Khayyam J. Math. 8 (2022), no. 2, 120-127
DOI: 10.22034/KJM.2022.267072.2136

Khayyam Journal of Mathematics
emis.de/journals/KJM
kjm-math.org

ALMOST SEPARABLE SPACES

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Communicated by L. Hola

Abstract. We introduce the notion of almost separable space, which is a generalization of separable space. We show that like separability, almost separability is c-productive and that the converse is true under some restrictions. We discuss about the cardinality of the set of all real valued continuous functions on an almost separable space. Finally, we establish a Baire category like theorem on pseudocompact spaces.

1. Introduction

Let X be any topological space and let $C(X)$ be the set of all real valued continuous functions on X. A subset A of X is called almost dense in X if for any $f \in C(X)$, the condition $f(A) = \{0\}$ implies that $f(X) = \{0\}$. A topological space X is called almost separable if it has a countable almost dense subset. A dense subset is always almost dense. In completely regular spaces, dense and almost dense sets are identical, but the converse is not true. We give an example of a noncompletely regular space in which dense and almost dense sets are same. We show that almost separability is c-productive and that under some restrictions, the converse is also true. Finally, we study relationships among almost separability, sequential separability, and strongly sequential separability.

Definition 1.1 ([1,3,4])**.** A space X is called sequentially separable if there exists a countable subset D of X such that, for every $x \in X$, there exists a sequence from D converging to x.

Date: Received: 12 January 2021; Revised: 8 January 2022; Accepted: 18 February 2022.
*Corresponding author.
2020 *Mathematics Subject Classification.* Primary 39B82; Secondary 44B20.
Key words and phrases. Almost separable space, pseudocompact space, functionally Hausdorff space, sequentially separable space, Strongly sequentially separable space.

Definition 1.2 ([1])**.** A space X is called strongly sequentially separable if it is separable and every dense countable subspace is sequentially dense. A subset D of X is called sequentially dense if for every x in X, there exists a sequence from D converging to x.

2. Almost dense subsets of a topological space

Definition 2.1. A subset A of a topological space X is called almost dense if for any $f \in C(X)$, the condition $f(A) = \{0\}$ implies that $f(X) = \{0\}$.

Definition 2.2. A subset A of a topological space X is called zero set if $A = Z(f)$, for some $f \in C(X)$, where $Z(f) = \{x \in X : f(x) = 0\}$. The complement of a zero set is called cozero set.

Theorem 2.3. *Dense subsets are always almost dense.*

Proof. Let A be a dense subset of a topological space (X, τ). We show that A is an almost dense subset of (X, τ). Let $f \in C(X)$ with $f(A) = \{0\}$. Since $f \in C(X)$, then $f(\overline{A}) \subseteq \overline{f(A)}$. Therefore $f(\overline{A}) = f(X) \subseteq \overline{f(A)} = \overline{\{0\}} = \{0\}$. Thus, $f(X) = \{0\}$. Hence A is an almost dense subset of X. $\square$

Theorem 2.4. *In a completely regular space X, almost dense subsets of X are dense in X.*

Proof. Let A be an almost dense subset of X. If possible, let A be not dense in X and let $x_\circ \in X \setminus \overline{A}$. Then there exists $f \in C(X)$ such that $f(x_\circ) = 1$ and $f(\overline{A}) = 0$, by the complete regularity property of X. Thus $f(A) = \{0\}$, but $f(X) \neq \{0\}$. This shows that A is not almost dense in X, which contradicts with our assumption. Hence A is dense in X. $\square$

We cite an example of a noncompletely regular space where every almost dense set is dense. To show this, we need the following theorem.

