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UNCERTAINTY PRINCIPLES ON NILPOTENT LIE GROUPS

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Abstract. We prove Hardy's type uncertainty principle on connected nilpotent Lie groups for the Fourier transform. An analogue of Hardy's theorem for the Gabor transform has been established for connected and simply connected nilpotent Lie groups. Finally Beurling's theorem for the Gabor transform is discussed for groups of the form $\mathbb{R}^n \times K$, where K is a compact group.

1. Introduction

Heisenberg uncertainty principle relates the uncertainties in the measurement of position and moment of microscopic particles. In harmonic analysis, the uncertainty principle relates the behavior of a function like support or decay with that of its Fourier transform. For $f \in L^1(\mathbb{R})$, the Fourier transform $\widehat{f}$ on $\mathbb{R}$ is given by

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)\, e^{-2\pi i \xi x}\, dx.$$

One of the uncertainty principles states that a nonzero integrable function f on $\mathbb{R}$ and its Fourier transform $\widehat{f}$ cannot both simultaneously decay rapidly. The following theorem of Hardy makes the above statement more precise.

Theorem 1.1 ([15]). *Let f be a measurable function on $\mathbb{R}$ such that*

(i) $|f(x)| \leq Ce^{-a\pi x^2}$ *for all* $x \in \mathbb{R}$,

(ii) $|\widehat{f}(\xi)| \leq Ce^{-b\pi \xi^2}$ *for all* $\xi \in \mathbb{R}$,

where a, b, and C are positive constants. If $ab > 1$, then $f = 0$ a.e.

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Several analogues of the above result have been proved in the setting of $\mathbb{R}^n$, Heisenberg group $\mathbb{H}_n$ [26], Heisenberg motion group $\mathbb{H}_n \ltimes K$ [5], locally compact abelian groups, various classes of solvable locally compact groups [3], Euclidean motion group [24], and nilpotent Lie groups [2,18,23]. A generalization of Hardy's theorem is Beurling's theorem, which can be stated as follows.

Theorem 1.2 ([17]). *Let f be a square integrable function on $\mathbb{R}$ satisfying*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |f(x)| \, |\widehat{f}(\xi)| e^{2\pi|x\cdot\xi|} \, dx \, d\xi < \infty.$$

Then $f = 0$ a.e.

Several analogues of Beurling's theorem for the Fourier transform has been proved for exponential solvable Lie groups [1] and various classes of nilpotent Lie groups [4, 22, 23, 27, 31]. Uncertainty principles like Heisenberg uncertainty inequality and qualitative uncertainty principle have been investigated for the Fourier transform (see [6,9,28,29]). For a detailed survey of the uncertainty principles for the Fourier transform, we refer to [13].
The transformation of a signal using the Fourier transform loses the information about time, and it is very difficult to tell where a certain frequency has occurred. Thus, in order to tackle such problems, *a joint time-frequency analysis* was utilized. *Gabor transform* is turned out to be one such tool. The approach used in this technique is cutting the signal into segments using a smooth window function and then computing the Fourier transform separately on each smaller segment. In this manner, the Gabor transform provides the local aspect of the Fourier transform with time resolution equal to the size of the window. It results in a two-dimensional representation of the signal.

Let $\psi \in L^2(\mathbb{R})$ be a fixed function usually called a *window function*. The Gabor transform of a function $f \in L^2(\mathbb{R})$ with respect to the window function ψ is defined by $G_\psi f : \mathbb{R} \times \widehat{\mathbb{R}} \to \mathbb{C}$ as

$$G_\psi f(t, \xi) = \int_{\mathbb{R}} f(x) \, \overline{\psi(x-t)} \, e^{-2\pi i \xi x} \, dx,$$

for all $(t, \xi) \in \mathbb{R} \times \widehat{\mathbb{R}}$.
In [10], the Gabor transform on a second countable, locally compact, unimodular group G of type I has been studied. The Heisenberg uncertainty inequality was proved in [7,30] for the Gabor transform for the groups of the form $K \ltimes \mathbb{R}^n$, where K is a separable unimodular locally compact group of type I and connected, simply connected nilpotent Lie groups. Qualitative uncertainty principle was proved for the Gabor transform for several classes of locally compact groups, including low dimensional nilpotent Lie groups [25]. Later, Hardy's uncertainty principle for the Gabor transform was proved for locally compact abelian groups having noncompact identity component and groups of the form $\mathbb{R}^n \times K$, where K is a compact group having irreducible representations of bounded dimension [8]. In [11], the spherical Gabor transform using the properties of Gelfand pairs and the spherical Fourier transform, has been studied and Lieb inequality, Donoho–Stark's uncertainty principles, and Beckner's uncertainty principles were proved.

In this paper, analogues of above uncertainty principles on nilpotent Lie groups for the Fourier and Gabor transforms have been studied. Results obtained have been organized as follows: In section 3, Hardy's type results for the Fourier transform have been established for connected nilpotent Lie groups. Section 4 deals with an analogue of Hardy's theorem for the Gabor transform. In the last section, we prove Beurling's theorem for the Gabor transform for locally compact abelian groups with noncompact connected component and groups of the form $\mathbb{R}^n \times K$, where K is a compact group.

2. Preliminaries

For a second countable, locally compact, unimodular group G of type I, dx will denote the Haar measure on G. Let $\widehat{G}$ be the dual space of G consisting of all irreducible unitary representations of G equipped with Plancherel measure $d\pi$. For $f \in L^1 \cap L^2(G)$, the Fourier transform $\widehat{f}$ of f is an operator-valued function on $\widehat{G}$ defined as

$$\widehat{f}(\pi) = \int_G f(x)\ \pi(x)^* dx.$$

Moreover, by the Plancherel theorem [12, Theorem 7.36], $\widehat{f}(\pi)$ is a Hilbert-Schmidt operator and satisfies the following property:

$$\int_G |f(x)|^2 dx = \int_{\widehat{G}} \|\widehat{f}(\pi)\|_{\mathrm{HS}}^2\ d\pi. \tag{2.1}$$

For each $(x,\pi) \in G \times \widehat{G}$, we define $\mathcal{H}_{(x,\pi)} = \pi(x)\mathrm{HS}(\mathcal{H}_\pi)$, where $\pi(x)\mathrm{HS}(\mathcal{H}_\pi) = \{\pi(x)T : T \in \mathrm{HS}(\mathcal{H}_\pi)\}$. Then $\mathcal{H}_{(x,\pi)}$ forms a Hilbert space with the inner product given by

$$\langle \pi(x)T, \pi(x)S\rangle_{\mathcal{H}_{(x,\pi)}} = \mathrm{tr}\,(S^*T) = \langle T, S\rangle_{\mathrm{HS}(\mathcal{H}_\pi)}.$$

Also, $\mathcal{H}_{(x,\pi)} = \mathrm{HS}(\mathcal{H}_\pi)$ for all $(x,\pi) \in G \times \widehat{G}$. Let $\mathcal{H}^2(G \times \widehat{G})$ denote the direct integral of $\{\mathcal{H}_{(x,\pi)}\}_{(x,\pi)\in G\times\widehat{G}}$ with respect to the product measure $dx\ d\pi$. Then $\mathcal{H}^2(G \times \widehat{G})$ forms a Hilbert space with the inner product given by

$$\langle F, K\rangle_{\mathcal{H}^2(G\times\widehat{G})} = \int_{G\times\widehat{G}} \mathrm{tr}\,[F(x,\pi)K(x,\pi)^*]\ dx\ d\pi.$$

Let $f \in C_c(G)$, the space of all continuous complex-valued functions on G with compact support, and let ψ be a fixed function in $L^2(G)$. For $(x,\pi) \in G \times \widehat{G}$, the continuous *Gabor Transform* [10, Definition 3.1] of f with respect to the window function ψ can be defined as a measurable field of operators on $G \times \widehat{G}$ by

$$G_\psi f(x,\pi) := \int_G f(y)\ \overline{\psi(x^{-1}y)}\ \pi(y)^*\ dy. \tag{2.2}$$

One can verify that $G_\psi f(x,\pi)$ is a Hilbert–Schmidt operator for all $x \in G$ and for almost all $\pi \in \widehat{G}$. We can extend G_ψ uniquely to a bounded linear operator from $L^2(G)$ into a closed subspace of $\mathcal{H}^2(G \times \widehat{G})$, which will again be denoted by

G_ψ. As in [10, Corollary 3.4], for $f_1, f_2 \in L^2(G)$ and window functions ψ_1 and ψ_2, we have

$$\langle G_{\psi_1} f_1, G_{\psi_2} f_2\rangle = \langle \psi_2, \psi_1\rangle \langle f_1, f_2\rangle. \tag{2.3}$$

For detailed study of the Gabor transform on second countable, locally compact, unimodular group G of type I, one can refer to [10].

