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OSCILLATIONS OF HIGHER-ORDER IMPULSIVE PARTIAL DIFFERENTIAL EQUATIONS WITH DISTRIBUTED DELAY

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Abstract. We consider a class of boundary value problems associated with even order nonlinear impulsive neutral partial functional differential equations with continuous distributed deviating arguments and damping term. Necessary and sufficient conditions are obtained for the oscillation of all solutions using impulsive differential inequalities and integral averaging scheme with the Robin boundary condition. Examples illustrating the results are also given.

1. Introduction and preliminaries

The theory of impulsive differential equations is much richer than the corresponding theory of differential equations without impulse effects. The theory of impulsive differential equations marks its beginning in [15] by Mil'man and Myshkis. The first investigation on the oscillation theory of impulsive differential equations was published in 1989 [2]. The first paper on impulsive partial differential equations, [1], was published in 1991.

The oscillation of impulsive and nonimpulsive partial differential equations has been extensively studied in the literature; we refer the readers to the papers [5, 11, 17–20] and the references therein cited. We also refer to the papers [9, 10] for oscillatory and/or nonoscillatory solutions to models from mathematical biology and physics formulated by partial differential equations such that their long time behavior is connected to the external source, idealized by nonlocal and/or taxis-driven terms. Consequently, it is required to study with

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impulse effect on higher-order partial differential equations. In the monographs, Wu [22] and Yoshida [24] provided several fundamental theories and applications of partial functional differential equations to population ecology, generic repression, climate models, viscoelastic materials, control problems, coupled oscillators, beam equations, and structured population models. There is a strong interest in these mathematical models for formulating this higher-order problem. In this effort, we begin oscillation criteria for even order impulsive neutral partial differential equations that are not formally studied. Thus the main results of this paper are the generalization of the results studied in [3,14] with additional force components along the system such as impulse and distributed delay. Distributed delay is a broad case of constant delay, which can be found in the monographs [7,8].

Consider the higher-order impulsive neutral delay partial differential equations with distributed delay of the form

$$\left.\begin{array}{l} \dfrac{\partial^m}{\partial t^m}\left(u(x,t)+c(t)u(x,\tau(t))\right)+\int_a^b q(t,\xi)u(x,\sigma(t,\xi))d\eta(\xi) \\ \quad = a(t)\Delta u(x,t)+\int_a^b b(t,\xi)\Delta u(x,\rho(t,\xi))d\eta(\xi), \qquad t\neq t_k,\,(x,t)\in\Omega\times[0,+\infty)\equiv G, \\ \dfrac{\partial^{(i)}u(x,t_k^+)}{\partial t^{(i)}} = I_k^{(i)}\left(x,t_k,\dfrac{\partial^{(i)}u(x,t_k)}{\partial t^{(i)}}\right), \quad k=1,2,\ldots,\; i=0,1,2,\ldots,m-1, \end{array}\right\} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a piecewise smooth boundary $\partial\Omega$ and Δ is the Laplacian in the Euclidean space $\mathbb{R}^N$.

Equation (1.1) is supplemented by the following Robin boundary condition:

$$\alpha(x)\frac{\partial u(x,t)}{\partial\gamma}+\beta(x)u(x,t) = 0, \qquad (x,t)\in\partial\Omega\times[0,+\infty), \tag{1.2}$$

where γ is the outer surface normal vector to $\partial\Omega$ and $\alpha,\beta\in C\left(\partial\Omega,[0,+\infty)\right)$, $\alpha^2(x)+\beta^2(x)\neq 0$.

In what follows, we assume that the following hypotheses hold:

- (H_1) $c(t)\in C^m([0,+\infty),[0,+\infty))$, $a(t)\in PC([0,+\infty),[0,+\infty))$, where PC denotes the class of functions, which are piecewise continuous in t with discontinuities of first kind only at $t=t_k$, $k=1,2,\ldots$, and left continuous at $t=t_k$, $k=1,2,\ldots$, $\tau(t)\in C([0,+\infty),\mathbb{R})$, $\lim_{t\to+\infty}\tau(t)=+\infty$, $\tau(t)\leq t$, $q(t,\xi)\in C([0,+\infty)\times[a,b],[0,+\infty))$.
- (H_2) $b(t,\xi)\in C([0,+\infty)\times[a,b],[0,+\infty))$, $\sigma(t,\xi),\rho(t,\xi)\in C([0,+\infty)\times[a,b],\mathbb{R})$, $\rho(t,\xi)\leq t$, $\sigma(t,\xi)\leq t$ for $\xi\in[a,b]$, $\sigma(t,\xi)$ and $\rho(t,\xi)$ are nondecreasing with respect to t and ξ, respectively, and $\liminf_{t\to+\infty,\ \xi\in[a,b]}\sigma(t,\xi)=\liminf_{t\to+\infty,\ \xi\in[a,b]}\rho(t,\xi)=+\infty$.
- (H_3) There exists a function $\theta(t)\in C([0,+\infty),[0,+\infty))$ satisfying $\theta(t)\leq\sigma(t,a)$, $\theta'(t)>0$ and $\lim_{t\to+\infty}\theta(t)=+\infty$, $\eta(\xi):[a,b]\to\mathbb{R}$ is nondecreasing, and the integral is a Stieltjes integral in (1.1).

(H$_4$) $\dfrac{\partial^{(i)}u(x,t)}{\partial t^{(i)}}$ are piecewise continuous in t with discontinuities of the first kind only at $t = t_k$, $k = 1, 2, \ldots$, and left continuous at $t = t_k$, $\dfrac{\partial^{(i)}u(x,t_k)}{\partial t^{(i)}} = \dfrac{\partial^{(i)}u(x,t_k^-)}{\partial t^{(i)}}$, $k = 1, 2, \ldots$, $i = 0, 1, 2, \ldots, m-1$.