Theorem 2.5. *Let Y be a dense subset of a topological space X (may or may not be a completely regular space). If Y has a base consisting of cozero subsets of X for open sets, then every almost dense set in X is dense in X.*

Proof. Suppose that A is an almost dense subset of X. We show that A is dense in X. Let U be a nonempty open set in X. Then $U \cap Y \neq \emptyset$ as Y is dense in X. If $A \cap U = \emptyset$, then $A \cap U \cap Y = \emptyset$. Let $y \in U \cap Y$. Then there exists $f \in C(X)$ such that $y \in X \setminus Z(f) \subseteq U \cap Y$. Consequently, $f(A) = \{0\}$, but $f(X) \neq \{0\}$, which is a contradiction, as A is almost dense in X. Hence $A \cap U \neq \emptyset$. $\square$

Now we give an example of a noncompletely regular space where almost dense subsets are dense in X.

Example 2.6. Let $K = \{\frac{1}{n} : n \in \mathbb{N}\}$ and let $\beta = \{(a, b) : a < b, a, b \in \mathbb{R}\} \cup \{(a, b) \setminus K : a < b, a, b \in \mathbb{R}\}$ be a base for the K-topology τ_K on $\mathbb{R}$. Then $(\mathbb{R}, \tau_K)$ is not regular and hence not completely regular. We would like to show that an almost dense set in $(\mathbb{R}, \tau_K)$ is dense in $(\mathbb{R}, \tau_K)$. Let $Y = \mathbb{R} \setminus \{0\}$. The set Y is open, dense in $(\mathbb{R}, \tau_K)$, and the relative topology of Y is the usual topology of Y, which is completely regular and as a result, possess a base consisting of cozero sets of $(\mathbb{R}, \tau_K)$. We invoke Theorem 2.5 to conclude that almost dense sets in $(\mathbb{R}, \tau_K)$ are dense in $(\mathbb{R}, \tau_K)$.

In a normal space X, an almost dense subset of X may not be dense. This can be shown in the following example.

Example 2.7. Let $X = \{a, b\}$ and let $\tau = \{\phi, X, \{a\}\}$ be a topology on X. Then X is a normal space. Moreover, $\{b\}$ is an almost dense set in X but not dense in X.

Theorem 2.8. *Let $f : X \longrightarrow Y$ be a continuous, surjective function. If A is an almost dense subset of X, then $f(A)$ is an almost dense subset of Y.*

Proof. Let $f : X \longrightarrow Y$ be a continuous surjective function and let A be an almost dense subset of X. To show that $f(A)$ is an almost dense subset of Y, let $g \in C(Y)$ such that $g(f(A)) = \{0\}$. Then $g \circ f \in C(X)$ such that $g \circ f(A) = \{0\}$. As A is almost dense in X, then we have $g \circ f(X) = \{0\}$. Since f is onto, we have $g(Y) = \{0\}$. Hence $f(A)$ is almost dense in Y. $\square$

Theorem 2.9. *Let τ_1 and τ_2 be two topologies on X such that τ_2 is finer than τ_1. Then if a subset A of X is almost dense in (X, τ_2), then A is almost dense in (X, τ_1).*

Proof. We denote by $C(X, \tau_i)$ the set of all real valued continuous functions on X with respect to the topology τ_i for $i = 1, 2$. Let A be almost dense in (X, τ_2). Let $f \in C(X, \tau_1)$ with $f(A) = \{0\}$. Since $f \in C(X, \tau_1)$ and τ_2 is finer than τ_1, we have $f \in C(X, \tau_2)$. Now A is almost dense in (X, τ_2) and $f \in C(X, \tau_2)$ with $f(A) = 0$. Thus $f(X) = \{0\}$. Hence A is almost dense in (X, τ_1). $\square$

Theorem 2.10. *If A is almost dense in X and B is almost dense in Y, then $A \times B$ is almost dense in $X \times Y$.*

Proof. Let $f : X \times Y \mapsto \mathbb{R}$ be continuous such that $f = 0$ on $A \times B$. Fix $a \in A$, and look at $f_a(y) = f(a, y), y \in Y$. Hence $f_a : Y \mapsto \mathbb{R}$ is continuous and $f_a(y) = 0$ for all $y \in B$. The hypothesis implies $f_a(y) = 0$ for all $y \in Y$. Since a is arbitrary, we obtain $f(A \times Y) = \{0\}$. Take any $(x, y) \in X \times Y$. Then $f_y : X \to \mathbb{R}$ defined by $f_y(z) = f(z, y), z \in X$, is a continuous map. Now $f_y(a) = 0$ for all $a \in A$. Then $f_y(x) = 0$ for all $x \in X$, that is, $f(x, y) = 0$. Since $y \in Y$ is arbitrary, this shows $f = 0$ on $X \times Y$. Hence $A \times B$ is almost dense in $X \times Y$. $\square$