3. Nilpotent Lie group

For a connected nilpotent Lie group G with its simply connected covering group $\widetilde{G}$, let Γ be a discrete subgroup of $\widetilde{G}$ such that $G = \widetilde{G}/\Gamma$. Denoting $\mathfrak{g}$ by the Lie algebra of G and $\widetilde{G}$, let $\mathcal{B} = \{X_1, X_2, \ldots, X_n\}$ be a strong Malcev basis of $\mathfrak{g}$ through the ascending central series of $\mathfrak{g}$. The norm function on $\mathfrak{g}$ is defined as the Euclidean norm of X with respect to the basis $\mathcal{B}$. Indeed, for $X = \sum_{j=1}^n x_j X_j \in \mathfrak{g}$ with $x_j \in \mathbb{R}$,

$$\|X\| = \Big(\sum_{j=1}^n x_j^2\Big)^{1/2}.$$

Define a “norm function” on G by setting

$$\|x\| = \inf\{\|X\| : X \in \mathfrak{g} \text{ such that } \exp_G X = x\}.$$

The composed map, $\mathbb{R}^n \to \mathfrak{g} \to \widetilde{G}$ given by

$$(x_1, \ldots, x_n) \to \sum_{j=1}^n x_j X_j \to \exp_{\widetilde{G}}\Big(\sum_{j=1}^n x_j X_j\Big)$$

is a diffeomorphism and maps the Lebesgue measure on $\mathbb{R}^n$ to the Haar measure on $\widetilde{G}$. In this manner, we identify the Lie algebra $\mathfrak{g}$, as a set with $\mathbb{R}^n$. Also, measurable (integrable) functions on $\widetilde{G}$ can be viewed as such functions on $\mathbb{R}^n$.

Let $\mathfrak{g}^*$ be the vector space dual of $\mathfrak{g}$ and let $\{X_1^*, \ldots, X_n^*\}$ be the basis of $\mathfrak{g}^*$, which is dual to $\{X_1, \ldots, X_n\}$. Then $\{X_1^*, \ldots, X_n^*\}$ is a Jordan–Hölder basis for the coadjoint action of G on $\mathfrak{g}^*$. We shall identify $\mathfrak{g}^*$ with $\mathbb{R}^n$ via the map

$$\xi = (\xi_1, \ldots, \xi_n) \to \sum_{j=1}^n \xi_j X_j^*,$$

and on $\mathfrak{g}^*$, the Euclidean norm relative to the basis $\{X_1^*, \ldots, X_n^*\}$ is defined as

$$\Big\|\sum_{j=1}^n \xi_j X_j^*\Big\| = \Big(\sum_{j=1}^n \xi_j^2\Big)^{1/2} = \|\xi\|.$$

Let $\mathscr{U}$ denote the Zariski open subset of $\mathfrak{g}^*$ of generic elements under the coadjoint action of $\widetilde{G}$ with respect to the basis $\{X_1^*, \ldots, X_n^*\}$. Suppose that S is the set of jump indices, $T = \{1, \ldots, n\} \setminus S$, and that $V_T = \mathbb{R}\text{-span}\{X_i^* : i \in T\}$.

Then $\mathcal{W} = \mathscr{U} \cap V_T$ is a cross-section for the generic orbits, and $\mathcal{W}$ supports the Plancherel measure on $\widetilde{G}$. Every element of a connected nilpotent Lie group G with noncompact center can be uniquely written as (t, z, y), $t \in \mathbb{R}$, $z \in \mathbb{T}^d$, and

$y \in Y$, where $Y = \exp(\sum_{j=d+2}^{n} \mathbb{R}X_j)$. We now prove a generalization of the result proved in [2].

Theorem 3.1. *Let G be a connected nilpotent Lie group with noncompact center and let $f : G \to \mathbb{C}$ be a measurable function satisfying*

(i) $|f(t,z,y)| \le C(1+|t|^2)^N e^{-\pi\alpha t^2}\phi(y)$ *for all* $(t,z,y) \in G$ *and for some* $\phi \in L^1 \cap L^2(Y)$.

(ii) $\|\pi_\xi(f)\|_{HS} \le C(1+\|\xi\|^2)^N e^{-\pi\beta\|\xi\|^2}$ *for all* $\xi \in \mathcal{W}$,

where α, β, and C are positive real numbers and N is a nonnegative integer. If $\alpha\beta > 1$, then $f = 0$ a.e.

Before proving this main result, we shall first prove some lemmas. Let K be a compact central subgroup of G and let χ be a character of K. For $f \in L^1(G)$, define $f_\chi : G \to \mathbb{C}$ by

$$f_\chi(t,z,y) = \int_K f(t,zk,y)\ \overline{\chi(k)}\ dk.$$

Lemma 3.2. *Let G be a connected nilpotent Lie group with a compact central subgroup K and let f be a measurable function on G satisfying conditions (i) and (ii) of Theorem 3.1. Then the function f_χ also satisfies these conditions.*

Proof. On normalizing the Haar measure on central subgroup K, we obtain

$$\begin{aligned}|f_\chi(t,z,y)| &\le \int_K C(1+t^2)^N\ e^{-\alpha\pi t^2}\varphi(y)\ dk\\ &= C(1+t^2)^N\ e^{-\alpha\pi t^2}\varphi(y).\end{aligned}$$

Also, $\pi_\xi(f_\chi) = \pi_\xi(f)\int_K \chi(k)\ \pi_\xi(k)\ dk$. If $\pi_\xi|_K$ is a multiple of some character of K, which is different from χ, then by orthogonality relation of compact groups, we have

$$\int_K \chi(k)\ \pi_\xi(k)\ dk = 0.$$

Thus, $\|\pi_\xi(f_\chi)\| \le C(1+\|\xi\|^2)^N e^{-\beta\pi\|\xi\|^2}$. □

Denote by G^c, the maximal compact subgroup of G. Then G^c is connected, contained in $Z(G)$, and G/G^c is simply connected.

Lemma 3.3. *Let G be a connected nilpotent Lie group. Suppose that Theorem 3.1 holds for all quotient subgroups $H = G/C$, where C is a closed subgroup of $G^c = Z(G)^c$ such that either $Z(G)^c = C$ or $Z(G)^c/C = \mathbb{T}$. Then Theorem 3.1 also holds for G.*

Proof. Let $K = Z(G)^c$ and let $f : G \to \mathbb{C}$ be a measurable function that satisfies the conditions of Theorem 3.1. For χ in $\widehat{K}$, consider $K_\chi = \{k \in K : \chi(k) = 1\}$ and $H = G/K_\chi$. Then f_χ is constant on the cosets of the subgroup K_χ and also by Lemma 3.2, it follows that the function f_χ satisfies the Hardy's type decay conditions. Since $H^c = K/K_\chi = \mathbb{T}$ or $H^c = \{e\}$, using the hypothesis, we get $f_\chi = 0$ a.e. As $\chi \in \widehat{K}$ is arbitrarily chosen, we have $f = 0$ a.e. □

For a second countable, locally compact group G containing $\mathbb{R}$ as a closed central subgroup, let S denote a Borel cross-section for the cosets of $\mathbb{R}$ in G. The inverse image of Haar measure on $G/\mathbb{R}$ under the map $s \to \mathbb{R}s$ from $S \to G/\mathbb{R}$ is denoted by ds.

Lemma 3.4. *Let G and S be as defined above and let $f : G \to \mathbb{C}$ be a measurable function satisfying $|f(ts)| \leq (1+|t|^2)^N e^{-\alpha\pi t^2}\phi(s)$, for some $\alpha > 0$ and $\phi \in L^2(S)$. Define a function g on $\mathbb{R}$ such that $g(t) = \int_S (f_s * f_s^*)(t)\ ds$, where*

$$f_s * f_s^*(t) = \int_{\mathbb{R}} f_s(z)\overline{f_s(z-t)}dz.$$

Then $|g(t)| \leq C_1 e^{-\gamma\pi\frac{t^2}{2}}$, for some $C_1 > 0$ and $0 < \gamma < \alpha$.