(H$_5$) $I_k^{(i)}\left(x, t_k, \dfrac{\partial^{(i)}u(x,t_k)}{\partial t^{(i)}}\right) \in PC(\bar{\Omega} \times [0, +\infty) \times \mathbb{R}, \mathbb{R})$, $k = 1, 2, \ldots$, $i = 0, 1, 2, \ldots, m-1$, and there exist positive constants $a_k^{(i)}$ and $b_k^{(i)}$ such that

$$a_k^{(i)} \leq \frac{I_k^{(i)}\left(x, t_k, \dfrac{\partial^{(i)}u(x,t_k)}{\partial t^{(i)}}\right)}{\dfrac{\partial^{(i)}u(x,t_k)}{\partial t^{(i)}}} \leq b_k^{(i)},$$

for $i = 0, 1, 2, \ldots, m-1$, $k = 1, 2, \ldots$.

Definition 1.1 ([24])**.** A solution u of (1.1) is a function $u \in C^m(\bar{\Omega} \times [t_{-1}, +\infty), \mathbb{R}) \cap C(\bar{\Omega} \times [\hat{t}_{-1}, +\infty), \mathbb{R})$ that satisfies (1.1), where

$$t_{-1} := \min\left\{0, \min_{\xi \in [a,b]}\left\{\inf_{t \geq 0} \rho(t,\xi)\right\}\right\} \quad \text{and}$$
$$\hat{t}_{-1} := \min\left\{0, \inf_{t \geq 0} \tau(t), \min_{\xi \in [a,b]}\left\{\inf_{t \geq 0} \sigma(t,\xi)\right\}\right\}.$$

Definition 1.2. The solution u of problem (1.1) with boundary condition (1.2) is said to be *oscillatory* in the domain G if for any positive number ℓ, there exists a point $(x_0, t_0) \in \Omega \times [\ell, +\infty)$ such that $u(x_0, t_0) = 0$.

Definition 1.3. A function $V(t)$ is said to be eventually positive (negative) if there exists $t_1 \geq t_0$ such that $V(t) > 0$ $(V(t) < 0)$ for all $t \geq t_1$.

Lemma 1.4 ([23])**.** *Assume that $\lambda_0 > 0$ is the smallest eigenvalue of the problem*

$$\left.\begin{array}{rclcl} \Delta\omega(x) + \lambda\omega(x) & = & 0 & \text{in} & \Omega \\ \alpha(x)\dfrac{\partial\omega(x)}{\partial\gamma} + \beta(x)\omega(x) & = & 0 & \text{on} & \partial\Omega \end{array}\right\} \tag{1.3}$$

and $\Phi(x) > 0$ is the corresponding eigenfunction of λ_0. Then $\lambda_0 = 0$, $\Phi(x) = 1$ as $\beta = 0$ $(x \in \Omega)$ and $\lambda_0 > 0$, $\Phi(x) > 0$ $(x \in \Omega)$ as $\beta(x) \not\equiv 0$ $(x \in \partial\Omega)$.

Lemma 1.5 ([6])**.** *Let $y(t)$ be a positive and n times differentiable function on $[0, +\infty)$. If $y^{(n)}(t)$ has constant sign and not identically zero on any ray $[t_1, +\infty)$ for $t_1 > 0$, then there exist $t_y \geq t_1$ and an integer l $(0 \leq l \leq n)$, with $n + l$ even for $y(t)y^{(n)}(t) \geq 0$ or $n + l$ odd for $y(t)y^{(n)}(t) \leq 0$ and for $t \geq t_y$, $y(t)y^{(k)}(t) > 0$, $0 \leq k \leq l$; $(-1)^{k-l}y(t)y^{(k)}(t) > 0$, $l \leq k \leq n$.*

Lemma 1.6 ([16])**.** *Suppose that the conditions of Lemma 1.5 are satisfied and that $y^{(n-1)}(t)y^{(n)}(t) \leq 0$, $t \geq t_y$. Then there exist constants $\mu \in (0,1)$ and $M > 0$ such that for sufficiently large t, $|y'(\mu t)| \geq Mt^{n-2}\left|y^{(n-1)}(t)\right|$.*

Lemma 1.7 ([4])**.** *If X and Y are nonnegative, then*

$$X^{\mu} - \mu X Y^{\mu-1} + (\mu - 1)Y^{\mu} \geq 0, \quad \mu > 1,$$
$$X^{\mu} - \mu X Y^{\mu-1} - (1 - \mu)Y^{\mu} \leq 0, \quad 0 < \mu < 1,$$

where the equality holds if and only if $X = Y$.

For each positive solution u of problem (1.1) with boundary condition (1.2), we combine the functions $V(t)$, $A(t)$, and $B(t)$ defined by

$$V(t) = \int_{\Omega} u(x,t)\Phi(x)dx, \qquad A(t) = g_0 \int_a^b q(t,\xi)d\eta(\xi), \quad \text{and}$$
$$B(t) = M(\theta(t))^{m-2}\theta'(t),$$

respectively, where $g_0 = 1 - c(\sigma(t,\xi))$.

This work is planned as follows: In Section 2, we discuss the oscillation of problem (1.1) with boundary condition (1.2). In Section 3, we present two examples to illustrate the main results.

2. Main results

In this section, we establish the oscillation criteria of problem (1.1) with boundary condition (1.2). Lemma 1.4 is very useful for establishing our main results.

Theorem 2.1. *Assume that $\beta(x) \not\equiv 0$ for $x \in \partial\Omega$. All solutions of (1.1) with boundary condition (1.2) are oscillatory if and only if all solutions of the equation*

$$\left.\begin{array}{l} [V(t) + c(t)V(\tau(t))]^{(m)} + \int_a^b q(t,\xi)V(\sigma(t,\xi))d\eta(\xi) \\ \quad +\lambda_0 a(t)V(t) + \lambda_0 \int_a^b b(t,\xi)V(\rho(t,\xi))d\eta(\xi) = 0, \quad t \neq t_k, \\ a_k^{(i)} \leq \dfrac{\dfrac{\partial^{(i)}V(t_k^+)}{\partial t^{(i)}}}{\dfrac{\partial^{(i)}V(t_k)}{\partial t^{(i)}}} \leq b_k^{(i)}, \quad k = 1,2,\ldots, \quad i = 0,1,2,\ldots,m-1, \end{array}\right\} \tag{2.1}$$

are oscillatory, where λ_0 is the smallest eigenvalue of (1.3).

