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ON ESTIMATING SOME DISTANCES INVOLVING OPERATOR ENTROPIES VIA RIEMANNIAN METRIC

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Abstract. We focus on geometric properties for relative operator entropy and its extensions for positive definite matrices by considering the Riemannian metric. In particular, we prove that the Tsallis relative entropy $T_p(A|B)$ lies inside the sphere centered at the geometric mean of A and B with the radius equal to the half of the Riemannian distance between A and B. Some numerical examples are given in the aim to verify the validity of the reverse of some results.

1. Introduction

Let $\mathcal{M}_n$ be the algebra of $n \times n$ matrices over $\mathbb{R}$, and let $\mathbb{P}_n$ denote the cone of symmetric, positive definite elements of $\mathcal{M}_n$. The identity matrix will be denoted by I. On $\mathbb{P}_n$, a partially order is defined by setting $A \leq B$ to mean that $B - A \geq 0$, that is, $B - A$ is a positive semi-definite matrix for two matrices A and B.

To determine the difference between two states of a dynamic system, the concept of entropy is largely used for this purpose. Its generalization for definite positive matrices A and B was defined by Kamei and Fujii [6, 7] as

$$S(A|B) = A^{\frac{1}{2}} \log\left(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}\right) A^{\frac{1}{2}}, \tag{1.1}$$

known in the literature by the relative operator entropy of A and B. The formula (1.1) extends the operator entropy defined by Nakamura and Umegaki [12] and the relative operator entropy introduced by Umegaki [15]. After that, two parametric versions of $S(A|B)$ were pointed by some authors. The first parametric

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extension was given by Furuta [9] as follows:

$$S_p(A|B) = A^{\frac{1}{2}}\big(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\big)^p \log(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\big)A^{\frac{1}{2}}, \qquad p \in \mathbb{R}. \tag{1.2}$$

The second parametric version of $S(A|B)$ was defined by Yanagi, Kuriyama, and Furuichi [16] using the Tsallis relative entropy for matrices in the following manner:

$$T_p(A|B) = \frac{A\sharp_p B - A}{p}, \qquad p \in [-1,1] \setminus \{0\}, \tag{1.3}$$

where $A\sharp_p B := A^{\frac{1}{2}}\left(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\right)^p A^{\frac{1}{2}}$, for all $p \in \mathbb{R}$. It is the p-weighted geometric mean of A and B, which coincides, when taking $p = 1/2$, with the well-known geometric mean that will be simply denoted in the rest of paper by $A\sharp B$.
It is important to cite here the next properties from [7]

$$\lim_{p\to 0} S_p(A|B) = S_0(A|B) = S(A|B)$$

and

$$\lim_{p\to 0} T_p(A|B) = S(A|B).$$

We can confirm that (1.2) and (1.3) generalize the relative operator entropy defined in (1.1). For more properties about the two parametric extensions of the relative operator entropy, one can see, for instance, these references [5, 8, 10, 13]. In the definitions of operator entropies, some means arise. This connection enables many authors to establish many inequalities for these operators by the use of a fundamental result in Kubo–Ando theory [11]. In fact, for the representing function $f_\sigma(x) = 1\sigma x$ for an operator mean σ acting on positive matrices, the scalar inequality $f_{\sigma_1}(x) \le f_{\sigma_2}(x)$ $(x > 0)$ is equivalent to the operator form $A\sigma_1 B \le A\sigma_2 B$ for all positive definite matrices A and B.

In this paper, we investigate some properties of the relative operator entropies recalled in (1.1), (1.2), and (1.3). Our investigation will take a geometrical aspect via the Riemannian metric denoted by δ_R and defined for any matrices A and B in $\mathbb{P}_n$ as follows:

$$\delta_R(A,B) = \left\| \log(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}) \right\|_2,$$

where $\|X\|_2 = \left(\mathrm{Tr}(X^T X)\right)^{\frac{1}{2}}$ is the well-known Frobenius norm of a matrix X.
Thus,

$$\delta_R(A,B) = \left(\sum_{i=1}^{i=n} \log^2 \lambda_i(A^{-1}B)\right)^{\frac{1}{2}},$$

where $\lambda_i(A^{-1}B)$ denotes the eigenvalues of the matrix $A^{\frac{-1}{2}}BA^{\frac{-1}{2}}$.

