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ORLICZ–LACUNARY CONVERGENT DOUBLE SEQUENCES AND THEIR APPLICATIONS TO STATISTICAL CONVERGENCE

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Abstract. We introduce and study Orlicz lacunary convergent double sequences over n-normed spaces. We define the notion of g_2-statistical convergence in double sequence spaces and study some topological and algebraic properties of these newly formed sequence spaces. Some inclusion relations are also established in this paper. Finally, we study some applications of statistical convergence of these sequence spaces.

1. Introduction and preliminaries

The initial work on double sequences was established by Bromwich [2]. Recently, Zeltser [27] studied both the theory of topological double sequence spaces and the theory of summability of double sequences. The double lacunary statistical convergence was presented by Patterson and Savas [22]. A double sequence $x = (x_{ij})$ has a Pringsheim limit L (denoted by $P - \lim x = L$) provided that for given $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $|x_{ij} - L| < \epsilon$ whenever $i, j > n$ (see [18]). We denote more briefly as P-convergent. The double sequence $x = (x_{ij})$ is bounded if there exists a positive number K such that $|x_{ij}| < K$ for all i, j. Lorentz [14] introduced the notion of F-summability and discussed its relation to various summability methods defined by matrices. The concept of almost convergent sequences and strongly almost convergent sequences was given in [9,15]. The concept of strongly almost convergent sequences was introduced by Maddox [16]. Recently, Kòrus [11, 12] studied Λ^2-strong convergence of numerical sequences

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and uniform convergence of double sine integrals with generated monotone coefficients. Maddox [16] defined a generalization of strong almost convergence. Detailed study on almost convergence can be seen in [1, 15, 26].

A double sequence $x = (x_{ij})$ of real numbers is said to be almost convergent to a limit ξ if

$$\lim_{r,s\to\infty} \sup_{m,n\geq 0} \left| \frac{1}{rs} \sum_{i=m}^{m+r-1} \sum_{j=n}^{n+s-1} (x_{ij} - \xi) \right| = 0. \text{ (see [18])} \tag{1.1}$$

The space of all almost convergent double sequences is denoted by $\mathcal{F}$,

$$\mathcal{F} = \Big\{ x = (x_{ij}) : \lim_{r,s} |h_{rsmn}(x) - \xi| = 0, \text{ unformily in } m, n \Big\},$$

where

$$h_{rsmn}(x) = \frac{1}{(r+1)(s+1)} \sum_{i=0}^{r} \sum_{j=0}^{s} x_{i+m,j+n}. \tag{1.2}$$

The space of all strongly almost convergent double sequences is denoted by $[\mathcal{F}]$. The theory of 2-normed spaces was first developed by Gähler [7], and aspects of n-normed spaces have been studied extensively; see [8, 10]. To know more about the definition of Orlicz sequence spaces, one can see [13, 21].

The concept of statistical convergence was introduced independently by Fast [6], Buck [3], and Schoenberg [25] for real and complex sequences. Later, the idea was associated with summability theory by Connor [4], Duman and Orhan [5], Mohiuddine [17], Mursaleen et al. [19, 20], and many others. Statistical convergence is closely related to the concept of convergence in probability. As defined by Mursaleen and Edely [19], a real double sequence $x = (x_{ij})$ is said to be statistically convergent to L if for each $\varepsilon > 0$,

$$P - \lim_{r,s} \frac{1}{rs} |\{(i,j) : i < r \text{ and } j < s, |x_{ij} - l| \geq \varepsilon\}| = 0.$$

In this case, we write $S^2 - \lim_{i,j} x_{ij} = l$ and the set of all statistical convergent double sequences is denoted by S^2.

The double sequence $\theta = \{(k_v, l_\eta)\}$ is called double lacunary if there exist two increasing sequences of integers such that

$$k_0 = 0, \; t_v = k_v - k_{v-1} \to \infty \text{ as } v \to \infty$$

and

$$l_0 = 0, \; \bar{h}_\eta = l_\eta - l_{\eta-1} \to \infty \text{ as } \eta \to \infty.$$

Let $k_{v,\eta} = k_v l_\eta$, let $t_{v,\eta} = t_v \bar{t}_\eta$, and let $\theta_{v,\eta}$ be determined by $I'_{v,\eta} = \{(k,l) : k_{v-1} < k \leq k_v \text{ and } l_{\eta-1} < l \leq l_\eta\}$, $q_v = \frac{k_v}{k_{v-1}}$, $\bar{q}_\eta = \frac{l_\eta}{l_{\eta-1}}$, and $q_{v,\eta} = q_v \bar{q}_\eta$ (see [22]). For more details about sequence spaces and paranormed spaces, see [23, 24] and reference therein.

Let $\mathcal{O} = (O_{kl})$ be a Museilak–Orlicz function, let $u = (u_{kl})$ be a double sequence of positive real numbers, and let $p = (p_{kl})$ be a bounded sequence of positive real numbers. We denote the space of all sequences defined over $(X, ||\cdot, \ldots, \cdot||)$ by $w(n - X)$. Now, we define the following sequence spaces for some ρ and for every nonzero $z_1, \ldots, z_{n-1} \in X$:

$$\big[\mathcal{O}, u, \mathcal{F}, p, \|\cdot, \ldots, \cdot\|\big] = \Big\{x \in w(n-X) : \lim_{r,s\to\infty} \sum_{k,l=1,1}^{\infty,\infty} \Big[u_{kl}O_{kl}\Big(\Big\|\frac{h_{rsmn}(x-\xi)}{\rho}, y_1, \ldots, y_{n-1}\Big\|\Big)\Big]^{p_{kl}} = 0, \text{ uniformly in } m, n \geq 1\Big\},$$

and

$$\big[\mathcal{O}, u, [\mathcal{F}], p, ||\cdot, \ldots, \cdot||\big] = \Big\{x \in w(n-X) : \lim_{r,s\to\infty} \sum_{k,l=1,1}^{\infty,\infty} \Big[u_{kl}O_{kl}\Big(h_{rsmn}\Big(\Big\|\frac{x-\xi}{\rho}, y_1, \ldots, y_{n-1}\Big\|\Big)\Big)\Big]^{p_{kl}} = 0, \text{ uniformly in } m, n \geq 1\Big\},$$

where $h_{rsmn}(x)$ is defined as in (1.2). We write $\big[\mathcal{O}, u, \mathcal{F}, p, \|\cdot, \ldots, \cdot\|\big] - \lim x = \xi$. Also,