Proof. For each $t \in \mathbb{R}$ and $0 < \gamma < \alpha$, we have

$$\begin{aligned}
|g(t)| &= \Big|\int_S \int_{\mathbb{R}} f(zs)\ \overline{f((z-t)s)}\ dz\ ds\Big| \\
&\leq \int_S \int_{\mathbb{R}} |f(zs)|\ |f((z-t)s)|\ dz\ ds \\
&\leq \int_S \phi(s)^2 ds \int_{\mathbb{R}} (1+|z|^2)^N (1+|z-t|^2)^N e^{-\pi\alpha(z^2+(z-t)^2)}\ dz \\
&\leq \|\phi\|_2^2\ \int_{\mathbb{R}} \sum_{k,j=0}^{N} \binom{N}{k}\binom{N}{j} z^{2k}(z-t)^{2j} e^{-(\alpha-\gamma)\pi z^2} e^{-\gamma\pi z^2} \\
&\qquad\qquad \times e^{-(\alpha-\gamma)\pi(z-t)^2} e^{-\gamma\pi(z-t)^2} dz.
\end{aligned}$$

The function $z \to \binom{N}{k} z^{2k} e^{-(\alpha-\gamma)\pi z^2}$ is bounded on $\mathbb{R}$, say by K_k.

Set $K = \max\{K_k : 0 \leq k \leq N\}$. Thus, it follows that

$$|g(t)| \leq K(N+1)\ \|\phi\|_2^2\ \sum_{j=0}^{N} \binom{N}{j} \int_{\mathbb{R}} (z-t)^{2j} e^{-\gamma\pi z^2} e^{-(\alpha-\gamma)\pi(z-t)^2} e^{-\gamma\pi(z-t)^2} dz.$$

Using the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
|g(t)| &\leq K(N+1)\ \|\phi\|_2^2\ \sum_{j=0}^{N} \binom{N}{j} \left(\int_{\mathbb{R}} (z-t)^{4j} e^{-2(\alpha-\gamma)\pi(z-t)^2} dz\right)^{1/2} \\
&\qquad \times \left(\int_{\mathbb{R}} e^{-2\gamma\pi z^2} e^{-2\gamma\pi(z-t)^2} dz\right)^{1/2} \\
&= K(N+1)\ \|\phi\|_2^2\ \sum_{j=0}^{N} \binom{N}{j} B_j \left(\int_{\mathbb{R}} e^{-2\gamma\pi(\frac{t^2}{2}+\frac{1}{2}(2z-t)^2)} dz\right)^{1/2} \\
&= K(N+1)\ \|\phi\|_2^2\ e^{-\gamma\pi\frac{t^2}{2}} \sum_{j=0}^{N} \binom{N}{j} B_j \int_{\mathbb{R}} e^{-\pi\gamma\frac{1}{2}(2z-t)^2)} dz \\
&= K(N+1)\ \|\phi\|_2^2\ e^{-\gamma\pi\frac{t^2}{2}} \sum_{j=0}^{N} \binom{N}{j} B_j \int_{\mathbb{R}} e^{-2\pi\gamma z^2} dz
\end{aligned}$$

$$= \frac{1}{\sqrt{2\gamma}} K(N+1)\ \|\phi\|_2^2\ e^{-\gamma\pi\frac{t^2}{2}} \sum_{j=0}^{N} \binom{N}{j} B_j$$

$$= C_1 e^{-\gamma\pi\frac{t^2}{2}},$$

where $C_1 = \frac{K(N+1)}{\sqrt{2\gamma}} \|\phi\|_2^2 \sum_{j=0}^{N} \binom{N}{j} B_j$ and $B_j = \left(\int_{\mathbb{R}} z^{4j} e^{-2(\alpha-\gamma)\pi z^2} dz\right)^{\frac{1}{2}}$. □

We shall now prove Hardy's type theorem for the Fourier transform for connected nilpotent Lie groups having noncompact center. Consider $V_k = [\xi_1 - \frac{1}{2k}, \xi_1 + \frac{1}{2k}]$ for every natural number k, and fix a real number ξ_1. For $m > 2k$, choose a C^∞ function $v_{k,m}$ on real line such that the support of $v_{k,m}$ is contained in V_k, $v_{k,m} = 1$ on $[\xi_1 - 1/2k + 1/m, \xi_1 + 1/2k - 1/m]$ and $0 \le v_{k,m} \le 1$. By the Plancherel inversion theorem, there exists $u_{k,m} \in L^1(\mathbb{R})$ such that $\widehat{u_{k,m}} = v_{k,m}$. For $f \in L^1(G)$, consider $f_{k,m} = u_{k,m} * f$ and define $F_{k,m} : G \to \mathbb{C}$ by

$$F_{k,m}(x) = \int_{\mathbb{T}} (f_{k,m} * f_{k,m}^*)(xz)\ dz,\ x \in G.$$

Next, we modify [2, Lemma 3.1] in order to prove Theorem 3.1.

Lemma 3.5. *Let $f : G \to \mathbb{C}$ be a measurable function satisfying condition (i) of Theorem 3.1. Then*

$$\lim_{k,m\to\infty} kF_{k,m}(e) = 0.$$

Proof. For fix $z, w \in \mathbb{T}$ and $y \in Y$, define

$$E_{k,m}(z, w, y) = \int_{\mathbb{R}} f(t, z, y) \left(\int_{\mathbb{R}} u_{k,m}(s)\overline{(u_{k,m} * f)(t + s, w, y)} ds\right) dt.$$

Then as proved in [2, Lemma 3.1], we have

$$F_{k,m}(e) = \int_Y \int_{\mathbb{T}^2} E_{k,m}(z, w, y) dz\ dw\ dy \tag{3.1}$$

and

$$\begin{aligned} E_k(z, w, y) &= \lim_{m\to\infty} E_{k,m}(z, w, y) \\ &= \int_{\mathbb{R}} f(t, z, y) \int_{\xi_1 - 1/2k}^{\xi_1 + 1/2k} \widehat{u_{k,m}}(s) \widehat{u_{k,m}}(t, s) \overline{\widehat{f}(t + s, w, y)} ds\ dt. \end{aligned}$$

Now $\chi_{V_k}(t + s) = 0$ for all $s \in [\xi_1 - \frac{1}{2k}, \xi_1 + \frac{1}{2k}]$ whenever $t \notin [\frac{-1}{k}, \frac{1}{k}]$, and if $t \in [\frac{-1}{k}, \frac{1}{k}]$, then

$$\chi_{V_k}(t + \cdot) = \chi_{[\xi_1 - t - 1/2k, \xi_1 - t + 1/2k]} \le \chi_{[\xi_1 - 3/2k, \xi_1 + 3/2k]}.$$

Using condition (i) of hypothesis of Theorem 3.1, we compute

$$\begin{aligned} |E_k(z, w, y)| &\le \int_{-1/k}^{1/k} |f(t, z, y)| \left(\int_{\xi_1 - 3/2k}^{\xi_1 + 3/2k} |\widehat{f}(t + s, w, y)| ds\right) dt \\ &\le \frac{3}{k} \|\widehat{f}\|_\infty \int_{-1/k}^{1/k} |f(t, z, y)| dt \end{aligned}$$

$$\leq \frac{3C}{k}\|\widehat{f}\|_\infty \phi(y) \int_{-1/k}^{1/k} (1+t^2)^N e^{-\alpha\pi t^2}\, dt$$
$$\leq \frac{3C}{k^2} 2^{(N+1)} \|\widehat{f}\|_\infty\, \phi(y). \tag{3.2}$$

Therefore, from (3.1) and (3.2), it follows that

$$\begin{aligned}\lim_{m\to\infty} |F_{k,m}(e)| &\leq \int_Y \int_{\mathbb{T}^2} |E_k(z,w,y)| dz\, dw\, dy \\ &\leq \frac{3C}{k^2} 2^{(N+1)} \|\widehat{f}\|_\infty \int_Y \varphi(y)\, dy.\end{aligned}$$

Hence, $\lim\limits_{k,m\to\infty} F_{k,m}(e) = 0$. □

It may be observed that the proof of Theorem 3.1 now follows from the technique used in [2, Theorem 1.1]. For the sake of completeness, we briefly sketch the proof. For fix $\xi_2 \in \mathbb{R}$, from [2], we have

$$\widehat{g}(\xi_2) = \lim_{k\to\infty} \int_{V_k} \left(\int_{X_{\eta_2}} |Pf(\eta)| \cdot \|\pi_\eta(f)\|^2_{\mathrm{HS}}\, d\eta' \right)$$

and

$$\begin{aligned}&\int_{X_{\eta_2}} |Pf(\eta)| \cdot \|\pi_\eta(f)\|^2_{\mathrm{HS}}\, d\eta' \\ &\leq C \sum_{n\in\mathbb{Z}^*} \left(\int_{V_{T}''} |Pf(\eta)|(1+\|\eta\|^2)^N exp(-2\beta(n^2+\eta_2^2+\|\eta''\|^2)) d\eta'' \right),\end{aligned}$$

where $V_T'' = \sum_{i\in T, i>2} \mathbb{R}X_i^*$. Let $0<\delta<\beta$. Since Pf is a polynomial function in η, there exists a constant $K>0$ such that for all $\eta \in \mathcal{W}$

$$|Pf(\eta)|(1+\|\eta\|^2)^N \exp(-2(\beta-\delta)\|\eta\|^2) \leq K.$$

As proved in [2], we have

$$|\widehat{g}(\xi_2)| \leq D \exp(-2\delta\xi_2^2)$$

for all $\xi_2 \in \mathbb{R}$ and $D>0$. By Lemma 3.4, for all $t\in\mathbb{R}$, we have

$$|g(t)| \leq C_1 e^{-\gamma t^2/2}$$

for some $C_1>0$ and $0<\gamma<\alpha$. Since $\alpha\beta>1$, we can choose γ and δ such that $\gamma\delta>1$. Then by Hardy's theorem for $\mathbb{R}$, we get $g=0$ a.e. Indeed, g is the integral of a positive definite function $f_s * f_s^*$ on $\mathbb{R}$, which implies that $f=0$ a.e. and this completes the proof.