It will be useful in this paper to recall the next two properties from [14] that hold for every three matrices A, B, and C in $\mathbb{P}_n$:

$$\delta_R(C\,A\,C, C\,B\,C) = \delta_R(A,B) \tag{1.4}$$

and

$$\delta_R(A^\alpha, B^\alpha) \le \alpha\, \delta_R(A,B), \quad \text{for all } \ \alpha \in]0,1]. \tag{1.5}$$

Very recently, many researches invest this metric in establishing nice and interesting properties concerning some parametric means. For precise details, we refer the reader to [1, 3, 4] and the references therein.

In the next section, we will focus on stating our main results with respect to the Riemannian metric.

2. Main results

In the ongoing section, we will explore some geometrical properties for the relative operator entropy and for its generalizations with respect to the Riemannian metric. We begin by quoting the following lemma in which we recall some elementary results of real analysis area.

Lemma 2.1. *Let x be a positive number. Then we have the next results:*

(i) *The function $p \longmapsto \dfrac{x^p - 1}{p}$ is increasing on $(0, 1]$.*

(ii) *The function $p \longmapsto (1+p)^{\frac{1}{p}}$ is decreasing on $(0, 1]$ and $\sup_{p \in (0,1]} (1+p)^{\frac{1}{p}} = e$.*

Proof. The statements are a routine exercise. □

To establish our first result, the following lemma stated by Carlen [2, Theorem 2.10] concerning monotonicity for trace functions will be used.

Lemma 2.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. If f is monotone, then $A \mapsto Tr(f(A))$ is also monotone on $\mathbb{P}_n$. In other words, for all $A, B \in \mathbb{P}_n$ such that $A \leq B$, we have*

$$Tr(f(A)) \leq Tr(f(B)).$$

The next proposition provides a comparison between distances with respect to the Riemannian metric when a certain order is given between three positive definite matrices.

Proposition 2.3. *Let A, B, and C be three matrices from $\mathbb{P}_n$ such that $A \leq B \leq C$. Then*

$$\delta_R(A, B) \leq \delta_R(A, C). \tag{2.1}$$

Proof. Since $A \leq B \leq C$, so $I \leq A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \leq A^{-\frac{1}{2}} C A^{-\frac{1}{2}}$. Combining this with the fact that the function $x \mapsto \log^2 x$ is increasing and continuous on $[1, \infty)$, by virtue of Lemma 2.2, we obtain

$$\mathrm{Tr}\Big(\log^2(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) \Big) \leq \mathrm{Tr}\Big(\log^2(A^{-\frac{1}{2}} C A^{-\frac{1}{2}}) \Big),$$

or equivalently

$$\Big\| \log^2(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) \Big\|_2 \leq \Big\| \log^2(A^{-\frac{1}{2}} C A^{-\frac{1}{2}}) \Big\|_2,$$

which is exactly the desired result. □

Hereafter for any given two positive definite matrices A and B, we will constantly set $C = A^{-1/2}\, B\, A^{-1/2}$.