$$\big[\mathcal{O}, u, [\mathcal{F}], p, \|\cdot, \ldots, \cdot\|\big] \subset \big[\mathcal{O}, u, \mathcal{F}, p, \|\cdot, \ldots, \cdot\|\big] \subset \big[M, u, l^{\infty}, p, \|\cdot, \ldots, \cdot\|\big]$$

holds by the following inequality:

$$\begin{aligned}\Big\|\frac{h_{rsmn}(x-\xi)}{\rho}, y_1, \ldots, y_{n-1}\Big\| &= \Big\|\frac{\frac{1}{(r+1)(s+1)}\sum_{i=0}^{r}\sum_{j=0}^{s}(x_{i+p,j+q}-\xi)}{\rho}, y_1, \ldots, y_{n-1}\Big\| \\ &\leq \frac{1}{(r+1)(s+1)}\sum_{i=0}^{r}\sum_{j=0}^{s}\Big\|\frac{x_{i+m,j+n}-\xi}{\rho}, y_1, \ldots, y_{n-1}\Big\| \\ &= h_{rsmn}\Big(\Big\|\frac{x-\xi}{\rho}, y_1, \ldots, y_{n-1}\Big\|\Big).\end{aligned}$$

The spaces of lacunary almost and strongly almost convergent double sequences in n-normed spaces are defined as follows:

$$\begin{aligned}&\big[\mathcal{O}, u, \mathcal{F}_{\theta}, p, ||\cdot, \ldots, \cdot||\big] \\ &\quad = \Big\{x = (x_{ij}) \in w(n-X) : \\ &\qquad \lim_{v,\eta\to\infty} \sum_{k,l=1,1}^{\infty,\infty} \Big[u_{kl}O_{kl}\Big\|\frac{1}{t_{v\eta}}\sum_{i,j\in I_{v\eta}}\Big(\frac{x_{i+m,j+n}-\xi}{\rho}, y_1, \ldots, y_{n-1}\Big)\Big\|\Big]^{p_{kl}} = 0, \\ &\qquad\qquad \text{uniformly in } m, n \geq 1\Big\}\end{aligned}$$

and

$$\begin{aligned}&\big[\mathcal{O}, u, [\mathcal{F}_{\theta}], p, \|\cdot, \ldots, \cdot\|\big] \\ &\quad = \Big\{x = (x_{ij}) \in w(n-X) :\end{aligned}$$

$$\lim_{v,\eta\to\infty}\frac{1}{t_{v\eta}}\sum_{i,j\in I_{v\eta}}\sum_{k,l=1,1}^{\infty,\infty}\Big[u_{kl}O_{kl}\Big(\Big\|\frac{x_{i+m,j+n}-\xi}{\rho},y_1,\ldots,y_{n-1}\Big\|\Big)\Big]^{p_{kl}}=0,$$

$$\text{uniformly in } m,n\geq 1\Big\}.$$

1.1. **Main results.**

Lemma 1.1. *For given $\varepsilon>0$, let $x=(x_{ij})$ be a strongly almost convergent double sequence. Then there exist r_0,s_0,m_0, and n_0 such that*

$$\frac{1}{rs}\sum_{i=m}^{m+r-1}\sum_{j=n}^{n+s-1}\sum_{k,l=1,1}^{\infty,\infty}\Big[u_{kl}O_{kl}\Big(\Big\|\frac{x_{ij}-\xi}{\rho},y_1,\ldots,y_{n-1}\Big\|\Big)\Big]^{p_{kl}}<\varepsilon$$

for all $p_{kl}\geq 1$, $r\geq r_0$, $s\geq s_0$, $m\geq m_0$, $n\geq n_0$ and for every nonzero $y_1,\ldots,y_{n-1}\in X$. Then $x\in\big[\mathcal{O},u,[\mathcal{F}],p,||\cdot,\ldots,\cdot||\big]$.

Proof. Given $\varepsilon>0$, take $r_0',\ s_0',\ m_0,\ n_0$ such that

$$\frac{1}{rs}\sum_{i=m}^{m+r-1}\sum_{j=n}^{n+s-1}\sum_{k,l=1,1}^{\infty,\infty}\Big[u_{kl}O_{kl}\Big(\Big\|\frac{x_{ij}-\xi}{\rho},y_1,\ldots,y_{n-1}\Big\|\Big)\Big]^{p_{kl}}<\frac{\varepsilon}{2}\tag{1.3}$$

for all $r\geq r_0',\ s\geq s_0'\ ,m\geq m_0,\ n\geq n_0$. Now we have to prove only that there exist r_0'' and s_0'' such that for $r>r_0'',\ s>s_0''$, $0\leq m\leq m_0$, $0\leq n\leq n_0$,

$$\frac{1}{rs}\sum_{i=m}^{m+r-1}\sum_{j=n}^{n+s-1}\sum_{k,l=1,1}^{\infty,\infty}\Big[u_{kl}O_{kl}\Big(\Big\|\frac{x_{ij}-\xi}{\rho},y_1,\ldots,y_{n-1}\Big\|\Big)\Big]^{p_{kl}}<\varepsilon.\tag{1.4}$$

By choosing $r_0=\max(r_0',\ r_0'')$ and $s_0=\max(s_0',\ s_0'')$, (1.4) will holds for $r\geq r_0$, $s\geq s_0$, and for all $m,\ n$. Let m_0 and n_0 be fixed. Then

$$\sum_{i=0}^{m_0-1}\sum_{j=0}^{n_0-1}\sum_{k,l=1,1}^{\infty,\infty}\Big[u_{kl}O_{kl}\Big(\Big\|\frac{x_{ij}-\xi}{\rho},y_1,\ldots,y_{n-1}\Big\|\Big)\Big]^{p_{kl}}=K\text{ (say)}.\tag{1.5}$$

$$\sum_{i=m}^{m_0-1}\sum_{j=n_0}^{n+s-1}\sum_{k,l=1,1}^{\infty,\infty}\Big[u_{kl}O_{kl}\Big(\Big\|\frac{x_{ij}-\xi}{\rho},y_1,\ldots,y_{n-1}\Big\|\Big)\Big]^{p_{kl}}=C\text{ (say)}.\tag{1.6}$$

$$\sum_{i=m_0}^{m+r-1}\sum_{j=n}^{n_0-1}\sum_{k,l=1,1}^{\infty,\infty}\Big[u_{kl}O_{kl}\Big(\Big\|\frac{x_{ij}-\xi}{\rho},y_1,\ldots,y_{n-1}\Big\|\Big)\Big]^{p_{kl}}=D\text{ (say)}.\tag{1.7}$$