Corollary 2.11. *Let $A_1, A_2, \ldots, A_n$ be almost dense in $X_1, X_2, \ldots, X_n$, respectively. Then $\prod_{i=1}^n A_i$ is almost dense in $\prod_{i=1}^n X_i$.*

Proposition 2.12. *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of topological spaces and let $A_\alpha \subset X_\alpha, \alpha \in \Lambda$, be almost dense in X_α. Then $\prod_{\alpha \in \Lambda} A_\alpha$ is almost dense in $\prod_{\alpha \in \Lambda} X_\alpha$.*

Proof. For $a = (a_\alpha)_{\alpha \in \Lambda} \in \prod_{\alpha \in \Lambda} A_\alpha$, define

$$D = \{(x_\alpha)_{\alpha \in \Lambda} \in \prod_{\alpha \in \Lambda} X_\alpha : \{\alpha : x_\alpha \neq a_\alpha\} \text{is finite}\}.$$

Let $f : \prod_{\alpha\in\Lambda} X_\alpha \mapsto \mathbb{R}$ be continuous such that $f = 0$ on $\prod_{\alpha\in\Lambda} A_\alpha$. For any finite subset $I \subset \Lambda$, $\prod_{\alpha\in I} X_\alpha \times \{a_\alpha : \alpha \in \Lambda \setminus I\}$ is homeomorphic to $\prod_{\alpha\in I} X_\alpha$ and $\prod_{\alpha\in I} A_\alpha$ is almost dense in $\prod_{\alpha\in I} X_\alpha$ (by Corollary 2.11). Hence $\prod_{\alpha\in I} A_\alpha \times \{a_\alpha : \alpha \in \Lambda \setminus I\}$ is almost dense in $\prod_{\alpha\in I} X_\alpha \times \{a_\alpha : \alpha \in \Lambda \setminus I\}$. Consequently $f(\prod_{\alpha\in\Lambda} X_\alpha \times \{a_\alpha : \alpha \in \Lambda \setminus I\}) = \{0\}$. Now $D = \cup[\prod_{\alpha\in I} X_\alpha \times \{\{a_\alpha : \alpha \in \Lambda \setminus I\} : I \subseteq \Lambda \text{ finite } \}]$. This implies that $f(D) = \{0\}$. □

We now invoke the following important result on product spaces.

Theorem 2.13. *Let $\{Y_\alpha : \alpha \in \Lambda\}$ be a family of topological spaces and let $a = (x_\alpha)_{\alpha\in\Lambda}$ be a fixed element of $\prod_{\alpha\in\Lambda} Y_\alpha$. The set $E = \{(y_\alpha)_{\alpha\in\Lambda} \in \prod_{\alpha\in\Lambda} Y_\alpha : \{\alpha \in \Lambda : y_\alpha \neq x_\alpha\}$ is finite$\}$ is dense in $\prod_{\alpha\in\Lambda} Y_\alpha$.*

Because of this theorem, D is dense in $\prod_{\alpha\in\Lambda} X_\alpha$ and $f(D) = 0$, which imply $f = 0$ on $\prod_{\alpha\in\Lambda} X_\alpha$. As a result, $\prod_{\alpha\in\Lambda} A_\alpha$ is almost dense in $\prod_{\alpha\in\Lambda} X_\alpha$.

Theorem 2.14. *If a topological space X contains a connected almost dense subset, then X is connected.*

Proof. Let A be a connected almost dense subset of X. If possible, let X be disconnected. Then there exists a continuous, onto function $f : X \mapsto \{0, 1\}$. Let $Y = \{x \in X : f(x) = 0\}$. Then $X \setminus Y = f^{-1}\{1\}$. Since A is connected, then either $A \subseteq Y$ or $A \subseteq X \setminus Y$.