Theorem 2.4. *Let $A, B \in \mathbb{P}_n$ such that $e\,A \leq B$. For all $p \in [0, \frac{e-1}{e}]$, the next inequality is satisfied*

$$\delta_R(A, S_p(A|B)) \leq \delta_R(A, B). \tag{2.2}$$

Proof. Let $p \in [0, \frac{e-1}{e}]$. By virtue of (1.4), inequality (2.2) is satisfied if and only if the next one holds true

$$\delta_R(I, C^p \log C) \leq \delta_R(I, C). \tag{2.3}$$

Consider $x \geq e$. By studying the variations of the function

$$x \longmapsto \psi(x) := x^{p-1} \log x - 1,$$

on (e, ∞), we conclude that $x_0 = \exp\left(\dfrac{1}{1-p}\right)$ is a point of maximum for ψ. Hence

$$\sup_{x \in (e,\infty)} \psi(x) = \psi(x_0) = \frac{1}{e(1-p)} - 1.$$

Since $p \in \left[0\,,\, \dfrac{e-1}{e}\right]$, so $\dfrac{1}{e(1-p)} - 1 \leq 0$. Hence, we can write

$$x^{p-1} \log x \leq 1.$$

This last inequality combined with the fact that the real function $x \longmapsto x^p \log x$ is increasing on $[e, \infty)$ leads to

$$1 \leq e^p \leq x^p \log x \leq x.$$

Consequently, if we suppose that $e\,A \leq B$, then $e\,I \leq C$, and we obtain

$$I \leq C^p \log C \leq C.$$

Employing Proposition 2.3, we deduce the desired inequality (2.3). □

For the particular case when $p = 0$, we obtain the following result.

Corollary 2.5. *Let A and B be two definite positive matrices such that $e\,A \leq B$. Then*

$$\delta_R(A, S(A|B)) \leq \delta_R(A, B). \tag{2.4}$$

Theorem 2.6. *Let A and B be two positive definite matrices such that $e\,A \leq B$. Then, for all $p \in (0, 1]$, the next inequality holds:*

$$\delta_R\big(A, T_p(A|B)\big) \leq \delta_R(A, B). \tag{2.5}$$

Proof. Take $p \in (0, 1]$. We have the next equivalence

$$\delta_R\big(A, T_p(A|B)\big) \leq \delta_R(A, B) \Longleftrightarrow \delta_R\Big(I, \frac{C^p - I}{p}\Big) \leq \delta_R(I, C). \tag{2.6}$$

If we consider $e\,A \leq B$, then by the result *ii)* in Lemma 2.1, we get $(1+p)^{\frac{1}{p}} I \leq C$ for all $p \in (0, 1]$. Therefore,

$$I \leq \frac{C^p - I}{p},$$

and we can deduce that

$$I \leq \frac{C^p - I}{p} \leq C - I \leq C.$$

Applying Proposition 2.3, we deduce the right side in the equivalence (2.6) and the proof is ended. □

Remark 2.7. By Theorems 2.4 and 2.6, we deduce that for a convenient parameter p, both parametric relative operator entropies $S_p(A|B)$ and $T_p(A|B)$ lie inside the sphere centered at A with the radius equal to the Riemannian distance between A and B.

Now, the monotonicity of the distances $\delta_R(A, S_p(A|B))$ and $\delta_R(A, T_p(A|B))$ with respect to the parameter p is pointed out in the next two theorems.

Theorem 2.8. *Let $A, B \in \mathbb{P}_n$ be two positive definite matrices such that $e\,A \leq B$. The map $p \longmapsto \delta_R(A, S_p(A|B))$ is increasing on $[0,1]$, that is, for all p and q with $0 \leq p \leq q \leq 1$, we have*

$$\delta_R(A, S_p(A|B)) \leq \delta_R(A, S_q(A|B)). \tag{2.7}$$

Proof. The condition $B \geq eA$ implies that $\log C \geq I$.
Furthermore, if $x > e$, then the function $p \longmapsto x^p$ is strictly increasing on $[0,1]$. Hence, for every $p, q \in [0,1]$ such that $p \leq q$, we obtain the next inequalities

$$I \leq \log C \leq C^p \log C \leq C^q \log C.$$

This chain of inequalities added to Proposition 2.3 gives

$$\delta_R(I, C^p \log C) \leq \delta_R(I, C^q \log C),$$

or equivalently

$$\delta_R(A, S_p(A|B)) \leq \delta_R(A, S_q(A|B)).$$

□

Remark 2.9. From the result of Theorem 2.8, it is natural to wonder if the monotonicity of the map $p \longmapsto \delta_R(A, S_p(A|B))$ on $[0,1]$ persists even if we do not take $e\,A \leq B$. A negative answer will be highlighted later.