Now taking $0\leq m\leq m_0$, $0\leq n\leq n_0$, $r>m_0$, and $s>n_0$, we have

$$\begin{aligned}
&\tfrac{1}{rs}\sum_{i=m}^{m+r-1}\sum_{j=n}^{n+s-1}\sum_{k,l=1,1}^{\infty,\infty}\Big[u_{kl}O_{kl}\Big(\Big\|\frac{x_{ij}-\xi}{\rho},y_1,\ldots,y_{n-1}\Big\|\Big)\Big]^{p_{kl}}\\
&=\frac{1}{rs}\Big(\sum_{i=m}^{m_0-1}+\sum_{i=m_0}^{m+r-1}\Big)\Big(\sum_{j=n}^{n_0-1}+\sum_{j=n_0}^{n+s-1}\Big)\sum_{k,l=1,1}^{\infty,\infty}\Big[u_{kl}O_{kl}\Big(\Big\|\frac{x_{ij}-\xi}{\rho},y_1,\ldots,y_{n-1}\Big\|\Big)\Big]^{p_{kl}}\\
&\leq\frac{K}{rs}+\frac{C}{rs}+\frac{D}{rs}+\frac{1}{rs}\sum_{i=m_0}^{m_0+r-1}\sum_{j=n_0}^{n_0+s-1}\sum_{k,l=1,1}^{\infty,\infty}\Big[u_{kl}O_{kl}\Big(\Big\|\frac{x_{ij}-\xi}{\rho},y_1,\ldots,y_{n-1}\Big\|\Big)\Big]^{p_{kl}}
\end{aligned}$$

(by (1.5), (1.6), (1.7))

$$\leq \frac{K}{rs}+\frac{C}{rs}+\frac{D}{rs}+\frac{\varepsilon}{2} \quad \text{(by (1.3))}.$$

Taking r and s sufficiently large, we can make

$$\frac{K}{rs}+\frac{C}{rs}+\frac{D}{rs}+\frac{\varepsilon}{2}<\varepsilon,$$

which yields (1.4). □

Theorem 1.2. *Let $p_{kl}\geq 1$ for all k,l. Then for every θ, we have*

$$\big[\mathcal{O},u,[\mathcal{F}_\theta],p,\|\cdot,\ldots,\cdot\|\big]=\big[\mathcal{O},u,[\mathcal{F}],p,\|\cdot,\ldots,\cdot\|\big].$$

Proof. Let $\{x_{ij}\}\in\big[\mathcal{O},u,[\mathcal{F}_\theta],p,\|\cdot,\ldots,\cdot\|\big]$. Then for given $\varepsilon>0$, there exist $r_0,\ s_0$, and ξ such that

$$\frac{1}{t_{v\eta}}\sum_{i=m}^{m+t_v-1}\sum_{j=n}^{n+\bar{t}_\eta-1}\sum_{k,l=1,1}^{\infty,\infty}\Big[u_{kl}O_{kl}\Big(\Big\|\frac{x_{ij}-\xi}{\rho},y_1,\ldots,y_{n-1}\Big\|\Big)\Big]^{p_{kl}}<\varepsilon$$

for $v\geq v_0,\ \eta\geq\eta_0$, and $m=M_{v-1}+1+a,\ n=N_{\eta-1}+1+a,\ \ a\geq 0$. Let $r\geq t_v$ and let $s\geq\bar{t}_\eta$. Then write $r=bt_v+\theta$ and $s=c\bar{t}_\eta+\theta$, where b and c are integers. Since $b,\ c\geq 1$, we have

$$\frac{1}{rs}\sum_{i=m}^{m+r-1}\sum_{j=n}^{n+s-1}\sum_{k,l=1,1}^{\infty,\infty}\Big[u_{kl}O_{kl}\Big(\Big\|\frac{x_{ij}-\xi}{\rho},y_1,\ldots,y_{n-1}\Big\|\Big)\Big]^{p_{kl}}$$

$$\leq \frac{1}{rs}\sum_{i=m}^{m+(c+1)t_v-1}\sum_{j=n}^{n+(b+1)\bar{t}_\eta-1}\sum_{k,l=1,1}^{\infty,\infty}\Big[u_{kl}O_{kl}\Big(\Big\|\frac{x_{ij}-\xi}{\rho},y_1,\ldots,y_{n-1}\Big\|\Big)\Big]^{p_{kl}}$$

$$=\frac{1}{rs}\sum_{u'=0}^{s}\sum_{i=m+u't_v}^{m+(u'+1)t_v-1}\sum_{u'=0}^{c}\sum_{j=n+v\bar{t}_\eta}^{n+(v'+1)\bar{t}_\eta-1}\sum_{k,l=1,1}^{\infty,\infty}\Big[u_{kl}O_{kl}\Big(\Big\|\frac{x_{ij}-\xi}{\rho},y_1,\ldots,y_{n-1}\Big\|\Big)\Big]^{p_{kl}}$$

$$\leq \frac{(c+1)(b+1)}{rs}t_v\bar{t}_\eta\varepsilon$$

$$\leq \frac{4cbt_v\bar{t}_\eta\varepsilon}{rs}(c,\ b\geq 1).$$

For $\frac{t_v\bar{t}_\eta}{rs}\leq 1$, since $\frac{cbt_v\bar{t}_\eta}{rs}\leq 1$, we have

$$\frac{1}{rs}\sum_{i=m}^{m+r-1}\sum_{j=n}^{n+s-1}\sum_{k,l=1,1}^{\infty,\infty}\Big[u_{kl}O_{kl}\Big(\Big\|\frac{x_{ij}-\xi}{\rho},y_1,\ldots,y_{n-1}\Big\|\Big)\Big]^{p_{kl}}\leq 4\varepsilon.$$

By using Lemma 1.1, $\big[\mathcal{O},u,[\mathcal{F}_\theta],p,||\cdot,\ldots,\cdot||\big]\subseteq\big[\mathcal{O},u,[\mathcal{F}],p,||\cdot,\ldots,\cdot||\big]$. □

Lemma 1.3. *Suppose for given $\varepsilon>0$, there exist $r_0,\ s_0$, and $m_0,\ n_0$ such that*

$$\sum_{k,l=1,1}^{\infty,\infty}\Big[u_{kl}O_{kl}\Big\|\frac{1}{rs}\sum_{i=m}^{m+r-1}\sum_{j=n}^{n+s-1}\Big(\frac{x_{ij}-\xi}{\rho},y_1,\ldots,y_{n-1}\Big)\Big\|\Big]^{p_{kl}}<\varepsilon$$

for all $n\geq r_0,\ m\geq s_0,\ p\geq m_0,\ q\geq n_0$, for every nonzero $y_1,\ldots,y_{n-1}\in X$, and for some $\rho>0$. Then $x\in\big[\mathcal{O},\mathcal{F},u,p,\|\cdot,\ldots,\cdot\|\big]$.