Proof. **(i) Sufficiency:** Assume, for the sake of contradiction, that there is a nonoscillatory solution u of (1.1) with boundary condition (1.2), which has no zero in $\Omega \times [t_0, +\infty)$ for some $t_0 \geq 0$. Without loss of generality, we assume that $u(x,t) > 0$, where $(x,t) \in \Omega \times [t_0, +\infty)$and $t_0 \geq 0$. Because of conditions (H_2) and (H_3), there exists $t_1 > t_0 > 0$ such that $\tau(t) \geq t_0$, $\sigma(t,\xi) \geq t_0$, and $\rho(t,\xi) \geq t_0$ for $(t,\xi) \in [t_1, +\infty) \times [a,b]$. Then $u(x,\tau(t)) > 0$ for $(x,t) \in \Omega \times [t_1, +\infty)$, $u(x,\sigma(t,\xi)) > 0$ for $(x,t,\xi) \in \Omega \times [t_1, +\infty) \times [a,b]$, and $u(x,\rho(t,\xi)) > 0$ for $(x,t,\xi) \in \Omega \times [t_1, +\infty) \times [a,b]$.

For $t \geq t_0,\ t \neq t_k,\ k = 1, 2, \dots,$ multiplying both sides of equation (1.1) with $\Phi(x) > 0$ and integrating with respect to x over the domain Ω, we obtain

$$\left.\begin{aligned} &\frac{d^m}{dt^m}\left(\int_\Omega u(x,t)\Phi(x)dx + \int_\Omega c(t)u(x,\tau(t))\Phi(x)dx\right) \\ &+\int_\Omega \int_a^b q(t,\xi)u(x,\sigma(t,\xi))\Phi(x)d\eta(\xi)dx \\ &\qquad = a(t)\int_\Omega \Delta u(x,t)\Phi(x)dx + \int_\Omega \int_a^b b(t,\xi)\Delta u(x,\rho(t,\xi))\Phi(x)d\eta(\xi)dx. \end{aligned}\right\} \tag{2.2}$$

From Green's formula and boundary condition (1.2), it follows that

$$\begin{aligned} \int_\Omega \Delta u(x,t)\Phi(x)dx = &\int_{\partial\Omega}\left[\Phi(x)\frac{\partial u(x,t)}{\partial\gamma} - u(x,t)\frac{\partial\Phi(x)}{\partial\gamma}\right]dS + \int_\Omega u(x,t)\Delta\Phi(x)dx \\ = &\int_{\partial\Omega}\left[\Phi(x)\frac{\partial u(x,t)}{\partial\gamma} - u(x,t)\frac{\partial\Phi(x)}{\partial\gamma}\right]dS \\ &-\lambda_0\int_\Omega u(x,t)\Phi(x)dx, \quad t \geq t_1, \end{aligned}$$

where dS is the surface element on $\partial\Omega$. If $\alpha(x) \equiv 0,\ x \in \partial\Omega$, then from (1.2), we have $\beta(x) \not\equiv 0$, $u(x,t) = 0$, and $(x,t) \in \partial\Omega \times [0, +\infty)$. Hence,

$$\int_{\partial\Omega}\left(\Phi(x)\frac{\partial u(x,t)}{\partial\gamma} - u(x,t)\frac{\partial\Phi(x)}{\partial\Omega}\right)dS \equiv 0, \quad t \geq t_1, \quad t \neq t_k.$$

If $\alpha(x) \not\equiv 0$, then $x \in \partial\Omega$. Note that $\partial\Omega$ is piecewise smooth, that $\alpha, \beta \in C(\partial\Omega, [0, +\infty))$, and that $\alpha^2(x) + \beta^2(x) \neq 0$. Without loss of generality, we can assume that $\alpha(x) > 0$, $x \in \partial\Omega$. Then by (1.2) and (1.3), we have

$$\begin{aligned} &\int_{\partial\Omega}\left(\Phi(x)\frac{\partial u(x,t)}{\partial\gamma} - u(x,t)\frac{\partial\Phi(x)}{\partial\gamma}\right)dS \\ &= \int_{\partial\Omega}\left(-\Phi(x)\frac{\beta(x)}{\alpha(x)}u(x,t) + \frac{\beta(x)}{\alpha(x)}\Phi(x)u(x,t)\right)dS = 0, \quad t \geq t_1. \end{aligned}$$

Using Lemma 1.4, we obtain

$$\int_\Omega \Delta u(x,t)\Phi(x)dx = -\lambda_0\int_\Omega u(x,t)\Phi(x)dx = -\lambda_0 V(t), \quad t \geq t_1, \tag{2.3}$$

and

$$\int_\Omega \Delta u(x,\rho(t,\xi))\Phi(x)dx = -\lambda_0\int_\Omega u(x,\rho(t,\xi))\Phi(x)dx = -\lambda_0 V(\rho(t,\xi)), \quad t \geq t_1. \tag{2.4}$$

It is easy to see that

$$\begin{aligned} \int_\Omega\int_a^b q(t,\xi)u(x,\sigma(t,\xi))\Phi(x)d\eta(\xi)dx = &\int_a^b q(t,\xi)\int_\Omega u(x,\sigma(t,\xi))\Phi(x)dxd\eta(\xi) \\ = &\int_a^b q(t,\xi)V(\sigma(t,\xi))d\eta(\xi). \end{aligned} \tag{2.5}$$