We conclude this section by remarking that if G is a connected nilpotent Lie group that has no square integrable irreducible representation and all the coadjoint orbits in $\mathfrak{g}^*$ are flat, then Hardy's type theorem holds for G. Let K be any compact central subgroup of G. Then $H = G/K$ has no square integrable irreducible representation and also satisfies the flat orbit condition. By Lemma 3.3, it is enough to prove Hardy's type theorem for such group H satisfying $H^c = \mathbb{T}$. Then H must have a noncompact center and by Theorem 3.1,

H satisfies Hardy's type theorem. Also in view of [2, Proposition 4.1], it is easy to see that Theorem 3.1 does not hold for nilpotent Lie groups having an irreducible square integrable representation in particular reduced Weyl–Heisenberg group, low-dimensional nilpotent Lie groups $G_{5,1}/\mathbb{Z}$, $G_{5,3}/\mathbb{Z}$, and $G_{5,6}/\mathbb{Z}$. For more details of such groups, one may refer to [20].

4. Analogue of Hardy's theorem for the Gabor transform

In this section, we deal with an analogue of Hardy's theorem for the Gabor transform.

Lemma 4.1. *Let G be a second countable locally compact group. For $f, \psi \in L^2(G)$ and $x \in G$, define $f_\psi^x : G \to \mathbb{C}$ such that*

$$f_\psi^x(y) = f(y)\ \overline{\psi(x^{-1}y)}.$$

If $f_\psi^x = 0$ a.e. for almost all $x \in G$, then either $f = 0$ a.e. or $\psi = 0$ a.e.

Proof. Let us assume that ψ is a nonzero function in $L^2(G)$. There exists a subset M of G with measure zero such that for all $x \in G\backslash M$, $f_\psi^x = 0$ a.e. Indeed $G\backslash M$ is dense in G and G is second countable, so we can take a sequence $(x_j)_{j\in\mathbb{N}}$ contained in $G \setminus M$, which is dense in G. Let

$$V = \left\{ t \in G : |\psi(t)| > \frac{1}{2||\psi||_\infty} \right\}.$$

Then V is a nonempty open subset of G and $\bigcup_{j\in\mathbb{N}} x_j V = G$. Consider the function

$$h(t) = \sum_{j\in\mathbb{N}} \frac{1}{2^j} |\psi(x_j^{-1}t)|, \qquad t \in G.$$

Clearly h is a strictly positive function on G. Moreover,

$$\begin{aligned} 0 \le \int_G |f(t)|h(t)\ dt &= \int_G \sum_{j\in\mathbb{N}} \frac{1}{2^j} |f(t)||\psi(x_j^{-1}t)|\ dt \\ &= \sum_{j\in\mathbb{N}} \frac{1}{2^j} \int_G |f_\psi^{x_j}(t)|\ dt = 0. \end{aligned}$$

Hence, $\int_G |f(t)|h(t)\ dt = 0$, which implies that $f \cdot h = 0$ a.e. Since h is strictly positive, it follows that $f = 0$ a.e. □

Theorem 4.2. *Let f be a measurable function on $\mathbb{R}^n$ such that $|f(x)| \le Ce^{-\alpha\pi\|x\|^2}$ for all $x \in \mathbb{R}^n$ and let ψ be a window function. Also assume that for almost all $y \in \mathbb{R}^n$,*

$$|G_\psi f(y, \xi)| \le \eta_y\ e^{-\beta\pi\|\xi\|^2} \qquad \text{for all } \xi \in \mathbb{R}^n,$$

where α, β, C, and η_y are positive scalars and η_y depends upon y. If $\alpha\beta > 1$, then either $f = 0$ a.e. or $\psi = 0$ a.e.

Proof. For each $y \in \mathbb{R}^n$, define the function $F_y : \mathbb{R}^n \to \mathbb{C}$ such that

$$F_y(x) = f_\psi^y * (f_\psi^y)^*(x).$$

Then for each $\xi \in \mathbb{R}^n$, we have

$$\widehat{F_y}(\xi) = |\widehat{f_\psi^y(\xi)}|^2 = |G_\psi f(y,\xi)|^2 \leq \eta_y^2 \ e^{-2\beta\pi\|\xi\|^2}.$$

Also, for each $x \in \mathbb{R}^n$, we obtain

$$\begin{aligned}
|F_y(x)| &\leq \int_{\mathbb{R}^n} |f_\psi^y(t)| \ |f_\psi^y(t-x)| \ dt \\
&= \int_{\mathbb{R}^n} |f(t)| \ |\psi(t-y)| \ |f(t-x)| \ |\psi(t-x-y)| \ dt \\
&\leq \int_{\mathbb{R}^n} C^2 \ e^{-\alpha\pi\|t\|^2} e^{-\alpha\pi\|t-x\|^2} |\psi(t-y)| \ |\psi(t-x-y)| \ dt \\
&= C^2 \int_{\mathbb{R}^n} e^{-\alpha\pi(\frac{\|x\|^2}{2}+\frac{1}{2}(\|2t-x\|^2))} |\psi(t-y)| \ |\psi(t-y-x)| \ dt \\
&\leq C^2 \ e^{-\alpha\pi\frac{\|x\|^2}{2}} \int_{\mathbb{R}^n} |\psi(t-y)||\psi(t-y-x)| \ dt \\
&= C^2 \ e^{-\alpha\pi\frac{\|x\|^2}{2}} (|\psi| * |\psi|^*)(x) \\
&\leq C^2 \ e^{-\alpha\pi\frac{\|x\|^2}{2}} \| \ |\psi| * |\psi|^*\|_\infty.
\end{aligned}$$

Taking $C_1 = \max\{\eta_y^2, \ C^2 \ \| \ |\psi| * |\psi|^*\|_\infty\}$, then

$$|F_y(x)| \leq C_1 e^{-\alpha\pi\frac{\|x\|^2}{2}} \text{ for all } x \in \mathbb{R}^n$$

and

$$|\widehat{F_y(\xi)}| \leq C_1 e^{-2\beta\pi\|\xi\|^2} \text{ for all } \xi \in \mathbb{R}^n.$$

Using Hardy's theorem for $\mathbb{R}^n$, it follows that $F_y = 0$ for almost all $y \in \mathbb{R}^n$ which further implies that $f_\psi^y = 0$ for almost all $y \in \mathbb{R}^n$. Therefore, using Lemma 4.1, either $f = 0$ a.e. or $\psi = 0$ a.e. □

Theorem 4.3. *Let G be a connected and simply connected nilpotent Lie group with noncompact center. Suppose that $\psi \in C_c(G)$ and that $f \in L^2(G)$ satisfies*

$$\|G_\psi f(x,\pi_\xi)\|_{HS} \leq C_x \ e^{-\pi\beta\|\xi\|^2},$$

where C_x is a positive scalar depending on x. If $\beta > 0$, then either $f = 0$ a.e. or $\psi = 0$ a.e.