Case 1: If $A \subseteq Y$, then $f(A) = \{0\}$, but $f(X) \neq \{0\}$, which contradicts with the fact that A is an almost dense set in X.

Case 2: If $A \subseteq X \setminus Y$, then considering $g = 1 - f$ implies $g(A) = \{0\}$ but $g(X) \neq \{0\}$, which is a contradiction as A is an almost dense set in X. Hence X is connected. □

Of course the converse of Theorem 2.14 is not true that is possible for a connected space X to have a disconnected almost dense subset. As an illustration, consider the following example:

Let $X = \mathbb{R}$ with the topology consisting of all subsets of $\mathbb{R}$ containing 0. Then X is obviously connected. The subspace $A = \{1, 2\}$ is almost dense in X but not connected.

Theorem 2.15. *A subset A of X is almost dense in X if and only if every nonempty cozero set intersects A.*

Proof. Let A be an almost dense subset of X and let $X \setminus Z(f)$ be a nonempty cozero set in X. Let us assume that $A \cap (X \setminus Z(f)) = \emptyset$. Then $f(A) = \{0\}$. Since A is almost dense in X, then $f(X) = \{0\}$, which contradicts with the fact that $X \setminus Z(f)$ is nonempty. Hence $A \cap (X \setminus Z(f)) \neq \emptyset$.

Conversely, let A intersect every nonempty cozero set of X. Suppose that A is not an almost dense subset of X. Then there exists a continuous function $f : X \mapsto \mathbb{R}$ such that $f(A) = \{0\}$ but $f(X) \neq \{0\}$. Then $X \setminus Z(f)$ is a nonempty cozero set and A does not intersect the nonempty cozero set $X \setminus Z(f)$. □

As is well known, in a topological space, a nonempty subset is dense if and only if it intersects every nonempty open set. Theorem 2.15 presents the analogous result in the case of almost denseness property. We have seen that in a topological space X, if $A \subseteq X$ is dense and U is a nonempty open set, then not only $A \cap U \neq \emptyset$, but also $A \cap U$ is dense in U. Now the following question arises:

Let A be almost dense in a topological space X and let $\emptyset \neq U \subset X$ be a cozero set. Then $A \cap U \neq \emptyset$ is no doubt, but is $A \cap U$ almost dense in U? The answer does not seem to be clear!

3. Almost separable spaces

Definition 3.1. A space X is called almost separable if it contains a countable almost dense subset.

Theorem 3.2. *Each separable space is almost separable.*

The converse of the above theorem is not true.

Example 3.3. Let $X = \mathbb{R}$ and let τ_c be the cocountable topology on X. For any countable set A in X, $X \setminus A$ is open and $A \cap (X \setminus A) = \emptyset$. Therefore A is not dense in X. Therefore X is not separable.

We want to show that $\mathbb{Q}$ is almost dense in X. Let $f \in C(X)$ and let $f(\mathbb{Q}) = \{0\}$. Since every real valued continuous function on X is constant, then $f(X) = \{0\}$. Therefore $\mathbb{Q}$ is almost dense in X. Therefore X contains a countable almost dense subset. Hence X is an almost separable space. In fact, any countable subset of X is almost dense in X.

Theorem 3.4. *Finite product of almost separable spaces is almost separable.*

Proof. It follows from Corollary 2.11. □

Theorem 3.5. *Let Y be almost dense subset of X and let Y be almost separable as a subspace. Then X is almost separable.*

Proof. Let A be a countable almost dense subset of Y. Let $f \in C(X)$ such that $f(A) = \{0\}$. Then $f|_Y \in C(Y)$ and $f|_Y(A) = \{0\}$. These imply that $f|_Y(Y) = \{0\}$ as A is almost dense in Y. Since Y is almost dense in X, then $f(X) = \{0\}$. Therefore A is a countable almost dense subset of X. Hence X is an almost separable space. □

In the case of infinite products, like the case of separable spaces, the following result is true.