Theorem 2.10. *Let A and B be two positive definite matrices such that $e\,A \leq B$. For every real numbers p and q from $(0,1]$ such that $p \leq q$, we have*

$$\delta_R\big(A, T_p(A|B)\big) \leq \delta_R\big(A, T_q(A|B)\big). \tag{2.8}$$

Proof. The inequality $\delta_R\big(A, T_p(A|B)\big) \leq \delta_R\big(A, T_q(A|B)\big)$ is equivalent to

$$\delta_R\Big(I, \frac{C^p - I}{p}\Big) \leq \delta_R\Big(I, \frac{C^q - I}{q}\Big).$$

Choosing $e\,A \leq B$, by virtue of Lemma 2.1, we get, for all $p, q \in (0,1]$ with $p \leq q$, the next inequalities

$$I \leq \frac{C^p - I}{p} \leq \frac{C^q - I}{q}.$$

Thus, from Proposition 2.3, we deduce the desired result. □

Remark 2.11. When the condition $e\,A \le B$ is not satisfied, then the inequalities (2.2), (2.4), (2.5), (2.7), and (2.8) are no longer true. The next counterexample illustrates this situation.

Counterexample 2.1. Consider the next two positive definite matrices

$$A = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}.$$

Computing with MATLAB software, we find the following values:

- $\delta_R(A, B) = 0,3406 < \delta_R(A, S(A|B)) = 2,1093.$
- $\delta_R(A, B) < \delta_R\big(A, S_{1/2}(A|B)\big) = 1.9517 < \delta_R\big(A, S_{1/4}(A|B)\big) = 2,0302.$
- $\delta_R(A, B) < \delta_R\big(A, T_{1/2}(A|B)\big) = 2,0295 < \delta_R\big(A, T_{1/4}(A|B)\big) = 2.0695.$

In the following results, we estimate distances involving the geometric mean and the the relative entropy operators.

Theorem 2.12. *Let A and B be two positive definite matrices. If $B \ge A$, then*

$$\delta_R(A\sharp B, S(A|B)) \le \delta_R(B, S(A|B)). \tag{2.9}$$

Proof. If $B \ge A$, then $C \ge I$. We can deduce

$$\log C \le C^{\frac{1}{2}} \le C,$$

or,

$$C^{\frac{-1}{4}}(\log C)C^{\frac{-1}{4}} \le I \le C^{\frac{1}{2}}. \tag{2.10}$$

From (2.10) and by the use of Proposition 2.3, we get

$$\delta_R\left(C^{\frac{-1}{4}}(\log C)C^{\frac{-1}{4}}, I\right) \le \delta_R\left(C^{\frac{-1}{4}}(\log C)C^{\frac{-1}{4}}, C^{\frac{1}{2}}\right),$$

which is equivalent to the following equation:

$$\delta_R(\log C, C^{\frac{1}{2}}) \le \delta_R(\log C, C).$$

Hence we can deduce inequality (2.9). □

Theorem 2.13. *Let A and B be two definite positive matrices such that $kA \le B$ for $1 < k$. We have for all $\frac{1}{2} \le p \le 1$,*

$$\delta_R\left(\frac{2(\sqrt{k}-1)}{\sqrt{k}}\, A\sharp B, T_p(A|B)\right) \le \frac{1}{2}\,\delta_R\left(\frac{4(\sqrt{k}-1)^2}{k}\, A, B\right). \tag{2.11}$$

Proof. If $k\,A \le B$, then $\dfrac{2(\sqrt{k}-1)}{\sqrt{k}}\, I \le 2(I - C^{-1/2})$.