Proof. For given $\varepsilon > 0$, take $r_0',\ s_0',\ m_0,\ n_0$ such that

$$\sum_{k,l=1,1}^{\infty,\infty}\Big[u_{kl}O_{kl}\Big\|\frac{1}{rs}\sum_{i=m}^{m+r-1}\sum_{j=n}^{n+s-1}\Big(\frac{x_{ij}-\xi}{\rho},y_1,\ldots,y_{n-1}\Big)\Big\|\Big]^{p_{kl}}<\frac{\varepsilon}{2} \tag{1.8}$$

for all $r \geq r_0',\ s \geq s_0',\ m \geq m_0,\ n \geq n_0$. By Lemma 1.1, it is enough to prove that there exist r_0'' and s_0'' such that for $r \geq r_0'',\ s \geq s_0''$, $0 \leq m \leq m_0$, $0 \leq n \leq n_0$,

$$\sum_{k,l=1,1}^{\infty,\infty}\Big[u_{kl}O_{kl}\Big\|\frac{1}{rs}\sum_{i=0}^{p+n-1}\sum_{j=0}^{q+m-1}\Big(\frac{x_{ij}-\xi}{\rho},y_1,\ldots,y_{n-1}\Big)\Big\|\Big]^{p_{kl}}<\varepsilon.$$

Since m_0 and n_0 are fixed, let

$$\sum_{i=0}^{m_0-1}\sum_{j=0}^{n_0-1}\sum_{k,l=1,1}^{\infty,\infty}\Big[u_{kl}O_{kl}\Big\|\Big(\frac{x_{ij}-\xi}{\rho},y_1,\ldots,y_{n-1}\Big)\Big\|\Big]^{p_{kl}}=K'\text{ (say)}.$$

$$\sum_{i=m}^{m_0-1}\sum_{j=n_0}^{n+s-1}\sum_{k,l=1,1}^{\infty,\infty}\Big[u_{kl}O_{kl}\Big(\Big\|\frac{x_{ij}-\xi}{\rho},y_1,\ldots,y_{n-1}\Big\|\Big)\Big]^{p_{kl}}=C'\text{ (say)}.$$

$$\sum_{i=m_0}^{m+r-1}\sum_{j=n}^{n_0-1}\sum_{k,l=1,1}^{\infty,\infty}\Big[u_{kl}O_{kl}\Big(\Big\|\frac{x_{ij}-\xi}{\rho},y_1,\ldots,y_{n-1}\Big\|\Big)\Big]^{p_{kl}}=D'\text{ (say)}.$$

Let $0 \leq m \leq m_0$, $0 \leq n \leq n_0$ and $r > m_0$, $s > n_0$. Then

$$\begin{aligned}
&\sum_{k,l=1,1}^{\infty,\infty}\Big[u_{kl}O_{kl}\Big\|\frac{1}{rs}\sum_{i=m}^{m+r-1}\sum_{j=n}^{n+s-1}\Big(\frac{x_{ij}-\xi}{\rho},y_1,\ldots,y_{n-1}\Big)\Big\|\Big]^{p_{kl}}\\
&\quad\leq\sum_{k,l=1,l}^{\infty,\infty}\Big[u_{kl}O_{kl}\Big\|\frac{1}{rs}\sum_{i=m}^{m_0-1}\sum_{j=n}^{n_0-1}\Big(\frac{x_{ij}-\xi}{\rho},y_1,\ldots,y_{n-1}\Big)\Big\|\Big]^{p_{kl}}\\
&\qquad+\sum_{k,l=1,l}^{\infty,\infty}\Big[u_{kl}O_{kl}\Big\|\frac{1}{rs}\sum_{i=m}^{m_0-1}\sum_{j=n_0}^{n+s-1}\Big(\frac{x_{ij}-\xi}{\rho},y_1,\ldots,y_{n-1}\Big)\Big\|\Big]^{p_{kl}}\\
&\qquad+\sum_{k,l=1,l}^{\infty,\infty}\Big[u_{kl}O_{kl}\Big\|\frac{1}{rs}\sum_{i=m_0}^{m+r-1}\sum_{j=n}^{n_0-1}\Big(\frac{x_{ij}-\xi}{\rho},y_1,\ldots,y_{n-1}\Big)\Big\|\Big]^{p_{kl}}\\
&\qquad+\sum_{k,l=1,1}^{\infty,\infty}\Big[u_{kl}O_{kl}\Big\|\frac{1}{rs}\sum_{i=m_0}^{m+r-1}\sum_{j=n_0}^{n+s-1}\Big(\frac{x_{ij}-\xi}{\rho},y_1,\ldots,y_{n-1}\Big)\Big\|\Big]^{p_{kl}}\\
&\quad\leq\frac{K'}{rs}+\frac{C'}{rs}+\frac{D'}{rs} \qquad\qquad (1.9)\\
&\qquad+\sum_{k,l=1,1}^{\infty,\infty}\Big[u_{kl}O_{kl}\Big\|\frac{1}{rs}\sum_{i=m_0}^{m_0+r+m-m_0-1}\sum_{j=n_0}^{n_0+s+n-n_0-1}\Big(\frac{x_{ij}-\xi}{\rho},y_1,\ldots,y_{n-1}\Big)\Big\|\Big]^{p_{kl}}.
\end{aligned}$$

Let $r - m_0 > r'_0$ and let $s - n_0 > s'_0$. Then for $0 \le m < m_0$, $0 \le n < n_0$, we have $r + m - m_0 \ge r'_0$ and $s + n - n_0 \ge s'_0$. From (1.8), we have

$$\sum_{k,l=1,1}^{\infty,\infty}\Big[u_{kl}O_{kl}\Big\|\frac{1}{(r+m+m_0)(s+n+n_0)}\sum_{i=m_0}^{m_0+r+m-m_0}\sum_{j=n_0}^{n_0+s+n-n_0}\Big(\frac{x_{ij}-\xi}{\rho},y_1,\dots,y_{n-1}\Big)\Big\|\Big]^{p_{kl}} < \frac{\varepsilon}{2}. \tag{1.10}$$