Theorem 3.6. *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of almost separable topological spaces, in which card$(\Lambda) = c$. Then $\prod_{\alpha\in\Lambda} X_\alpha$ with the product topology is almost separable space.*

Proof. Let $A_\alpha \subseteq X_\alpha$ be a countable almost dense subset of X_α. To avoid triviality, we assume A_α to be countably infinite. Let $f_\alpha : \mathbb{N} \to A_\alpha$ be a bijection. Define $\prod_{\alpha\in\Lambda} f_\alpha : \mathbb{N}^\Lambda \to \prod_{\alpha\in\Lambda} A_\alpha$ as follows: $\prod_{\alpha\in\Lambda} f_\alpha(n_\alpha, \alpha \in \Lambda) = (f_\alpha(n_\alpha), \alpha \in \Lambda)$, where $(n_\alpha, \alpha \in \Lambda) \in \mathbb{N}^\Lambda$. We write f^Λ for $\prod_{\alpha\in\Lambda} f_\alpha$. We know $\mathbb{N}^\Lambda = \{g : \Lambda \to \mathbb{N}\}$ and $f^\Lambda(g) = (f_\alpha(g(\alpha)) : \alpha \in \Lambda) \in \prod_{\alpha\in\Lambda} A_\alpha$. It is easy to see that f^Λ is onto. Let $p_\alpha : \prod_{\alpha\in\Lambda} A_\alpha \to A_\alpha$ be the projection to the αth coordinate. Then $p_\alpha \circ f^\Lambda(g) = f_\alpha \circ g(\alpha), \alpha \in \Lambda$. This shows that $f^\Lambda : \mathbb{N}^\Lambda \to \prod_{\alpha\in\Lambda} A_\alpha$ is continuous and onto. It is well known that $\mathbb{N}^\Lambda$ is separable, and hence almost separable. Therefore $\prod_{\alpha\in\Lambda} A_\alpha$ is almost separable. Since $\prod_{\alpha\in\Lambda} A_\alpha$ is almost dense in $\prod_{\alpha\in\Lambda} X_\alpha$, $\prod_{\alpha\in\Lambda} X_\alpha$ is almost separable. $\square$

Definition 3.7. A topological space X is called functionally Hausdorff if, for any two distinct points $a, b \in X$, there exists $f \in C(X)$ such that $f(a) = 0$ and $f(b) = 1$.

Lemma 3.8. *If X a functionally Hausdorff space, then for any two distinct points $a, b \in X$, there exist two distinct cozero sets C and D in X such that $a \in C$ and $b \in D$.*

Proof. Since X is functionally Hausdorff, for $a, b \in X$ with $a \neq b$, there exists $f \in C(X)$ such that $f(a) = 0$ and $f(b) = 1$. Let $C = \{x \in X : f(x) < \frac{1}{2}\} = ((f - \frac{1}{2}) \wedge \underline{0})^{-1}(\mathbb{R} \setminus \{0\})$ and $D = \{x \in X : f(x) > \frac{1}{2}\} = ((f - \frac{1}{2}) \vee \underline{0})^{-1}(\mathbb{R} \setminus \{0\})$, where $(g \vee h)(x) = \max\{g(x), h(x)\}$ and $(g \wedge h)(x) = \min\{g(x), h(x)\}$ for all $x \in X$. Thus C and D are disjoint cozero sets in X, $a \in C$, and $b \in D$. $\square$

Theorem 3.9. *Let $X = \prod_{\alpha\in\Lambda} X_\alpha$ be an almost separable space, where each X_α is functionally Hausdorff and contains at least two points. Then each X_α is almost separable and $card(\Lambda) \leq c$.*

Proof. Let D be a countable, almost dense subset of X. Consider the projection function $p_\alpha : X \mapsto X_\alpha$ to the αth coordinate. Then the function $p_\alpha : X \mapsto X_\alpha$ is continuous. Since the continuous image of an almost separable space is almost separable, each X_α is almost separable.