From (2.2)–(2.5), we get

$$\left.\begin{array}{l}[V(t)+c(t)V(\tau(t))]^{(m)}+\int_a^b q(t,\xi)V(\sigma(t,\xi))d\eta(\xi)+\lambda_0 a(t)V(t)\\ +\lambda_0\int_a^b b(t,\xi)V(\rho(t,\xi))d\eta(\xi)=0,\quad t\geq t_1,\ t\neq t_k.\end{array}\right\}$$

For $t\geq t_0,\ t=t_k,\ k=1,2,\ldots,\ i=0,1,2,\ldots,m-1$, multiplying both sides of equation (1.1) with $\Phi(x)>0$, and then integrating with respect to x over the domain Ω, and from (H_5), we obtain

$$a_k^{(i)}\leq\frac{\dfrac{\partial^{(i)}u(x,t_k^+)}{\partial t^{(i)}}}{\dfrac{\partial^{(i)}u(x,t_k)}{\partial t^{(i)}}}\leq b_k^{(i)}.$$

According to $V(t)=\int_\Omega u(x,t)\Phi(x)dx$, we have

$$a_k^{(i)}\leq\frac{\dfrac{\partial^{(i)}V(t_k^+)}{\partial t^{(i)}}}{\dfrac{\partial^{(i)}V(t_k)}{\partial t^{(i)}}}\leq b_k^{(i)},$$

that is, $V(t)$ is a positive solution of (2.1), which contradicts the fact that all solutions of (2.1) are oscillatory.

(ii) Necessity: Suppose that (2.1) has a nonoscillatory solution $\tilde{V}(t)$. Without loss of generality, we assume that $\tilde{V}(t)>0$ for $t\geq t_*\geq 0$, where t_* is some large number. From (2.1), we have

$$\left.\begin{array}{l}\left[\tilde{V}(t)+c(t)\tilde{V}(\tau(t))\right]^{(m)}+\int_a^b q(t,\xi)\tilde{V}(\sigma(t,\xi))d\eta(\xi)+\lambda_0 a(t)\tilde{V}(t)\\ +\lambda_0\int_a^b b(t,\xi)\tilde{V}(\rho(t,\xi))d\eta(\xi)=0,\quad t\geq t_*,\ t\neq t_k,\ x\in\Omega,\\ a_k^{(i)}\leq\dfrac{\dfrac{\partial^{(i)}\tilde{V}(t_k^+)}{\partial t^{(i)}}}{\dfrac{\partial^{(i)}\tilde{V}(t_k)}{\partial t^{(i)}}}\leq b_k^{(i)}.\end{array}\right\}\quad(2.6)$$

For $t\geq t_0$, $t\neq t_k$, $k=1,2,\ldots$, multiplying both sides of (2.6) with $\Phi(x)>0$, we obtain

$$\left.\begin{array}{l}\dfrac{\partial^m}{\partial t^m}\left(\tilde{V}(t)\Phi(x)+c(t)\tilde{V}(\tau(t))\Phi(x)\right)+\int_a^b q(t,\xi)\tilde{V}(\sigma(t,\xi))\Phi(x)d\eta(\xi)\\ +\lambda_0 a(t)\tilde{V}(t)\Phi(x)+\lambda_0\int_a^b b(t,\xi)\tilde{V}(\rho(t,\xi))\Phi(x)d\eta(\xi)=0,\quad t\geq t_*,\ x\in\Omega.\end{array}\right\}\quad(2.7)$$

Let $\tilde{u}(x,t)=\tilde{V}(t)\Phi(x)$, $(x,t)\in\Omega\times[0,+\infty)$. From Lemma 1.4, we have $\Delta w(x)=-\lambda_0 w(x)$, $x\in\Omega$. Then (2.7) implies

$$\left.\begin{array}{l}\dfrac{\partial^m}{\partial t^m}\left(\tilde{u}(x,t)+c(t)\tilde{u}(x,\tau(t))\right)+\int_a^b q(t,\xi)\tilde{u}(x,\sigma(t,\xi))d\eta(\xi)\\ =\lambda_0 a(t)\Delta\tilde{u}(x,t)+\lambda_0\int_a^b b(t,\xi)\Delta\tilde{u}(x,\rho(t,\xi))d\eta(\xi),\quad t\geq t_*,\ x\in\Omega.\end{array}\right\}\quad(2.8)$$

For $t \geq t_0,\ t = t_k, k = 1, 2, \ldots,$ multiplying both sides of equation (2.6) with $\Phi(x) > 0$, we have

$$a_k^{(i)} \frac{\partial^{(i)}}{\partial t^{(i)}} \tilde{V}(t_k)\Phi(x) \leq \frac{\partial^{(i)}}{\partial t^{(i)}} \tilde{V}(t_k^+)\Phi(x) \leq b_k^{(i)} \frac{\partial^{(i)}}{\partial t^{(i)}} \tilde{V}(t_k)\Phi(x).$$

Since $\tilde{u}(x,t) = \tilde{V}(t)\Phi(x)$, $(x,t) \in \Omega \times [0, +\infty)$, we get

$$a_k^{(i)} \frac{\partial^{(i)}}{\partial t^{(i)}} \tilde{u}(x, t_k) \leq \frac{\partial^{(i)}}{\partial t^{(i)}} \tilde{u}(x, t_k^+) \leq b_k^{(i)} \frac{\partial^{(i)}}{\partial t^{(i)}} \tilde{u}(x, t_k),$$
$$\frac{\partial^{(i)}}{\partial t^{(i)}} \tilde{u}(x, t_k^+) = I_k^{(i)} \left(x, t_k, \frac{\partial^{(i)}}{\partial t^{(i)}} \tilde{u}(x, t_k) \right),$$

which means that $\tilde{u}(x,t) = \tilde{V}(t)\Phi(x)$, $(x,t) \in \Omega \times [t_*, +\infty)$ satisfies (1.1). On the other hand, from Lemma 1.4, we get

$$\alpha(x) \frac{\partial w(x)}{\partial \gamma} + \beta(x) w(x) = 0, \quad x \in \partial\Omega,$$

which implies

$$\alpha(x) \frac{\partial \tilde{u}(x,t)}{\partial \gamma} + \beta(x) \tilde{u}(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0, +\infty). \tag{2.9}$$