From (1.9) and (1.10), we have

$$\begin{aligned}\sum_{k,l=1,1}^{\infty,\infty}\Big[u_{kl}O_{kl}\Big\|\frac{1}{rs}\sum_{i=m}^{m+r-1}\sum_{j=n}^{n+s-1}\Big(\frac{x_{ij}-\xi}{\rho},y_1,\dots,y_{n-1}\Big)\Big\|\Big]^{p_{kl}} \\ \le \frac{K'}{rs}+\frac{C'}{rs}+\frac{D'}{rs}+\frac{(r+m-m_0)(s+n-n_0)}{rs}\frac{\varepsilon}{2} \\ \le \frac{K'}{rs}+\frac{C'}{rs}+\frac{D'}{rs}+\frac{\varepsilon}{2} \\ < \varepsilon,\end{aligned}$$

for sufficiently large r and s. □

Theorem 1.4.
(i) *For every* θ, $[\mathcal{O},u,\mathcal{F}_\theta,p,||\cdot,\dots,\cdot||]\cap[\mathcal{O},u,l^\infty,p,||\cdot,\dots,\cdot||] = [\mathcal{O},u,\mathcal{F},p,||\cdot,\dots,\cdot||]$.
(ii) *For every* θ, $[\mathcal{O},u,\mathcal{F}_\theta,p,||\cdot,\dots,\cdot||] \not\subset [\mathcal{O},u,l^\infty,p,||\cdot,\dots,\cdot||]$.

Proof. (i) Let $\{x_{ij}\} \in [\mathcal{O},u,\mathcal{F}_\theta,p,||\cdot,\dots,\cdot||]\cap[\mathcal{O},u,l^\infty,p,||\cdot,\dots,\cdot||]$ for every $\varepsilon > 0$. Then there exist v_0 and n_0 such that

$$\sum_{k,l=1,1}^{\infty,\infty}\Big[u_{kl}O_{kl}\Big\|\frac{1}{t_{v\eta}}\sum_{i=m}^{m+t_v-1}\sum_{j=n}^{n+\bar{t}_\eta-1}\Big(\frac{x_{ij}-\xi}{\rho},y_1,\dots,y_{n-1}\Big)\Big\|\Big]^{p_{kl}} < \frac{\varepsilon}{2} \tag{1.11}$$

for $v,\eta \ge v_0,\eta_0,\ m \ge m_0,\ n \ge n_0,\ m = M_{v-1}+1+a,\ n = N_{\eta-1}+1+a,\ \ a \ge 0$. Let $r \ge t_v$, let $s \ge \bar{t}_\eta$, and let $b,\ c\ \ge 1$. Then

$$\begin{aligned}&\sum_{k,l=1,1}^{\infty,\infty}\Big[u_{kl}O_{kl}\Big\|\frac{1}{rs}\sum_{i=m}^{m+r-1}\sum_{j=n}^{n+s-1}\sum_{k,l=1,1}^{\infty,\infty}O_{kl}\Big(\frac{x_{ij}-\xi}{\rho},y_1,\dots,y_{n-1}\Big)\Big\|\Big]^{p_{kl}} \\ &\le \sum_{k,l=1,1}^{\infty,\infty}\Big[u_{kl}O_{kl}\Big\|\frac{1}{rs}\sum_{\mu=0}^{c-1}\sum_{i=m+\mu t_v}^{m+(\mu+1)t_v-1}\sum_{\psi=0}^{b-1}\sum_{j=n+\psi\bar{t}_\eta}^{n+(\psi+1)\bar{t}_\eta-1}\Big(\frac{x_{ij}-\xi}{\rho},y_1,\dots,y_{n-1}\Big)\Big\|\Big]^{p_{kl}}+\frac{1}{rs} \\ &= \sum_{k,l=1,1}^{\infty,\infty}\Big[u_{kl}O_{kl}\sum_{i=m+ct_v}^{m+r-1}\sum_{j=n+b\bar{t}_\eta}^{n+s-1}\Big\|\Big(\frac{x_{ij}-\xi}{\rho},y_1,\dots,y_{n-1}\Big)\Big\|\Big]^{p_{kl}}.\end{aligned} \tag{1.12}$$

Since $\{x_{ij}\} \in [\mathcal{O},u,l^\infty,p,\|\cdot,\dots,\cdot\|]$ for all i and j, we have

$$\sum_{k,l=1,1}^{\infty,\infty}\Big[u_{kl}O_{kl}\Big\|\Big(\frac{x_{ij}-\xi}{\rho},y_1,\dots,y_{n-1}\Big)\Big\|\Big]^{p_{kl}} < K\ \ \text{(say)}.$$

From (1.11) and (1.12), we have

$$\sum_{k,l=1,1}^{\infty,\infty}\Big[u_{kl}O_{kl}\Big\|\frac{1}{rs}\sum_{i=m}^{m+r-1}\sum_{j=n}^{n+s-1}\Big(\frac{x_{ij}-\xi}{\rho},y_1,\ldots,y_{n-1}\Big)\Big\|\Big]^{p_{kl}}\leq\frac{1}{rs}cb.t_v\bar{t}_\eta\frac{\varepsilon}{2}+\frac{Kt_{v\eta}}{rs}$$

for $\frac{t_v\bar{t}_\eta}{rs}\leq 1$. Since $\frac{cbt_v\bar{t}_\eta}{rs}\leq 1$ and $\frac{Kt_{v\eta}}{rs}$ can be made less than $\frac{\varepsilon}{2}$, taking r and s sufficiently large, so

$$\sum_{k,l=1,1}^{\infty,\infty}\Big[u_{kl}O_{kl}\Big\|\frac{1}{rs}\sum_{i=m}^{m+r-1}\sum_{j=n}^{n+s-1}\Big(\frac{x_{ij}-\xi}{\rho},y_1,\ldots,y_{n-1}\Big)\Big\|\Big]^{p_{kl}}<\varepsilon$$

for $v,\eta\geq v_0,\eta_0,\ m\geq m_0,\ n\geq n_0$. Therefore,

$$\big[\mathcal{O},u,\mathcal{F}_\theta,p,||\cdot,\ldots,\cdot||\big]\cap\big[\mathcal{O},u,l_\infty,p,\|\cdot,\ldots,\cdot\|\big]\subseteq\big[\mathcal{O},u,\mathcal{F},p,\|\cdot,\ldots,\cdot\|\big].$$

(ii) Let $\{x_{ij}\}=(-1)^{ij}(ij)^{\mu}$, where ψ is constant with $0<\psi<1$. Then

$$\sum_{i=m}^{m+t_v-1}\sum_{j=n}^{n+\bar{t}_\eta-1}x_{ij},\quad m,\ n\geq 0.$$

Let $X=\mathbb{R}^n$. It is straightforward to verify that $\{x_{ij}\}\in\big[\mathcal{O},u,\mathcal{F}_\theta,p,\|\cdot,\ldots,\cdot\|\big]$ with $\xi=0$, but $\{x_{ij}\}$ is not bounded.