For every $\alpha \in \Lambda$, let $a_\alpha, b_\alpha \in X_\alpha$ with $a_\alpha \neq b_\alpha$. Since each X_α is functionally Hausdorff, there exist disjoint cozero sets C_α and D_α in X_α such that $a_\alpha \in C_\alpha$ and $b_\alpha \in D_\alpha$. Moreover, $p_\alpha^{-1}(C_\alpha)$ is a nonempty cozero set in X. Theorem 2.15 implies that $K_\alpha = D \cap p_\alpha^{-1}(C_\alpha) \neq \emptyset$.

Define the function $\phi : \Lambda \to \mathcal{P}(D)$ by $\phi(\alpha) = K_\alpha$. Now $p_\alpha^{-1}(C_\alpha) \cap p_\alpha^{-1}(D_\beta)$ is a nonempty cozero set in X. Then there exists $x \in D \cap p_\alpha^{-1}(C_\alpha) \cap p_\alpha^{-1}(D_\beta)$ by Theorem 2.15. Hence, $x \in K_\alpha$ and $x \notin K_\beta$. Therefore $K_\alpha \neq K_\beta$. Thus ϕ is injective. Hence $card(\Lambda) \leq card(\mathcal{P}(D)) = c$. $\square$

Theorem 3.10. *For an almost separable space X, the cardinality of $C(X)$ is less than or equal to c.*

Proof. Let A be a countable almost dense subset of X. Define a map $\phi : C(X) \to C(A)$ by $\phi(f) = f|_A$. We show that ϕ is an injective mapping. Let $\phi(f) = \phi(g)$, where $f, g \in C(X)$. Then $f|_A = g|_A$. Let $h = f - g$. Then $h \in C(X)$ and $h(A) = \{0\}$. Since A is almost dense in X, then $h(X) = \{0\}$. Therefore ϕ is injective. Since the cardinality of $C(A)$ is less that or equal to c, the cardinality of $C(X)$ is less than or equal to c. $\square$

Corollary 3.11. *If an almost separable space has an uncountable closed discrete subspace, then it is not normal.*

Proof. Let A be an uncountable closed discrete subset of an almost separable space X. Then any real valued function on A is continuous as A is discrete. Thus the cardinality of $C(A)$ is greater than or equal to 2^c. If X is normal, then by the Tietze's extension theorem, any real valued continuous function on A can be extended to a member of $C(X)$ as A is a closed subset of X. Then $card(C(A)) \geq 2^c$ implies that $card(C(X)) \geq 2^c$, which contradicts with Theorem 3.10. Hence X is not normal. □

Theorem 3.12. *Let X be a functionally Hausdorff, almost separable space. Then the cardinality of X is at most 2^c.*

Proof. Consider the function $\psi : X \to \mathcal{P}(C(X))$ by $\psi(x) = \{f \in C(X) : f(x) = 0\}$. Using the functionally Hausdorff property of X, we have to show that ψ is injective. By Theorem 3.10, the cardinality of $C(X)$ is c. Thus cardinality of X is at most 2^c. □

4. Baire category like theorem

A topological space X is called pseudocompact if every real valued continuous function on X is bounded. The concept of pseudocompact space was introduced in [2]. The following result was established in [2, Theorem 34] assuming the complete regularity of X. If we follow the steps of the proof of [2, Theorem 34], then we can observe that the result is true for an arbitrary topological space.

Theorem 4.1. *For a topological space X, the following conditions are equivalent:*
(i) X is pseudocompact.
(ii) If $\{F_n : n \in \mathbb{N}\}$ is a sequence of zero sets of X with finite intersection property, then $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.
(iii) If $\{U_n : n \in \mathbb{N}\}$ is a countable cover of X consisting of cozero sets, then there exists a finite subcover.