Hence $\tilde{u}(x,t) = \tilde{V}(t)\Phi(x) > 0$ is a nonoscillatory solution of (1.1) with boundary condition (1.2), which is a contradiction. □

Theorem 2.2. *If $\beta(x) \not\equiv 0$ for $x \in \partial\Omega$ and the impulsive differential inequality*

$$\left.\begin{array}{l} Z^{(m)}(t) + g_0 \int_a^b q(t,\xi) Z(\theta(t)) d\eta(\xi) \leq 0, \quad t \neq t_k \\ a_k^{(i)} \leq \dfrac{\dfrac{\partial^{(i)} Z(t_k^+)}{\partial t^{(i)}}}{\dfrac{\partial^{(i)} Z(t_k)}{\partial t^{(i)}}} \leq b_k^{(i)}, \quad k = 1, 2, \ldots, \quad i = 0, 1, 2, \ldots, m-1 \end{array}\right\} \tag{2.10}$$

has no eventually positive solution, then all solutions of (1.1) with boundary condition (1.2) are oscillatory in G.

Proof. Assume, for the sake of contradiction, that there is a nonoscillatory solution u of (1.1) with boundary condition (1.2), which has no zero in $\Omega \times [t_0, +\infty)$ for some $t_0 \geq 0$. Without loss of generality, we assume that $u(x,t) > 0$, $(x,t) \in \Omega \times [t_0, +\infty)$, $t_0 \geq 0$. By the assumption that there exists $t_1 > t_0$ such that $\tau(t) \geq t_0$, $\sigma(t,\xi) \geq t_0$, $\rho(t,\xi) \geq t_0$ for $(t,\xi) \in [t_1, +\infty) \times [a,b]$, then $u(x, \tau(t)) > 0$ for $(x,t) \in \Omega \times [t_1, +\infty)$, $u(x, \sigma(t,\xi)) > 0$ for $(x,t,\xi) \in \Omega \times [t_1, +\infty) \times [a,b]$ and $u(x, \rho(t,\xi)) > 0$ for $(x,t,\xi) \in \Omega \times [t_1, +\infty) \times [a,b]$.

For $t \geq t_0,\ t \neq t_k,\ k = 1, 2, \ldots,$ we obtain (2.1). In view of Lemma 1.4, we have

$$\left.\begin{array}{l} [V(t) + c(t)V(\tau(t))]^{(m)} + \int_a^b q(t,\xi) V(\sigma(t,\xi)) d\eta(\xi) \\ \quad = -\lambda_0 a(t) V(t) - \lambda_0 \int_a^b b(t,\xi) V(\rho(t,\xi)) d\eta(\xi) \leq 0, \quad t \geq t_1,\ t \neq t_k. \end{array}\right\} \tag{2.11}$$

Set $Z(t) = V(t) + c(t)V(\tau(t))$. Equation (2.11) can be written as

$$Z^{(m)}(t) + \int_a^b q(t,\xi)V(\sigma(t,\xi))d\eta(\xi) \le 0, \qquad t \ne t_k. \tag{2.12}$$

Furthermore, from Lemma 1.5, there exist $t_2 \ge t_1$ and an odd number l, $0 \le l \le m-1$, such that

$$Z^{(i)}(t) > 0, \;\; 0 \le i \le l, \qquad (-1)^{(i-1)}Z^{(i)}(t) > 0, \quad t \ge t_2, \quad l \le i \le m-1.$$

By choosing $i = 1$, we have $Z'(t) > 0$. Since $Z(t) \ge x(t) > 0$, $Z'(t) \ge 0$, we have $Z(\sigma(t,\xi)) \ge Z(\sigma(t,\xi) - \tau(t)) \ge x(\sigma(t,\xi) - \tau(t))$, and therefore $Z^{(m)}(t) + \int_a^b q(t,\xi)Z(\sigma(t,\xi))\,(1 - c(\sigma(t,\xi)))\,d\eta(\xi) \le 0$. From (2.12), we get

$$Z^{(m)}(t) + g_0 \int_a^b q(t,\xi)Z(\sigma(t,\xi))d\eta(\xi) \le 0. \tag{2.13}$$

From (H$_2$) and (H$_3$), we obtain $Z(\sigma(t,\xi)) \ge Z(\sigma(t,a)) > 0$, $\;\xi \in [a,b]$ and $\theta(t) \le \sigma(t,\xi) \le t$. Thus $Z(\theta(t)) \le Z(\sigma(t,a))$ for $t \ge t_2$. Hence (2.13) can be written as

$$Z^{(m)}(t) + g_0 \int_a^b q(t,\xi)Z(\theta(t))d\eta(\xi) \le 0. \tag{2.14}$$

For $t \ge t_0,\; t = t_k,\; k = 1, 2, \ldots$, from (2.1), we have

$$a_k^{(i)} \le \frac{\dfrac{\partial^{(i)}Z(t_k^+)}{\partial t^{(i)}}}{\dfrac{\partial^{(i)}Z(t_k)}{\partial t^{(i)}}} \le b_k^{(i)}.$$

That is, $Z(t)$ is an eventually positive solution of (2.10), which contradicts our hypothesis. □

Theorem 2.3. *Let $\beta(x) \not\equiv 0$ for some $x \in \partial\Omega$. If for some $t_0 > 0$, there exists a function $\varphi(t) \in C'([0,+\infty),(0,+\infty))$ that is nondecreasing with respect to t, such that*

$$\int_{t_0}^{+\infty} \prod_{t_0 \le t_k < s} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} \left[A(s)\varphi(s) - \frac{(\varphi'(s))^2}{4B(s)\varphi(s)}\right] ds = +\infty, \tag{2.15}$$

then all solutions of (1.1) with boundary condition (1.2) are oscillatory in G.

