 1$, then $r \in U(R)$. It follows that $a = 0 + r \in E(R) + U(R)$. Hence R is a clean ring. Conversely, let R be a clean ring. Let $a \in R$, $a = e + u$, where $e \in E(R)$ and $u \in U(R)$. Since R has only 0 and 1 idempotents, $a = u$ or $a = 1 + u$. In both cases, a is JR. Hence R is a JR-ring.

(2). Let R be a JR-ring and let $a \in R$. If $a = 0$, then a is unit regular. Consider $0 \neq a \in R$. Then $a = r + j$, where $r \in reg(R)$ and $j \in J(R)$. Since R is J-semisimple, $J(R) = 0$. On the same line from the proof of Proposition 3.9(1), we get $r \in U(R)$. It follows that $a \in U(R)$ and every unit is unit regular. Hence R is unit regular. Conversely, let R be unit regular. Then by [2, Theorem 5], R is a clean ring and so a JR-ring by Proposition 3.9(1). □

Corollary 3.10. *Let R be a J-semisimple ring with the only idempotents* 0 *and* 1. *If R is a JR-ring, then it is Dedekind finite.*

Proposition 3.11. *If all regular elements of a ring R are idempotents, then the following conditions are equivalent:*

(1) *R is a Boolean ring.*
(2) *R is a regular ring.*
(3) *R is a J-semisimple JR-ring.*

Proof. Clear. □

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