□

Theorem 1.5. *The sequence space* $\big[\mathcal{O},u,[\mathcal{F}],p,||\cdot,\ldots,\cdot||\big]$ *is a linear topological space total parnormed by*

$$\begin{aligned}g_2(x)&=\sup_{\substack{r,s\geq 1,\ m,n\geq 1\\ 0\neq y_1,\ldots,y_{n-1}\in X}}\Bigg(\frac{1}{rs}\sum_{i=m}^{m+r-1}\sum_{j=n}^{n+s-1}\sum_{k,l=1,1}^{\infty,\infty}\Big[u_{kl}O_{kl}\Big(\Big\|\frac{x_{ij}}{\rho},y_1,\ldots,y_{n-1}\Big\|\Big)\Big]^{p_{kl}}\Bigg)\\&=\sup_{\substack{r,s\geq 1,\ m,n\geq 1\\ 0\neq y_1,\ldots,y_{n-1}\in X}}\sum_{k,l=1,1}^{\infty,\infty}u_{kl}O_{kl}\Big[\Big(h_{rsmn}\Big(\Big\|\frac{x_{ij}}{\rho},y_1,\ldots,y_{n-1}\Big\|\Big)\Big)\Big]^{p_{kl}}.\end{aligned}$$

Proof. Clearly $g_2(x)=0$ if and only if $x=0,\ g_2(x)=g_2(-x)$ and g_2 is subadditive.

Let $(x^{(k)})$ in $\big[\mathcal{O},u,[\mathcal{F}],p,\|\cdot,\ldots,\cdot\|\big]$ such that $g_2(x^{(k)}-x)\to 0$ as $k\to\infty$ and let (ν_{kl}) be any sequence of scalars such that $\nu_{kl}\to\nu$ as $k,l\to\infty$. Since

$$g_2(x^{(k)})\leq g_2(x)+g_2(x^{(k)}-x)$$

holds by subadditivity of g_2, $g_2(x^{(k)})$ is bounded, we have

$$\begin{aligned}&g_2(\nu_{kl}x^{(k)}-\nu x)\\&=\sup_{\substack{r,s\geq 1,\ m,n\geq 1\\ 0\neq y_1,\ldots,y_{n-1}\in X}}\Bigg(\frac{1}{rs}\sum_{i=m}^{m+r-1}\sum_{j=n}^{n+s-1}\sum_{k,l=1,1}^{\infty,\infty}\Big[u_{kl}O_{kl}\Big(\Big\|\frac{\nu_{kl}x_{ij}^{(k)}-\mu x_{ij}}{\rho},y_1,\ldots,y_{n-1}\Big\|\Big)\Big]^{p_{kl}}\Bigg)\\&\leq|\nu_{kl}-\nu|\sup_{\substack{r,s\geq 1,\ m,n\geq 1\\ 0\neq y_1,\ldots,y_{n-1}\in X}}\Bigg(\frac{1}{rs}\sum_{i=m}^{m+r-1}\sum_{j=n}^{n+s-1}\sum_{k,l=1,1}^{\infty,\infty}\Big[u_{kl}O_{kl}\Big(\Big\|\frac{x_{ij}^{(k)}}{\rho},y_1,\ldots,y_{n-1}\Big\|\Big)\Big]^{p_{kl}}\Bigg)\end{aligned}$$

$$+ |\nu| \sup_{\substack{r,s\geq 1,\ m,n\geq 1\\ 0\neq y_1,\ldots,y_{n-1}\in X}} \left(\frac{1}{rs} \sum_{i=m}^{m+r-1} \sum_{j=n}^{n+s-1} \sum_{k,l=1,1}^{\infty,\infty} \left[u_{kl} O_{kl} \Big(\Big\| \frac{x_{ij}^{(k)} - x_{ij}}{\rho}, y_1, \ldots, y_{n-1} \Big\| \Big) \right]^{p_{kl}} \right)$$

$$= |\nu_{kl} - \nu| g_2(x^{(k)}) + |\nu| g_2(x^{(k)} - x) \to 0$$

as $k, l \to \infty$. □

Definition 1.6. A double sequence $x = (x_{ij})$ is said to be strongly p-Cesàro summable $(0 < p < \infty)$ to a limit ξ in $\big(\big[\mathcal{O}, [\mathcal{F}], u, p, \|\cdot, \ldots, \cdot\|\big], g_2\big)$ if

$$\lim_{k,l\to\infty} \frac{1}{kl} \sum_{i,j=1,1}^{k,l} (g_2(x_{ij} - \xi e))^p = 0,$$

and we write it as $x_{ij} \to \xi[C, g_2]_p$.

Definition 1.7. A double sequence $x = (x_{ij})$ is said to be statistically convergent (or g_2-statistically convergent) to a number ξ in $\big(\big[\mathcal{O}, u, [\mathcal{F}], p, \|\cdot, \ldots, \cdot\|\big], g_2\big)$ if for each $\varepsilon > 0$

$$\lim_{k,l\to\infty} \frac{1}{kl} |\{i \leq k,\ j \leq l : g_2(x_{ij} - \xi e) \geq \varepsilon\}| = 0,$$

where

$$g_2(x_{ij} - \xi e)$$
$$= \sup_{\substack{r,s\geq 1,\ m,n\geq 1\\ 0\neq y_1,\ldots,y_{n-1}\in X}} \left(\frac{1}{rs} \sum_{i=m}^{m+r-1} \sum_{j=n}^{n+s-1} \sum_{k,l=1,1}^{\infty,\infty} \left[u_{kl} O_{kl} \Big(\Big\| \frac{x_{ij} - \xi e}{\rho}, y_1, \ldots, y_{n-1} \Big\| \Big) \right]^{p_{kl}} \right).$$

By $S^2_{\left(\left[\mathcal{O},u,[\mathcal{F}],p,\|\cdot,\ldots,\cdot\|\right],g_2\right)}$ we denote the set of all $g_2(stat)$-convergent sequences in $\big(\big[\mathcal{O}, u, [\mathcal{F}], p, \|\cdot, \ldots, \cdot\|\big], g_2\big)$.