Proof. From Theorem 2.2, it is enough to prove that the impulsive differential inequality (2.10) has no eventually positive solution. Suppose that $Z(t) > 0$ is a solution of (2.10). Set

$$W(t) = \varphi(t)\frac{Z^{(m-1)}(t)}{Z(\theta(t))}, \quad t \ge t_0. \tag{2.16}$$

Clearly $W(t) \ge 0$ for $t \ge t_0$, and

$$W'(t) \;=\; \frac{\varphi'(t)}{\varphi(t)}W(t) + \frac{\varphi(t)z^{(m)}(t)}{Z(\theta(t))} - \frac{\varphi(t)Z^{(m-1)}(t)Z'(\theta(t))\theta'(t)}{Z^2(\theta(t))}.$$

Since $Z^{(m)}(t) \leq 0$, according to Lemma 1.6, we obtain

$$Z'^{m-2} Z^{(m-1)}(t). \tag{2.17}$$

Thus

$$\begin{aligned} W'(t) &\leq \frac{\varphi'(t)}{\varphi(t)} W(t) - A(t)\varphi(t) - \frac{B(t)}{\varphi(t)} W^2(t), \\ W(t_k^+) &\leq \frac{b_k^{(m-1)}}{a_k^{(0)}} W(t_k). \end{aligned}$$

Define

$$U(t) = \prod_{t_0 \leq t_k < t} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} W(t).$$

In fact, $W(t)$ is continuous on each interval $(t_k, t_{k+1}]$, and in the consideration of $W(t_k^+) \leq \dfrac{b_k^{(m-1)}}{a_k^{(0)}} W(t_k)$, it follows that for $t \geq t_0$,

$$U(t_k^+) = \prod_{t_0 \leq t_j \leq t_k} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} W(t_k^+) \leq \prod_{t_0 \leq t_j < t_k} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} W(t_k) = U(t_k)$$

and

$$U(t_k^-) = \prod_{t_0 \leq t_j \leq t_{k-1}} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} W(t_k^-) \leq \prod_{t_0 \leq t_j < t_k} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} W(t_k) = U(t_k),$$

which implies that $U(t)$ is continuous on $[t_0, +\infty)$. Moreover,

$$\begin{aligned} &U'(t) + \prod_{t_0 \leq t_k < t} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}} \right) \frac{U^2(t)B(t)}{\varphi(t)} + \prod_{t_0 \leq t_k < t} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} A(t)\varphi(t) - \frac{\varphi'(t)}{\varphi(t)} U(t) \\ &= \prod_{t_0 \leq t_k < t} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} \left[W'^2(t) \frac{B(t)}{\varphi(t)} - W(t) \frac{\varphi'(t)}{\varphi(t)} + A(t)\varphi(t) \right] \leq 0. \end{aligned}$$

That is,

$$U'(t) \leq -\prod_{t_0 \leq t_k < t} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}} \right) \frac{B(t)}{\varphi(t)} U^2(t) + \frac{\varphi'(t)}{\varphi(t)} U(t) - \prod_{t_0 \leq t_k < t} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} A(t)\varphi(t). \tag{2.18}$$

Taking

$$X = \sqrt{\prod_{t_0 \leq t_k < t} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}} \right) \frac{B(t)}{\varphi(t)}} U(t), \quad Y = \frac{\varphi'(t)}{2} \sqrt{\prod_{t_0 \leq t_k < t} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} \frac{1}{\varphi(t)B(t)}},$$

from Lemma 1.7, we have

$$\frac{\varphi'(t)}{\varphi(t)}U(t) - \prod_{t_0\le t_k<t}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)\frac{B(t)}{\varphi(t)}U^2(t) \le \frac{(\varphi'^2}{4B(t)\varphi(t)}\prod_{t_0\le t_k<t}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}.$$

Thus

$$U'(t) \le -\prod_{t_0\le t_k<t}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}\left[A(t)\varphi(t) - \frac{(\varphi'^2}{4B(t)\varphi(t)}\right]. \tag{2.19}$$

Integrating both sides from t_0 to t, we have

$$U(t) \le U(t_0) - \int_{t_0}^{t}\prod_{t_0\le t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}\left[A(s)\varphi(s) - \frac{(\varphi'^2}{4B(s)\varphi(s)}\right]ds.$$

Letting $t\to+\infty$, and taking into account the fact that (2.15) holds, we have $\lim_{t\to+\infty} U(t) = -\infty$, which contradicts with $U(t)\ge 0$. □

Theorem 2.4. *Let $\beta(x)\not\equiv 0$ for $x\in\partial\Omega$. Moreover, suppose that there exist functions $\varphi(t)$ and $\phi(s)\in C'([0,+\infty),(0,+\infty))$, where $\varphi(t)$ is nondecreasing with respect to t, and the functions $H(t,s), h(t,s)\in C'(D,\mathbb{R})$, where $D=\{(t,s)|t\ge s\ge t_0>0\}$, such that*

- (H_6) $H(t,t)=0,\quad t\ge t_0;\quad H(t,s)>0,\quad t>s\ge t_0,$
- (H_7) $H_t'(t,s)\ge 0,\quad H_s'(t,s)\le 0,$
- (H_8) $-\dfrac{\partial}{\partial s}[H(t,s)\phi(s)] - H(t,s)\phi(s)\dfrac{\varphi'(s)}{\varphi(s)} = h(t,s).$

If

$$\limsup_{t\to+\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\prod_{t_0\le t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}\left[A(s)\varphi(s)H(t,s)\phi(s) - \frac{1}{4}\frac{|h(t,s)|^2\varphi(s)}{B(s)H(t,s)\phi(s)}\right]ds = +\infty, \tag{2.20}$$

then all solutions of (1.1) with boundary condition (1.2) are oscillatory in G.