Proof. For $y = (y_2, y_3, \ldots, y_n) \in \mathbb{R}^{n-1}$, define a function $f_y : \mathbb{R} \to \mathbb{C}$ such that

$$f_y(x_1) = f(\exp(x_1 X_1 + \sum_{j=2}^n y_j X_j)).$$

For $z \in G$, define a function $F_z : \mathbb{R} \to \mathbb{C}$ given by

$$F_z(x_1) = \int_{\mathbb{R}^{n-1}} (f_\psi^z)_y * (f_\psi^z)_y^*(x_1) \ dy.$$

As $\psi \in C_c(G)$, therefore f_ψ^z has compact support. Moreover,

$$\begin{aligned} F_z(x_1) &= \int_{\mathbb{R}^{n-1}} (f_\psi^z)_y * (f_\psi^z)_y^*(x_1)\, dy \\ &= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} f_\psi^z(t,y)\overline{f_\psi^z(t-x_1,y)}\, dy\, dt \\ &= f_\psi^z * f_\psi^z(x_1,0). \end{aligned}$$

Therefore, F_z is a continuous function with compact support, say K. Choose $\alpha > 0$ such that $\alpha\beta > 1$. Since the function $x_1 \to \exp(-\alpha\pi x_1^2)$ attains minima on K, therefore $r \leq e^{-\pi\alpha x_1^2}$ for some $r > 0$. Also, there exists $C_1 > 0$ such that $|F_z(x_1)| \leq C_1$, for all $x_1 \in \mathbb{R}$. Choose $C' > 0$ satisfying $rC' > C_1$ and therefore for each $x \in K$, we obtain

$$|F_z(x_1)| \leq C_1 < rC' \leq C' e^{-\pi\alpha x_1^2},$$

and for $x_1 \in \mathbb{R} \setminus K$, we have $F_z(x_1) = 0$. Also $f_\psi^z \in L^1 \cap L^2(G)$ and

$$\|\pi_\xi(f_\psi^z)\|_{\mathrm{HS}} \leq \|G_\psi f(x,\pi_\xi)\|_{\mathrm{HS}} \leq C_x e^{-\pi\beta\|\xi\|^2}.$$

Using [18, Lemma 2], we get that $|\widehat{F_z}(\xi_1)| \leq c\, e^{-2\pi\beta\|\xi\|^2}$, for some $c > 0$. Therefore, using Hardy's theorem for the Fourier transform, the function $F_z = 0$ a.e. Since F_z is integral of a positive definite function $(f_\psi^z)_y * (f_\psi^z)_y^*$ on $\mathbb{R}$, therefore $(f_\psi^z)_y = 0$ a.e. This holds for all $z \in G$, which further gives that either $f = 0$ a.e. or $\psi = 0$ a.e. ☐

Corollary 4.4. *Let G be a connected and simply connected nilpotent Lie group. Let $\psi \in C_c(G)$ and $f \in L^2(G)$ such that*

$$\|G_\psi f(x,\pi_\xi)\|_{HS} \leq Ce^{-\pi(a\|x\|^2 + b\|\xi\|^2)/2}$$

for all $(x,\xi) \in G \times \mathcal{W}$, where $a, b,$ and C are positive real numbers. Then either $f = 0$ a.e. or $\psi = 0$ a.e.

5. Beurling theorem

In the next theorem, we prove a result of the Beurling type theorem.

Theorem 5.1. *Let G be a connected and simply connected nilpotent Lie group and let $\psi \in C_c(G)$ and $f \in L^2(G)$ be such that*

$$\int_G \int_{\mathcal{W}} \|G_\psi f(x,\pi_\xi)\|_{HS}\, e^{\pi(\|x\|^2 + \|\xi\|^2)} Pf(\xi)\, dx\, d\xi < \infty. \tag{5.1}$$

Then either $f = 0$ a.e. or $\psi = 0$ a.e.

Proof. From (5.1), there exists a zero set $M \subset G$ such that for all $x \in G \setminus M$, we have

$$\int_{\mathcal{W}} \|G_\psi f(x,\pi_\xi)\|_{\mathrm{HS}}\, e^{\pi(\|x\|^2 + \|\xi\|^2)} Pf(\xi)\, d\xi < \infty. \tag{5.2}$$

For $x \in G \setminus M$, we consider the function f_ψ^x and compute

$$\begin{aligned}
&\int_G \int_{\mathcal{W}} |f_\psi^x(z)| \, \|\widehat{f_\psi^x(\pi_\xi)}\|_{\mathrm{HS}} \, e^{2\pi\|z\|\|\xi\|} Pf(\xi) \, dz \, d\xi \\
&\leq \int_G \int_{\mathcal{W}} |f_\psi^x(z)| \, \|\widehat{f_\psi^x(\pi_\xi)}\|_{\mathrm{HS}} \, e^{\pi(\|z\|^2+\|\xi\|^2)} Pf(\xi) \, dz \, d\xi \\
&= \int_G \int_{\mathcal{W}} |f_\psi^x(z)| \, \|G_\psi f(x,\pi_\xi)\|_{\mathrm{HS}} \, e^{\pi(\|z\|^2+\|\xi\|^2)} Pf(\xi) \, dz \, d\xi \\
&= \int_G |f_\psi^x(z)| e^{\pi\|z\|^2} dz \int_{\mathcal{W}} \|G_\psi f(x,\pi_\xi)\|_{\mathrm{HS}} \, e^{\pi\|\xi\|^2} Pf(\xi) \, d\xi. \qquad (5.3)
\end{aligned}$$

Also,

$$\begin{aligned}
\int_G |f_\psi^x(z)| e^{\pi\|z\|^2} dz &= \int_G |f(z)| |\psi(x^{-1}z)| e^{\pi\|z\|^2} dz \\
&\leq \left(\int_G |f(z)|^2 dz \right)^{1/2} \left(\int_G |\psi(x^{-1}z)|^2 e^{2\pi\|z\|^2} dz \right)^{1/2}. \qquad (5.4)
\end{aligned}$$

As $\psi \in C_c(G)$, so $\psi \cdot e^{\pi\|\cdot\|^2} \in L^2(G)$ and hence $\int_G |f_\psi^x(z)| e^{\pi\|z\|^2} dz < \infty$. Thus, using (5.2), (5.3), and (5.4), we get

$$\int_G \int_{\mathcal{W}} |f_\psi^x(z)| \, \|\widehat{f_\psi^x(\pi_\xi)}\|_{\mathrm{HS}} \, e^{2\pi\|x\|\cdot\|\xi\|} Pf(\xi) \, dz \, d\xi < \infty.$$

Using the Beurling theorem for connected and simply connected nilpotent Lie groups [27], it follows that $f_\psi^x = 0$ a.e. for all $x \in G \setminus M$. Hence, by Lemma 4.1, either $f = 0$ a.e. or $\psi = 0$ a.e. $\square$

Using [1, Theorem 3.1], a careful reading of the proof of the above theorem shows the following result.

Theorem 5.2. *Let G be an exponential solvable Lie group with a nontrivial center, and let $\psi \in C_c(G)$ and $f \in L^2(G)$ such that*

$$\int_G \int_{\mathcal{W}} \|K_\xi G_\psi f(x,\pi_\xi)\|_{HS}^2 \, e^{\pi(\|x\|^2+\|\xi\|^2)} \, dx \, d\xi < \infty,$$

where K_ξ is a semi-invariant operator [1, 2.6]. Then either $f = 0$ a.e. or $\psi = 0$ a.e.

Remark 5.3. Let G be a connected nilpotent Lie group with a square integrable representation. Then as proved in [8, Theorem 5.1], there exist nonzero functions f and ψ in $L^2(G)$ such that for all $x \in G$ and $\xi \in \mathcal{W}$,

$$\|G_\psi f(x,\pi_\xi)\|_{\mathrm{HS}} \leq C e^{-\pi(a\|x\|^2+b\|\xi\|^2)/2},$$

where a and b are nonnegative real numbers with $ab > 1$ and C is a positive constant. For $a, b > 1$, it follows that

$$\int_G \int_{\mathcal{W}} \|G_\psi f(x,\pi_\xi)\|_{\mathrm{HS}} \, e^{\pi(\|x\|^2+\|\xi\|^2)/2} \, Pf(\xi) \, d\xi \, dx < \infty.$$

Thus, the analogue of Beurling theorem does not hold for G. Several examples of such type of group exist including Weyl–Heisenberg group, low-dimensional nilpotent Lie groups $G_{5,1}/\mathbb{Z}$, $G_{5,3}/\mathbb{Z}$, and $G_{5,6}/\mathbb{Z}$. More such examples can be obtained using the following result.