Definition 1.8. A double sequence $x = (x_{ij})$ is said to be statistically Cauchy in $\big(\big[\mathcal{O}, u, [\mathcal{F}], p, \|\cdot, \ldots, \cdot\|\big], g_2\big)$ (or $g_2(stat)$-Cauchy) if for every $\varepsilon > 0$ there exist two numbers $R = R(\varepsilon)$ and $S = S(\varepsilon)$ such that

$$\lim_{r,s\to\infty} \frac{1}{rs} |\{i \leq r,\ j \leq s : g_2(x_{ij} - x_{RS}) \geq \varepsilon\}| = 0.$$

Theorem 1.9. *If a double sequence is statistically convergent in* $\big(\big[\mathcal{O}, u, [\mathcal{F}], p, \|\cdot, \ldots, \cdot||\big], g_2\big)$, *then its limit is unique.*

Proof. For given $\varepsilon > 0$, we define the following sets as

$$K_1(\varepsilon) = \Big\{(i,j) \in \mathbb{N} \times \mathbb{N} : g_2(x_{ij} - \xi_1) \geq \frac{\varepsilon}{2}\Big\}$$

and

$$K_2(\varepsilon) = \Big\{(i,j) \in \mathbb{N} \times \mathbb{N} : g_2(x_{ij} - \xi_2) \geq \frac{\varepsilon}{2}\Big\}.$$

Suppose $g_2(stat) - \lim x = \xi_1$ and $g_2(stat) - \lim x = \xi_2$. Since $g_2(stat) - \lim x = \xi_1$, we have $\delta_2(K_1(\varepsilon)) = 0$. Similarly, since $g_2(stat) - \lim x = \xi_2$, we have $\delta_2(K_2(\varepsilon)) =$

0. Now let $K(\varepsilon) = K_1(\varepsilon) \cup K_2(\varepsilon)$. Then $\delta_2(K(\varepsilon)) = 0$, and therefore, $K^c(\varepsilon)$ is a nonempty set and $\delta_2(K^c(\varepsilon)) = 1$. If $(i,j) \in \mathbb{N} \times \mathbb{N} \setminus K(\varepsilon)$, then

$$g_2(\xi_1 - \xi_2) \leq g_2(x_{ij} - \xi_1) + g_2(x_{ij} - \xi_2) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get $g_2(\xi_1 - \xi_2) = 0$, and hence $\xi_1 = \xi_2$. □

Theorem 1.10. *Let $x = (x_{ij}) \in \big([\mathcal{O}, u, [\mathcal{F}], p, \|\cdot, \dots, \cdot\|], g_2\big)$ be statistically convergent to ξ if and only if there exists a set $K = \{(i_r, j_s) \in \mathbb{N} \times \mathbb{N} : i_1 < i_2 < \dots < i_r < \cdots,\ j_1 < j_2 < \dots < j_s < \cdots\}$ with $\delta_2(K) = 1$ such that $g_2(x_{i_r j_s} - \xi) \to 0$ as $i_r, j_s \to \infty$.*

Proof. Let $g_2(stat) - \lim x = \xi$. Now write for $v = 1, 2, \dots,$

$$K_s(\varepsilon) = \Big\{(r,s) \in \mathbb{N} \times \mathbb{N} : g_2(x_{i_r j_s} - \xi) \leq 1 + \frac{1}{v}\Big\}$$

and

$$M_v(\varepsilon) = \Big\{(r,s) \in \mathbb{N} \times \mathbb{N} : g_2(x_{i_r j_s} - \xi) > \frac{1}{v}\Big\}.$$

Then $\delta_2(K_s) = 0$. Also

$$M_1 \supset M_2 \supset \cdots \supset M_i \supset M_{i+1} \supset \cdots \tag{1.13}$$

and

$$\delta_2(M_v) = 1, \quad v = 1, 2, \dots. \tag{1.14}$$

As we know, $\{x_{i_r j_s}\}$ is g_2-convergent to ξ. Assume that $\{x_{ij}\}$ is not g_2-convergent to ξ. Therefore, there is $\varepsilon > 0$ such that $g_2(x_{i_r j_s} - \xi) \leq \varepsilon$ for infinitely many terms.
Let $M_\varepsilon = \Big\{(r,s) \in \mathbb{N} \times \mathbb{N} : g_2(x_{i_r j_s} - \xi) > \varepsilon\Big\}$ and let $\varepsilon > \frac{1}{v}$, $(v = 1, 2, \dots)$. Then $\delta_2(M_\varepsilon) = 0$, and by (1.13), $M_v \subset M_\varepsilon$. Hence, $\delta_2(M_v) = 0$, which contradicts (1.14) and we get that $\{x_{ij}\}$ is g_2-convergent to ξ.
Conversely, suppose that there exists a subset $K = \{(i_r, j_s) \in \mathbb{N} \times \mathbb{N} : i_1 < i_2 < \dots < i_r < \cdots,\ j_1 < j_2 < \dots < j_s < \cdots\}$ with $\delta_2(K) = 1$ such that $g_2 - \lim\limits_{r,s \to \infty} x_{i_r j_s} = \xi$. Then there exists $N(\varepsilon)$ such that

$$g_2(x_{ij} - \xi) < \varepsilon \qquad \text{for } i, j > N.$$

Let

$$K_\varepsilon = \Big\{(i,j) \in \mathbb{N} \times \mathbb{N} : g_2(x_{ij} - \xi) \geq \varepsilon\Big\}$$

and let $K' = \{(i_{N+1}, j_{N+1}), (i_{N+2}, j_{N+2}), \dots\}$. Then $\delta_2(K') = 1$ and $K_\varepsilon \subseteq \mathbb{N} \times \mathbb{N} \setminus K'$, which implies that $\delta_2(K_\varepsilon) = 0$. Hence $g_2(stat) - \lim x = \xi$. □

Theorem 1.11. *Let $g_2(stat) - \lim x = \xi_1$ and let $g_2(stat) - \lim y = \xi_2$. Then*
(i) $g_2(stat) - \lim(x \pm y) = \xi_1 \pm \xi_2$;
(ii) $g_2(stat) - \lim(\alpha x) = \alpha \xi_1, \ \ \alpha \in \mathbb{R}$.