Proof. Assume, for the sake of contradiction, that (1.1) with boundary condition (1.2) has a nonoscillatory solution $u(x,t)$. Without loss of generality, assume that $u(x,t)>0$, $(x,t)\in\Omega\times[0,+\infty)$. Proceeding as in the proof of Theorem 2.3, we have $u(x,\tau(t))>0$, $u(x,\sigma(t,\xi))>0$, $u(x,\rho(t,\xi))>0$, for $(x,t)\in\Omega\times[t_1,+\infty)$, $(x,t,\xi)\in\Omega\times[t_1,+\infty)\times[a,b]$, and

$$U'(t) \le -\prod_{t_0\le t_k<t}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)\frac{B(t)}{\varphi(t)}U^2(t) + \frac{\varphi'(t)}{\varphi(t)}U(t) - \prod_{t_0\le t_k<t}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}A(t)\varphi(t).$$

Multiplying the above inequality with $H(t,s)\phi(s)$ for $t \geq s \geq T$, and integrating from T to t, we get

$$\begin{aligned}\int_T^t \prod_{t_0\leq t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} & A(s)\varphi(s)H(t,s)\phi(s)ds \\ & \leq U(T)H(t,T)\phi(T) + \int_T^t |h(t,s)U(s)|\,ds \\ & \quad - \int_T^t \prod_{t_0\leq t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)\frac{B(s)}{\varphi(s)}U^2(s)H(t,s)\phi(s)ds.\end{aligned} \tag{2.21}$$

Put

$$X = \sqrt{\prod_{t_0\leq t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)\frac{B(s)}{\varphi(s)}H(t,s)\phi(s)}U(s),$$

$$Y = \frac{1}{2}|h(t,s)|\sqrt{\prod_{t_0\leq t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}\frac{\varphi(s)}{B(s)H(t,s)\phi(s)}}.$$

From Lemma 1.7, we attain for $t \geq T \geq t_0$ that

$$\begin{aligned}|h(t,s)U(s)| - \prod_{t_0\leq t_k<s} & \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)\frac{B(s)}{\varphi(s)}H(t,s)\phi(s)U^2(s) \\ & \leq \frac{1}{4}\frac{|h(t,s)|^2\varphi(s)}{B(s)H(t,s)\phi(s)}\prod_{t_0\leq t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}.\end{aligned} \tag{2.22}$$

In addition, from (2.21) and (2.22), we have

$$\begin{aligned}\int_T^t \prod_{t_0\leq t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} & A(s)\varphi(s)H(t,s)\phi(s)ds \\ & - \frac{1}{4}\int_T^t \frac{|h(t,s)|^2\varphi(s)}{B(s)H(t,s)\phi(s)}\prod_{t_0\leq t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} ds \\ & \leq U(T)H(t,T)\phi(T) \leq H(t,t_0)\phi(T)U(T), \quad t \geq T \geq t_0.\end{aligned} \tag{2.23}$$

Thus

$$\begin{aligned}&\frac{1}{H(t,t_0)}\int_{t_0}^t \prod_{t_0\leq t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}\left[A(s)\varphi(s)H(t,s)\phi(s) - \frac{1}{4}\frac{|h(t,s)|^2\varphi(s)}{B(s)H(t,s)\phi(s)}\right]ds \\ &\leq \int_{t_0}^T \prod_{t_0\leq t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} G(s)\varphi(s)\phi(s)ds + \phi(T)U(T).\end{aligned}$$

Letting $t \to +\infty$, we get

$$\limsup_{t\to+\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \prod_{t_0\le t_k<s} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} \times \left[A(s)\varphi(s)H(t,s)\phi(s) - \frac{1}{4}\frac{|h(t,s)|^2\varphi(s)}{B(s)H(t,s)\phi(s)}\right] ds \le \int_{t_0}^{T} \prod_{t_0\le t_k<s} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} A(s)\varphi(s)\phi(s)ds + \phi(T)U(T) < +\infty,$$

which contradicts (3.21). □

Remark 2.5. In Theorem 2.4, by choosing $\phi(s) = \varphi(s) \equiv 1$, we have the following corollary.

Corollary 2.6. *Assume that all the conditions of Theorem 2.4 hold, and that*

$$\limsup_{t\to+\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \prod_{t_0\le t_k<s} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} \left[A(s)H(t,s) - \frac{1}{4}\frac{|h(t,s)|^2}{B(s)H(t,s)}\right] ds = +\infty.$$

Then all solutions of (1.1) *with boundary condition* (1.2) *are oscillatory in* G.

Remark 2.7. From Theorem 2.4 and Corollary 2.6, we can obtain a variety of oscillatory criteria by different choices of the weighted function $H(t,s)$. For example, choosing $H(t,s) = (t-s)^{\mu-1}$, $t \ge s \ge t_0$, in which $\mu > 2$ is an integer, then $h(t,s) = (\mu-1)(t-s)^{\mu-2}$, $t \ge s \ge t_0$. From Corollary 2.6, we have the following result.

Corollary 2.8. *If there is an integer* $\mu > 2$ *such that*

$$\limsup_{t\to+\infty} \frac{1}{(t-t_0)^{\mu-1}} \int_{t_0}^{t} \prod_{t_0\le t_k<s} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} \times (t-s)^{\mu-1} \left[A(s) - \frac{1}{4B(s)}\frac{(\mu-1)^2}{(t-s)^2}\right] ds = +\infty,$$

then all solutions of (1.1) *with boundary condition* (1.2) *are oscillatory in* G.

3. Examples

We illustrate the significance of our results by the following examples.