By the monotonicity of the map $p \longmapsto \dfrac{C^p - I}{p}$ on $(0, 1]$, we have the next chain of inequalities when taking $1/2 \le p \le 1$,

$$2C^{\frac{-1}{4}}\big(C^{\frac{1}{2}} - I\big)C^{\frac{-1}{4}} \le C^{\frac{-1}{4}}\frac{C^p - I}{p}C^{\frac{-1}{4}} \le C^{\frac{1}{2}} - C^{\frac{-1}{2}} \le C^{\frac{1}{2}}.$$

We can deduce

$$\frac{2(\sqrt{k}-1)}{\sqrt{k}}\,I \le 2(I - C^{-1/2}) \le C^{\frac{-1}{4}}\frac{C^p - I}{p}C^{\frac{-1}{4}} \le C^{\frac{1}{2}} - C^{\frac{-1}{2}} \le C^{\frac{1}{2}}.$$

Proposition 2.3 enables us to state

$$\delta_R\left(\frac{2(\sqrt{k}-1)}{\sqrt{k}}\,I, C^{\frac{-1}{4}}\,\frac{C^p - I}{p}\,C^{\frac{-1}{4}}\right) \le \delta_R\left(\frac{2(\sqrt{k}-1)}{\sqrt{k}}I, C^{1/2}\right),$$

and from (1.5), we get

$$\delta_R\left(\frac{2(\sqrt{k}-1)}{\sqrt{k}}\,I, C^{\frac{-1}{4}}\,\frac{C^p - I}{p}\,C^{\frac{-1}{4}}\right) \le \frac{1}{2}\,\delta_R\left(\frac{4(\sqrt{k}-1)^2}{k}\,I, C\right).$$

Pre-multiplying and post-multiplying the left side of this last inequality by $C^{\frac{1}{4}}$ and the right one by $A^{\frac{1}{2}}$, we obtain

$$\delta_R\left(\frac{2(\sqrt{k}-1)}{\sqrt{k}}\,C^{1/2}, \frac{C^p - I}{p}\right) \le \frac{1}{2}\,\delta_R\left(\frac{4(\sqrt{k}-1)^2}{k}A, B\right).$$

Consequently,

$$\delta_R\left(\frac{2(\sqrt{k}-1)}{\sqrt{k}}\,A^{1/2}C^{1/2}A^{1/2}, \frac{A^{1/2}(C^p - I)A^{1/2}}{p}\right) \le \frac{1}{2}\,\delta_R\left(\frac{4(\sqrt{k}-1)^2}{k}A, B\right),$$

which coincides exactly with inequality (2.11). □

If we take $k = 4$ in the last theorem, then we obtain the following interesting result.

Corollary 2.14. *Let A and B be two definite positive matrices such that $4A \le B$. Then for all $\frac{1}{2} \le p \le 1$, we have*

$$\delta_R(A\sharp B, T_p(A|B)) \le \frac{1}{2}\delta_R(A, B). \tag{2.12}$$

Remark 2.15. Corollary 2.14 informs us that $T_p(A|B)$ lies inside the sphere centered at the geometric mean of A and B with the radius equal to the half of the Riemannian distance between A and B. With this interpretation, we can consider that Corollary 2.14 gives a refinement of the result in Theorem 2.6.

Remark 2.16. If $p \notin \left[\frac{1}{2}, 1\right]$, then inequality (2.12) is no longer valid. This fact is shown in the next counterexample.

Counterexample 2.2. Let us choose $p = 1/8$ and the next two positive definite matrices

$$A = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} \text{ and } B = \begin{pmatrix} 16,4 & 4,1 \\ 4,1 & 16,4 \end{pmatrix}.$$

Calculations with MATLAB give

$$\delta_R(A\sharp B, T_{1/8}(A|B)) = 1.3645 > \frac{1}{2}\delta_R(A, B) = 0,9977.$$

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