Proof. It is easy to prove, so we omit it. □

Theorem 1.12. *If $0 < p < \infty$ and $x_{ij} \to \xi[C, g_2]_p$, then (x_{ij}) is g_2-statistically convergent to ξ in $\big([\mathcal{O}, u, [\mathcal{F}], p, \|\cdot, \dots, \cdot\|], g_2\big)$.*

Proof. Let $x_{ij} \to \xi[C, g_2]_p$. Then

$$\frac{1}{kl}\sum_{i,j=1,1}^{k,l}(g_2(x_{ij}-\xi e))^p \geq \frac{1}{kl}\sum_{\substack{i,j=1 \\ g_2(x_{ij}-\xi e)\geq\varepsilon}}^{k,l}(g_2(x_{ij}-\xi e))^p \geq \frac{\varepsilon^p}{kl}|K_\varepsilon|.$$

Since $\lim_{k,l\to\infty}\frac{1}{kl}|K_\varepsilon| = 0$ so $\delta_2(K_\varepsilon) = 0$, where $K_\varepsilon = \{i \leq k,\ j \leq l : g_2(x_{ij}-\xi e) \geq \varepsilon\}$, the desired result. □

Theorem 1.13. *If $x = (x_{ij})$ is $g_2(stat)$-convergent to ξ in $\big([\mathcal{O}, [\mathcal{F}], u, p, ||\cdot, \ldots, \cdot||], g_2\big)$, then $x_{ij} \to \xi[C, g_2]_p$.*

Proof. Suppose that $x = (x_{ij})$ is g_2-statistically convergent to ξ in $\big([\mathcal{O}, [\mathcal{F}], u, p, \|\cdot, \ldots, \cdot\|], g_2\big)$. Then for $\varepsilon > 0$, we have $\delta_2(K_\varepsilon) = 0$, where $K_\varepsilon = \{i \leq k,\ j \leq l : g_2(x_{ij}-\xi e) \geq \varepsilon\}$. Since $x = (x_{ij}) \in \big[\mathcal{O}, u, l^\infty, p, \|\cdot, \ldots, \cdot\|\big]$, then there exists $K > 0$ such that $\Big[u_{kl}O_{kl}\Big(\Big\|\frac{x_{ij}-\xi e}{\rho}, y_1, \ldots, y_{n-1}\Big\|\Big)\Big]^{p_{kl}} \leq K$ for all i, j. Thus,

$$\begin{aligned} g_2(x_{ij}-\xi e) &= \sup_{\substack{r,s\geq 1,\ m,n\geq 1 \\ 0\neq y_1,\ldots,y_{n-1}\in X}}\left(\frac{1}{rs}\sum_{i=m}^{m+r-1}\sum_{j=n}^{n+s-1}\Big[u_{kl}O_{kl}\Big(\Big\|\frac{x_{ij}-\xi e}{\rho}, y_1, \ldots, y_{n-1}\Big\|\Big)\Big]^{p_{kl}}\right) \\ &\leq K. \end{aligned}$$

Hence

$$\begin{aligned} \frac{1}{kl}\sum_{i,j=1,1}^{k,l}(g_2(x_{ij}-\xi e))^p &= \frac{1}{kl}\sum_{\substack{i,j=1,1 \\ i,j\notin K_\varepsilon}}^{k,l}(g_2(x_{ij}-\xi e))^p + \frac{1}{kl}\sum_{\substack{i,j=1,1 \\ i,j\in K_\varepsilon}}^{k,l}(g_2(x_{ij}-\xi e))^p \\ &\leq \varepsilon^p + \frac{K^p}{kl}|K_\varepsilon|. \end{aligned}$$

□

Theorem 1.14. *It holds that $\Big(S^2_{\big(\big[\mathcal{O},u,[\mathcal{F}],p,\|\cdot,\ldots,\cdot\|\big],g_2\big)}\Big)_{reg} = ([C, g_2]_p)_{reg}$.*

Proof. By combining Theorems 1.11 and 1.12, we have the proof. □

Theorem 1.15. *Let $\big(\big[\mathcal{O}, u, [\mathcal{F}], p, \|\cdot, \ldots, \cdot\|\big], g_2\big)$ be a complete paranormed space. Then a sequence in $\big[\mathcal{O}, u, [\mathcal{F}], p, \|\cdot, \ldots, \cdot\|\big]$ is g_2-statistical convergent if and only if it is g_2-statistical Cauchy.*

Proof. Let $g_2(stat) - \lim x = \xi$. Then we get $\delta_2(A(\varepsilon)) = 0$, where

$$X(\varepsilon) = \Big\{(i, j) \in \mathbb{N}\times\mathbb{N} : g_2(x_{ij}-\xi) \geq \frac{\varepsilon}{2}\Big\}.$$

This implies

$$\delta_2(X^c(\varepsilon)) = \delta_2(\{(i, j) \in \mathbb{N}\times\mathbb{N} : g_2(x_{ij}-\xi)) < \varepsilon\}) = 1.$$

Let $k, l \in X^c(\varepsilon)$. Then $g_2(x_{kl}-\xi) < \frac{\varepsilon}{2}$. Now let

$$Y(\varepsilon) = \Big\{(i, j) \in \mathbb{N}\times\mathbb{N} : g_2(x_{kl}-x_{ij}) \geq \varepsilon\Big\}.$$

We need to show that $Y(\varepsilon) \subset X(\varepsilon)$. Let $(i,j) \in Y(\varepsilon)$. Then $g_2(x_{kl} - x_{ij}) \geq \varepsilon$, and therefore $g_2(x_{ij} - \xi) \geq \varepsilon$, that is, $(i,j) \in X(\varepsilon)$. Otherwise, if $g_2(x_{ij} - \xi) < \varepsilon$, then

$$\varepsilon \leq g_2(x_{ij} - x_{kl}) \leq g_2(x_{ij} - \xi) + g_2(x_{kl} - \xi) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which is impossible. Thus, $Y(\varepsilon) \subset X(\varepsilon)$, and therefore, $x = (x_{ij})$ is g_2-statistical convergent sequence.

Conversely, let $x = (x_{ij})$ be g_2-statistical Cauchy but not g_2-statistical convergent sequence. Then there exists $(t', w') \in \mathbb{N} \times \mathbb{N}$ such that

$$\delta_2(D(\varepsilon)) = \delta_2\Big(\Big\{(i,j) \in \mathbb{N} \times \mathbb{N} : g_2(x_{ij} - x_{t'w'}) \geq \varepsilon\Big\}\Big) = 0$$

and $\delta_2(G(\varepsilon)) = 0$, where

$$G(\varepsilon) = \Big\{(i,j) \in \mathbb{N} \times : g_2(x_{ij} - \xi) < \frac{\varepsilon}{2}\Big\},$$

that is, $\delta_2(G^c(\varepsilon)) = 1$, since $g_2(x_{ij} - x_{kl}) \leq 2g_2(x_{ij} - \xi) < \varepsilon$. If $g_2(x_{ij} - \xi) < \frac{\varepsilon}{2}$, then $\delta_2(D^c(\varepsilon)) = 0$, that is, $\delta_2(D(\varepsilon)) = 1$, which leads to a contradiction.

□

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