Proposition 5.4. *Let G be a group of the form $G = A \times K \times D$, where A is a connected nilpotent Lie group, K a compact group, and D a type I discrete group. If the Beurling theorem fails for A, then it also fails for G.*

Proof. Since the Beurling theorem fails for A, there exist nonzero functions $f, \psi \in L^2(A)$ such that

$$\int_A \int_{\mathcal{W}} \|G_\psi f(x, \pi_\xi)\|_{\mathrm{HS}}\ e^{\pi(\|x\|^2+\|\xi\|^2)/2} Pf(\xi)\ dx\ d\xi < \infty.$$

Define functions $F, \Psi : G \to \mathbb{C}$ by

$$F(x,k,t) = f(x)\chi_e(t) \qquad \text{and} \qquad \Psi(x,k,t) = \psi(x)\chi_e(t),$$

where e is the identity element of D. Let $\{e_i^\xi\}, \{e_i^\delta\}$, and $\{e_i^\gamma\}$ be orthonormal basis of Hilbert spaces corresponding to the representations π_ξ, δ and γ of A, K, and D, respectively. Then

$$\begin{aligned}&\langle G_\Psi F(x,k,t,\pi_\xi,\delta,\gamma) e_i^\xi \otimes e_m^\delta \otimes e_p^\gamma, e_j^\xi \otimes e_n^\delta \otimes e_q^\gamma\rangle \\ &\qquad\qquad = \begin{cases} \langle G_\psi f(x,\pi_\xi) e_i^\xi, e_j^\xi\rangle & \text{if } t = e \text{ and } \delta \equiv I, \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Also, using [19] or survey in [21], D is a bounded dimensional representation group. So, there exists a positive scalar M such that $\dim(\gamma) \le M$ for all $\gamma \in \widehat{D}$. Therefore, we have

$$\begin{aligned}&\|G_\Psi F(x,k,e,\pi_\xi,I,\gamma)\|_{\mathrm{HS}}^2 \\ &\quad \le \sum_{i,j}\sum_{m,n}\sum_{p,q} |\langle G_\Psi F(x,k,e,\pi_\xi,I,\gamma) e_i^\xi \otimes e_m^\delta \otimes e_p^\gamma, e_j^\xi \otimes e_n^\delta \otimes e_q^\gamma\rangle|^2 \\ &\quad = \sum_{i,j}\sum_{m,n}\sum_{p,q} |\langle G_\psi f(x,\pi_\xi) e_i^\xi, e_j^\xi\rangle|^2 \le M^2 \|G_\psi f(x,\pi_\xi)\|_{\mathrm{HS}}^2.\end{aligned}$$

Thus,

$$\begin{aligned}&\int_A \int_K \sum_{t\in D} \int_{\mathcal{W}} \sum_{\delta \in \widehat{K}} \int_{\widehat{D}} \|G_\psi f(x,k,t,\pi_\xi,\delta,\gamma)\|_{\mathrm{HS}} \\ &\qquad\qquad\qquad\qquad \times e^{\pi(\|x\|^2+\|\xi\|^2)/2} Pf(\xi) dx\ dk\ d\xi\ d\gamma \\ &\quad \le \int_A \int_K \int_{\mathcal{W}} \int_{\widehat{D}} \|G_\psi f(x,k,e,\pi_\xi,I,\gamma)\|_{\mathrm{HS}}\ e^{\pi(\|x\|^2+\|\xi\|^2)/2} Pf(\xi) dx\ dk\ d\xi\ d\gamma \\ &\quad = \int_A \int_{\mathcal{W}} \|G_\psi f(x,\pi_\xi)\|_{\mathrm{HS}}\ e^{\pi(\|x\|^2+\|\xi\|^2)/2} Pf(\xi) dx\ d\xi\ < \infty.\end{aligned}$$

Hence, the Beurling theorem fails for G. □

Remark 5.5. Let G be a compactly generated abelian group. Then by the structure theorem [16, Theorem 9.8], G is topologically isomorphic with $\mathbb{R}^n \times \mathbb{Z}^m \times K$ for some nonnegative integers n, m and some compact abelian group K. Let A be a connected nilpotent Lie group for which Beurling's theorem fails. Then there exist nonzero functions F and $\Psi \in L^2(A \times \mathbb{R}^n)$ such that either

$$\int_A \int_{\mathbb{R}^n} \int_{\mathcal{W}} \int_{\mathbb{R}^n} \|G_\psi f(x,t,\pi_\xi,\gamma_u)\|_{\mathrm{HS}}\ e^{\pi(\|x\|^2+\|t\|^2+\|\xi\|^2)} dx\ dt\ d\xi\ du < \infty \tag{5.5}$$

or

$$\int_A \int_{\mathbb{R}^n} \int_{\mathcal{W}} \int_{\mathbb{R}^n} \|G_\psi f(x,t,\pi_\xi,\gamma_u)\|_{\mathrm{HS}}\ e^{\pi(\|x\|^2+\|\xi\|^2+\|u\|^2)} dx\ dt\ d\xi\ du < \infty. \tag{5.6}$$

Consider the functions $F(x,t) = f(x)e^{-a\|t\|^2}$ and $\Psi(x,t) = \psi(x)e^{-a\|t\|^2}$ for some fixed $a \in \mathbb{R}^+$ and nonzero functions $f, \psi \in L^2(A)$ satisfying

$$\int_A \int_{\mathcal{W}} \|G_\psi f(x,\pi_\xi)\|_{\mathrm{HS}}\ e^{\pi(\|x\|^2+\|\xi\|^2)/2} Pf(\xi)\ dx\ d\xi < \infty.$$

Then, for $a > \pi$, functions F and Ψ satisfy (5.5) and for $a < \pi$, F and Ψ satisfy (5.6). Thus, by Proposition 5.4 and the structure theorem, it follows that if Beurling's theorem fails for the connected nilpotent Lie group A, then the above functions F and Ψ exist on $A \times G$, where G is a compactly generated abelian group.

Next we look at an analogue of Beurling's theorem for the Fourier transform on abelian groups. Let G be a second countable, locally compact, abelian group with dual group $\widehat{G}$. Using the structure theory of abelian groups [16], G decomposes into a direct product $G = \mathbb{R}^n \times S$, where $n \geq 0$ and S contains a compact open subgroup. Hence, the connected component of identity of G is noncompact if and only if $n \geq 1$. Let $G = \mathbb{R}^n \times S$ has a noncompact connected component of identity. The dual group $\widehat{G}$ is identified with $\widehat{G} = \widehat{\mathbb{R}^n} \times \widehat{S}$.

Theorem 5.6. *Let $f \in L^1 \cap L^2(\mathbb{R}^n \times S)$ be such that*

$$\int_{\mathbb{R}^n} \int_S \int_{\mathbb{R}^n} \int_{\widehat{S}} |f(x,s)||\widehat{f}(\xi,\gamma)| e^{2\pi|x\cdot\xi|}\ dx\ ds\ d\xi\ d\gamma < \infty. \tag{5.7}$$

Then $f = 0$ a.e.

Before proving the above theorem, we shall prove some lemmas.

Lemma 5.7. *Let $f \in L^1 \cap L^2(\mathbb{R}^n \times K)$, where K is a compact group satisfying*

$$\int_{\mathbb{R}^n} \int_K \int_{\mathbb{R}^n} \int_{\widehat{K}} |f(x,s)|\ \|\xi \otimes \gamma(f)\|_{HS}\ e^{2\pi|x\cdot\xi|}\ dx\ d\xi\ ds\ d\gamma < \infty.$$

Then $f = 0$ a.e.

Proof. For $\gamma \in \widehat{K}$, let $\mathcal{H}_\gamma$ be the Hilbert space of dimension d_γ with orthonormal basis $\{e_i^\gamma\}_{i=1}^{d_\gamma}$. For fixed e_i^γ and e_j^γ, define $f_\gamma : \mathbb{R}^n \to \mathbb{C}$ such that

$$f_\gamma(x) = \int_K f(x,k)\ \overline{\langle \gamma(k)^* e_i^\gamma, e_j^\gamma \rangle}\ dk.$$

For $\xi \in \mathbb{R}^n$, we obtain

$$\langle \xi \otimes \gamma(f) e_i^\gamma, e_j^\gamma \rangle = \int_{\mathbb{R}^n} \int_K f(x,k) e^{-2\pi i x\cdot\xi}\ \overline{\langle \gamma(k)^* e_i^\gamma, e_j^\gamma \rangle}\ dx\ dk$$

$$= \int_{\mathbb{R}^n} f_\gamma(x) e^{-2\pi i x\cdot\xi}\, dx = \widehat{f_\gamma}(\xi). \tag{5.8}$$

Thus, it follows that

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f_\gamma(x)|\, |\widehat{f_\gamma}(\xi)| e^{2\pi|x\cdot\xi|}\, dx\, d\xi \\ &\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_K |f(x,k)|\, \|\xi\otimes\gamma(f)\|_{\mathrm{HS}}\, e^{2\pi|x\cdot\xi|}\, dx\, dk\, d\xi < \infty.\end{aligned}$$

Hence, using the Beurling theorem for $\mathbb{R}^n$, we get $f_\gamma = 0$ a.e. For fixed $\gamma \in \widehat{K}$ and $\xi \in \mathbb{R}^n$, using (5.8), it follows that $\langle \xi\otimes\gamma(f)e_i^\gamma, e_j^\gamma\rangle = 0$ for all $1 \leq i, j \leq d_\gamma$. Since $\gamma \in \widehat{K}$ and $\xi \in \mathbb{R}^n$ are arbitrarily fixed and $f \in L^1 \cap L^2(G)$, therefore using (2.1), we conclude that $f = 0$ a.e. □