Theorem 4.2. *Let X be a topological space. Given a nonempty cozero set U and $x \in U$, there exist a cozero set V and a zero set F such that $x \in V \subseteq F \subset U$.*

Proof. Let $f : X \to [0,1]$ such that $U = f^{-1}(0,1] = X \setminus f^{-1}(\{0\})$. Since $x \in U, f(x) > 0$. Choose $\delta > 0$ such that $0 < f(x) - \delta < f(x)$. Let $V = f^{-1}(f(x) - \delta, 1])$ and let $F = f^{-1}[f(x)-\delta, 1]$. Then $x \in V \subseteq F \subseteq U$. Now $(f(x)-\delta, 1] \subseteq [0,1]$ is an open set and hence a cozero set $[f(x) - \delta, 1]$ is a closed subset of $[0,1]$ and hence a zero set. Then $V = f^{-1}(f(x)-\delta, 1]$ is a cozero set and $F = f^{-1}[f(x)-\delta, 1]$ is a zero set of X. □

Theorem 4.3 (Baire category like theorem)**.** *Let X be a pseudocompact space. If $\{U_n : n \in \mathbb{N}\}$ is a sequence of almost dense cozero sets of X, then $\bigcap_{n=1}^{\infty} U_n$ is a nonempty almost dense subset of X.*

Proof. Write $D = \bigcap_{n=1}^{\infty} U_n$. We show that $D \neq \emptyset$ and that D intersects every nonempty cozero set. Let V be a nonempty cozero set and let $x \in V$. Since U_1 is almost dense, $V \cap U_1 \neq \emptyset$ and is a cozero set. Let $x_1 \in V \cap U_1$. By Theorem 4.2, there exist cozero sets V_1 and F_1 such that $x_1 \in V_1 \subseteq F_1 \subset V \cap U_1$. Now the fact that $V_1 \neq \emptyset$ is a cozero set implies that $V_1 \cap U_2 \neq \emptyset$ and is a zero set. Let $x_2 \in V_1 \cap U_2$. Then there exist a nonempty cozero set V_2 and a zero set F_2 such that $x_2 \in V_2 \subseteq F_2 \subset V_1 \cap U_2 \subseteq V \cap U_1 \cap U_2$. Now $V_2 \neq \emptyset$ is a cozero set and $V_2 \cap U_3 \neq \emptyset$ and is a cozero set. Let $x_3 \in V_2 \cap U_3$. There exist a cozero set

V_3 and a zero set F_3 such that $x_3 \in V_3 \subseteq F_3 \subseteq V_2 \cap U_3 \subseteq V \cap U_1 \cap U_2 \cap U_3$. Proceeding in this way, we obtain a nonempty cozero set V_{n+1} and a zero set F_{n+1} such that $x_{n+1} \in V_{n+1} \subseteq F_{n+1} \subseteq V_n \cap U_{n+1} \subseteq V \cap U_1 \cap U_2 \cap U_3 \cap \cdots \cap U_{n+1}$ for all $n \geq 0$. Note that $F_{n+1} \subseteq F_n$ and F_n's are nonempty zero sets. Since X is pseudocompact, in virtue of Theorem 4.1, $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$. Hence $\bigcap_{n=1}^{\infty} F_n \subseteq \bigcap_{n=1}^{\infty}(V \cap U_1 \cap U_2 \cap U_3 \cap \cdots \cap U_n) = V \cap \bigcap_{n=1}^{\infty} U_n$, so $V \cap (\bigcap_{n=1}^{\infty} U_n) \neq \emptyset$. That is, $V \cap D \neq \emptyset$. Since V is an arbitrary nonempty cozero set, D is almost dense. $\square$

We know that separability is not a hereditary property. Niemytzky's plane is a well-known example. The same is true about almost separability as well. Niemytzky's plane provides an example in this case also.

We now finish our paper giving relations among the different types of separability notions. From [1, Section 1.2], we get the following relation:

Strong sequentially separable ⇒ Sequentially separable ⇒ Separable.

In this paper, Theorem 3.2 gives that separable implies almost separable. Combining these two, we get the following relations:

Strong sequentially separable ⇒ Sequentially separable ⇒ Separable ⇒ Almost separable.

Acknowledgement. The authors would like to thank to Professor Alan Dow for Theorem 2.4. Also the authors wish to thank Professor Asit Baran Raha and learned referee's for their valuable suggestions that improved the article.

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