Example 3.1. Consider the equation

$$\left.\begin{aligned}
&\frac{\partial^6}{\partial t^6}\Big(u(x,t)+\frac{2}{5}u(x,t-\tfrac{\pi}{2})\Big)+\frac{4}{5}\int_{-\pi/2}^{-\pi/4}u(x,t+2\xi)d\xi\\
&=\frac{11}{5}\Delta u(x,t)+\frac{8}{5}\int_{-\pi/2}^{-\pi/4}\Delta u(x,t+2\xi)d\xi,\quad t\neq t_k,\\
&u(x,t_k^+)=\frac{k}{k+1}u(x,t_k),\\
&\frac{\partial^{(i)}}{\partial t^{(i)}}u(x,t_k^+)=\frac{\partial^{(i)}}{\partial t^{(i)}}u(x,t_k),\quad i=1,2,3,4,5,\quad k=1,2,\ldots,
\end{aligned}\right\}\tag{3.1}$$

for $(x,t)\in(0,\pi)\times[0,+\infty)$, with the boundary condition

$$u(0,t)=u(\pi,t)=0,\quad t\neq t_k.\tag{3.2}$$

Here $\Omega=(0,\pi)$, $m=6$, $a_k^{(0)}=b_k^{(0)}=\dfrac{k}{k+1}$, $a_k^{(i)}=b_k^{(i)}=1$, $i=1,2,3,4,5$, $c(t)=\dfrac{2}{5}$, $\tau(t)=t-\dfrac{\pi}{2}$, $q(t,\xi)=\dfrac{4}{5}$, $\sigma(t,\xi)=\rho(t,\xi)=t+2\xi$, $a(t)=\dfrac{11}{5}$, $b(t,\xi)=\dfrac{8}{5}$, $[a,b]=[-\pi/2,-\pi/4]$, $\eta(\xi)=\xi$, $M=1$, $\theta(t)=t$, $\theta'(t)=1$, $\mu=3$, $t_0=1$, $t_k=2^k$, $g_0=\dfrac{4}{5}$, $A(s)=\dfrac{3\pi}{25}$, $B(s)=s^4$.
Clearly (H_1)-(H_5) hold, and moreover

$$\begin{aligned}
\lim_{t\to+\infty}\int_{t_0}^{t}\prod_{t_0\le t_k<s}\frac{a_k^{(0)}}{b_k^{(i)}}ds &= \int_1^{+\infty}\prod_{1<t_k<s}\frac{k}{k+1}ds\\
&=\int_1^{t_1}\prod_{1<t_k<s}\frac{k}{k+1}ds+\int_{t_1^+}^{t_2}\prod_{1<t_k<s}\frac{k}{k+1}ds\\
&\quad+\int_{t_2^+}^{t_3}\prod_{1<t_k<s}\frac{k}{k+1}ds\ +\cdots\\
&=1+\frac{1}{2}\times 2+\frac{1}{2}\times\frac{2}{3}\times 2^2+\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}\times 2^3+\cdots\\
&=\sum_{n=0}^{+\infty}\frac{2^n}{n+1}=+\infty.
\end{aligned}$$

Thus,

$$\limsup_{t\to+\infty}\frac{1}{(t-1)^2}\left\{\int_1^t\prod_{1<t_k<s}\frac{k}{k+1}(t-s)^2\left[\frac{3\pi}{25}-\frac{1}{s^4(t-s)^2}\right]ds\right\}=+\infty.$$

That is, all the conditions of Corollary 2.8 are satisfied, and therefore all solutions of (3.1)–(3.2) are oscillatory in G. In fact, $u(x,t)=\sin x\cos t$ is such a solution.

Example 3.2. Consider the equation

$$\left.\begin{aligned}&\frac{\partial^4}{\partial t^4}\left(u(x,t)+\frac{1}{2}u(x,t-\pi)\right)+\frac{3}{4}\int_{-\pi}^{0}u(x,t+\xi)d\xi\\&=\frac{1}{2}\Delta u(x,t)+\frac{3}{4}\int_{-\pi}^{0}\Delta u(x,t+\xi)d\xi,\quad t\neq t_k,\\&u(x,t_k^+)=\frac{k}{k+1}u(x,t_k),\\&\frac{\partial^{(i)}}{\partial t^{(i)}}u(x,t_k^+)=\frac{\partial^{(i)}}{\partial t^{(i)}}u(x,t_k),\quad i=1,2,3,\quad k=1,2,\dots,\end{aligned}\right\}\tag{3.3}$$

for $(x,t)\in(0,\pi)\times[0,+\infty)$, with the boundary condition

$$u_x(0,t)+u(0,t)=u_x(\pi,t)+u(\pi,t)=0,\quad t\neq t_k.\tag{3.4}$$

Here $\Omega=(0,\pi)$, $m=4$, $a_k^{(0)}=b_k^{(0)}=\dfrac{k}{k+1}$, $a_k^{(i)}=b_k^{(i)}=1$, $i=1,2,3$, $c(t)=\dfrac{1}{2}$, $\tau(t)=t-\pi$, $q(t,\xi)=\dfrac{3}{4}$, $\sigma(t,\xi)=\rho(t,\xi)=t+\xi$, $a(t)=\dfrac{1}{2}$, $b(t,\xi)=\dfrac{3}{4}$, $[a,b]=[-\pi,0]$, $\eta(\xi)=\xi$, $M=1$, $\theta(t)=t$, $\theta'(t)=1$, $\mu=3$, $t_0=1$, $t_k=2^k$, $g_0=\dfrac{1}{2}$, $A(s)=\dfrac{3\pi}{8}$, $B(s)=s^2$.

Clearly (H_1)-(H_5) hold, and moreover

$$\limsup_{t\to+\infty}\frac{1}{(t-1)^2}\left\{\int_1^t\prod_{1<t_k<s}\frac{k}{k+1}(t-s)^2\left[\frac{3\pi}{8}-\frac{1}{s^2(t-s)^2}\right]ds\right\}=+\infty.$$

That is, all the conditions of the Corollary 2.8 are satisfied, and therefore all solutions of (3.3)–(3.4) are oscillatory in G. In fact $u(x,t)=e^{-x}\cos t$ is such a solution.

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