Lemma 5.8. *Let $M = \mathbb{R}^n \times H$ be an open subgroup of an abelian group $G = \mathbb{R}^n \times S$. If $f \in L^1(G)$ satisfies (5.7), then so does $f|_M$.*

Proof. Since $\widehat{S/H}$ is compact and $\widehat{\widehat{S/H}}$ is identified with S/H [16, Theorem 24.2], we have

$$\int_{\widehat{S/H}} \overline{\eta(x)}\, d\eta = \begin{cases} 0 & \text{if } x \notin H, \\ 1 & \text{if } x \in H. \end{cases}$$

Thus,

$$\begin{aligned}\int_{\widehat{S/H}} \widehat{f}(\xi,\chi\eta)\, d\eta &= \int_{\mathbb{R}^n}\int_S f(x,s)e^{-2\pi i\xi x}\, \overline{\chi(s)}\, \left(\int_{\widehat{S/H}} \overline{\eta(s)} d\eta\right) dx\, ds \\ &= \int_{\mathbb{R}^n}\int_H f(x,s) e^{-2\pi i \xi x}\, \overline{\chi(s)}\, dx\, ds = \widehat{f|_M}(\xi, \chi|_M).\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_{\mathbb{R}^n\times H}\int_{\mathbb{R}^n\times\widehat{H}} |f|_M(x,h)|\, |\widehat{f}|_M(\xi,\chi)|\, e^{2\pi|x\cdot\xi|}\, dx\, dh\, d\xi\, d\chi \\ &= \int_{\mathbb{R}^n\times H}\int_{\mathbb{R}^n\times\widehat{H}} |f|_M(x,h)|\, \left|\int_{\widehat{S/H}} \widehat{f}(\xi,\chi\eta)\, d\eta\right|\, e^{2\pi|x\cdot\xi|}\, dx\, dh\, d\xi\, d\chi \\ &\leq \int_{\mathbb{R}^n\times H}\int_{\mathbb{R}^n}\int_{\widehat{H}}\int_{\widehat{S/H}} |f|_M(x,h)|\, |\widehat{f}(\xi,\chi\eta)|\, e^{2\pi|x\cdot\xi|}\, dx\, dh\, d\xi\, d\chi\, d\eta \\ &\leq \int_{\mathbb{R}^n\times S}\int_{\mathbb{R}^n\times\widehat{S}} |f(x,h)|\, |\widehat{f}(\xi,\chi\eta)|\, e^{2\pi|x\cdot\xi|}\, dx\, dh\, d\xi\, d\chi \; < \infty.\end{aligned}$$ □

Using Lemmas 5.7 and 5.8, we now prove Theorem 5.6.

Proof of Theorem 5.6. Let $s \in S$ be arbitrary. If $f \in L^1 \cap L^2(G)$ satisfies the condition of Theorem 5.6, then so does f_s, where $f_s(x,t) = f(x,st)$. Since S has a compact open subgroup K, therefore using Lemmas 5.7 and 5.8, we get $f_s|_{\mathbb{R}^n\times K} = 0$ a.e. Thus, we get $f = 0$ a.e. □

For $z \in G$ and $\omega \in \widehat{G}$, we define the *translation operator* T_z on $L^2(G)$ as

$$(T_z f)(y) = f(z^{-1}y)$$

and the *modulation operator* M_ω on $L^2(G)$ as

$$(M_\omega f)(y) = f(y)\ \omega(y),$$

where $f \in L^2(G)$ and $y \in G$. For $f, \psi \in L^2(G)$, the following property of the Gabor transform can be easily verified:

$$G_\psi(M_\omega T_z f)(x,\gamma) = (\omega^{-1}\gamma)(z^{-1})\ G_\psi f(z^{-1}x, \omega^{-1}\gamma) \tag{5.9}$$

for all $x, z \in G$ and $\gamma, \omega \in \widehat{G}$. In the next result, we give a Beurling theorem version for the Gabor transform on abelian groups by reducing it to the Fourier transform case.

Theorem 5.9. *Let $f \in L^2(G)$ and let ψ be a window function such that*

$$\int_{\mathbb{R}^n}\int_S\int_{\mathbb{R}^n}\int_{\widehat{S}} |G_\psi f(x,s,\xi,\sigma)|\ e^{\pi(\|x\|^2+\|\xi\|^2)/2}\ dx\ ds\ d\xi\ d\sigma < \infty.$$

Then either $f = 0$ a.e. or $\psi = 0$ a.e.

Proof. For $(x,k),(z,t) \in \mathbb{R}^n \times S$ and $(\xi,\gamma),(\zeta,\chi) \in \widehat{\mathbb{R}^n} \times \widehat{S}$, define

$$\begin{aligned}F_{(z,t,\zeta,\chi)}(x,k,\xi,\gamma) = e^{2\pi i\xi x}\ \gamma(k)\ & G_\psi(M_{\zeta,\chi}T_{z,t}f)(x,k,\xi,\gamma)\\ &\times G_\psi(M_{\zeta,\chi}T_{z,t}f)(-x,k^{-1},-\xi,\gamma^{-1}).\end{aligned}$$

The function $F_{(z,t,\zeta,\chi)}$ is continuous and is in $L^1 \cap L^2(\mathbb{R}^n \times S \times \widehat{\mathbb{R}^n} \times \widehat{S})$. Moreover, using [8, Lemma 3.2], we have

$$\widehat{F_{(z,t,\zeta,\chi)}}(\omega,\delta,y,v) = F_{(z,t,\zeta,\chi)}(-y,v^{-1},\omega,\delta). \tag{5.10}$$

Using (5.9), $F_{(z,t,\zeta,\chi)}(x,k,\xi,\gamma)$ can be written as

$$\begin{aligned}&F_{(z,t,\zeta,\chi)}(x,k,\xi,\gamma)\\ &= e^{2\pi i\xi x}\ \gamma(k)\ e^{-2\pi i(\xi-\zeta)z}\ (\chi^{-1}\gamma)(t^{-1})\ G_\psi f(x-z, t^{-1}k, \xi-\zeta, \chi^{-1}\gamma)\\ &\quad \times e^{-2\pi i(-\xi-\zeta)z}\ (\chi^{-1}\gamma^{-1})(t^{-1})\ G_\psi f(-x-z, t^{-1}k^{-1}, -\xi-\zeta, \chi^{-1}\gamma^{-1}).\end{aligned} \tag{5.11}$$

Applying (5.10) and (5.11), we have

$$\begin{aligned}&\int_{\mathbb{R}^n\times S}\int_{\mathbb{R}^n\times\widehat{S}}\int_{\mathbb{R}^n\times S}\int_{\mathbb{R}^n\times\widehat{S}} |F_{(z,t,\zeta,\chi)}(x,k,\xi,\gamma)|\ |\widehat{F_{(z,t,\zeta,\chi)}}(\omega,\delta,y,v)|\\ &\qquad\qquad \times e^{2\pi|x\cdot\omega+\xi\cdot y|}\ dx\ dk\ d\xi\ d\gamma\ d\omega\ d\delta\ dy\ dv\\ &\le \int_{\mathbb{R}^n\times S}\int_{\mathbb{R}^n\times\widehat{S}}\int_{\mathbb{R}^n\times S}\int_{\mathbb{R}^n\times\widehat{S}} |F_{(z,t,\zeta,\chi)}(x,k,\xi,\gamma)||F_{(z,t,\zeta,\chi)}(-y,v^{-1},\omega,\delta)|\\ &\qquad\qquad \times e^{\pi(\|x\|^2+\|\xi\|^2+\|\omega\|^2+\|y\|^2)}\ dx\ dk\ d\xi\ d\gamma\ d\omega\ d\delta\ dy\ dv\\ &= \left(\int_{\mathbb{R}^n\times S}\int_{\mathbb{R}^n\times\widehat{S}} |F_{(z,t,\zeta,\chi)}(x,k,\xi,\gamma)| e^{\pi(\|x\|^2+\|\xi\|^2)}\ dx\ dk\ d\xi\ d\gamma\right)^2\\ &= \Bigg(\int_{\mathbb{R}^n\times S}\int_{\mathbb{R}^n\times\widehat{S}} |G_\psi f(-x-z, t^{-1}k^{-1}, -\xi-\zeta, \gamma^{-1}\chi^{-1})|\\ &\qquad\qquad \times |G_\psi f(x-z, t^{-1}k, \xi-\zeta, \gamma\chi^{-1})| e^{(\|x\|^2+\|\xi\|^2)}\ dx\ dk\ d\xi\ d\gamma\Bigg)^2\end{aligned}$$

$$= \left(\int_{\mathbb{R}^n\times S}\int_{\mathbb{R}^n\times\widehat{S}} |G_\psi f(-x-2z, t^{-2}k^{-1}, -\xi-2\zeta, \gamma^{-1}\chi^{-2})| \right.$$
$$\left. \times\, |G_\psi f(x,k,\xi,\gamma)| e^{\pi(\|x+z\|^2+\|\xi+\zeta\|^2)}\, dx\, dk\, d\xi\, d\gamma\right)^2$$
$$= e^{2\pi(\|z\|^2+\|\zeta\|^2)}(H * H(-2z, t^{-2}, -2\xi, \gamma^{-2}))^2 < \infty,$$

where $H(x,s,\xi,\sigma) = |G_\psi f(x,s,\xi,\sigma)| e^{\pi(\|x\|^2+\|\xi\|^2)/2}$. Thus, using Theorem 5.6, it follows that $F_{(z,t,\zeta,\chi)} \equiv 0$ for all (z,t,ζ,χ). Since,

$$F_{(-z,t^{-1},-\zeta,\chi^{-1})}(0,e,0,I) = e^{4\pi i\zeta z}\ \chi(t)^2\ (G_\psi f(z,t,\zeta,\chi))^2,$$

therefore, $G_\psi f \equiv 0$, which using (2.3) implies that either $f=0$ a.e. or $\psi = 0$ a.e. □

We shall next prove an analogue of Beurling's theorem for the Gabor transform for the groups of the form $\mathbb{R}^n \times K$, when K is a compact group.

Theorem 5.10. *Let $f,\psi \in L^2(\mathbb{R}^n\times K)$, where K is a compact group such that*

$$\int_{\mathbb{R}^n}\int_{\mathbb{K}}\int_{\mathbb{R}^n}\sum_{\gamma\in\widehat{K}} \|G_\psi f(x,k,\xi,\gamma)\|_{HS}\ e^{\pi(\|x\|^2+\|\xi\|^2)/2}\ dx\ dk\ d\xi\ < \infty.$$

Then either $f=0$ a.e. or $\psi=0$ a.e.

Proof. Assume that $\psi \neq 0$. For $\omega,\gamma\in\widehat{K}$, let $\mathcal{H}_\omega$ and $\mathcal{H}_\gamma$ be the Hilbert spaces of dimensions d_ω and d_γ with orthonormal bases $\{e_i^\omega\}_{i=1}^{d_\omega}$ and $\{e_i^\gamma\}_{i=1}^{d_\gamma}$, respectively. For fixed e_r^γ, e_s^γ, we define $\tau : \mathbb{R}^n \to \mathbb{C}$ by

$$\tau(x) = \int_K \psi(x,k)\ \overline{\langle\gamma(k)^* e_r^\gamma, e_s^\gamma\rangle}\ dk.$$

Using the Hölder's inequality, it follows that $\tau\in L^2(\mathbb{R}^n)$. Fix $\gamma\in\widehat{K}$ for which $\tau\neq 0$. For $\sigma\in\widehat{K}$, we can write

$$\gamma(k)e_r^\gamma = \sum_{j=1}^{d_\gamma} C_{j,r}^k e_j^\gamma$$

and

$$\gamma\otimes\sigma = \sum_{\delta\in K_\sigma} m_\delta\ \delta, \tag{5.12}$$

where K_σ is a finite subset of $\widehat{K}$ and $C_{j,r}^k$'s and m_δ's are scalars (see [16]). For fixed e_p^ω and e_q^ω, we define $g : \mathbb{R}^n \to \mathbb{C}$ such that

$$g(x) = \int_K f(x,k)\ \ \overline{\langle\omega(k)^* e_p^\omega, e_q^\omega\rangle}\ dk.$$

Clearly, $g\in L^2(\mathbb{R}^n)$. Consider a function $\varphi : \mathbb{R}^n\times K \to \mathbb{C}$ defined by

$$\varphi(x,k) = \psi(x,k)\ \overline{\langle\gamma(k)^* e_r^\gamma, e_s^\gamma\rangle}.$$

Then $\varphi \in L^2(\mathbb{R}^n \times K)$ and $G_\varphi f(x,k,\xi,\sigma)$ is a Hilbert–Schmidt operator for all $(x,k) \in \mathbb{R}^n \times K$ and for almost all $(\xi,\sigma) \in \widehat{\mathbb{R}^n} \times \widehat{K}$.
For $\sigma \in \widehat{K}$ and fixed e_l^σ, e_m^σ, using [8], we have

$$\langle G_\varphi f(x,k,\xi,\sigma) e_l^\sigma, e_m^\sigma\rangle = \sum_{j=1}^{d_\gamma} \sum_{\delta \in K_\sigma} C_{j,r}^k \; m_\delta \; \langle G_\psi f(x,k,\xi,\delta) e_{l,j}^\delta, e_{m,s}^\delta\rangle.$$

Let $M_\sigma = \max\{|m_\delta| : \delta \in K_\sigma\}$. As $|K_\sigma| \leq d_\gamma d_\sigma < \infty$, we have $M_\sigma < \infty$. Using the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
\|G_\varphi f(x,k,\xi,\sigma)\|_{\mathrm{HS}}^2 &= \sum_{l,m=1}^{d_\sigma} |\langle G_\varphi f(x,k,\xi,\sigma) e_l^\sigma, e_m^\sigma\rangle|^2 \\
&\leq \sum_{l,m=1}^{d_\sigma} \Big(\sum_{j=1}^{d_\gamma} \sum_{\delta \in K_\sigma} |C_{j,r}^k \; m_\delta \; \langle G_\psi f(x,k,\xi,\delta) e_{l,j}^\delta, e_{m,s}^\delta\rangle| \Big)^2 \\
&\leq \sum_{l,m=1}^{d_\sigma} M_\sigma^2 \; |K_\sigma| \; d_\gamma \Big(\sum_{j=1}^{d_\gamma} \sum_{\delta \in K_\sigma} |\langle G_\psi f(x,k,\xi,\delta) e_{l,j}^\delta, e_{m,s}^\delta\rangle|^2 \Big) \\
&\leq \sum_{l,m=1}^{d_\sigma} M_\sigma^2 \; |K_\sigma| \; d_\gamma \; \sum_{j=1}^{d_\gamma} \sum_{\delta \in K_\sigma} \|G_\psi f(x,k,\xi,\delta)\|_{\mathrm{HS}}^2 \\
&\leq d_\sigma^2 \; M_\sigma^2 \; |K_\sigma| \; d_\gamma^2 \Big(\sum_{\delta \in K_\sigma} \|G_\psi f(x,k,\xi,\delta)\|_{\mathrm{HS}} \Big)^2.
\end{aligned}$$

Hence, it follows that

$$\|G_\varphi f(x,k,\xi,\sigma)\|_{\mathrm{HS}} \leq C_{\sigma,\gamma} \sum_{\delta \in K_\sigma} \|G_\psi f(x,k,\xi,\delta)\|_{\mathrm{HS}}, \tag{5.13}$$

where $C_{\sigma,\gamma} = d_\sigma \; M_\sigma \; |K_\sigma| \; d_\gamma$ is a constant depending on σ and γ. Now for every $\sigma \in \widehat{K}$, using (5.13), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} \int_K \int_{\mathbb{R}^n} \|G_\varphi f(x,k,\xi,\sigma)\|_{\mathrm{HS}} \; e^{\pi(\|x\|^2+\|\xi\|^2)/2} \; dx \; dk \; d\xi \\
&\leq C_{\sigma,\gamma} \int_{\mathbb{R}^n} \int_K \int_{\mathbb{R}^n} \sum_{\delta \in K_\sigma} \|G_\psi f(x,k,\xi,\delta)\|_{\mathrm{HS}} \; e^{\pi(\|x\|^2+\|\xi\|^2)/2} \; dx \; dk \; d\xi < \infty.
\end{aligned} \tag{5.14}$$

For $x, \xi \in \mathbb{R}^n$, the function $G_\tau g$ is given by

$$G_\tau g(x,\xi) = \int_K \langle G_\varphi f(x,k,\xi,\omega) e_p^\omega, e_q^\omega\rangle \; dk.$$

Thus,

$$|G_\tau g(x,\xi)| \leq \int_K \|G_\varphi f(x,k,\xi,\omega)\|_{\mathrm{HS}} \; dk.$$

On using (5.14), it follows

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |G_\tau g(x,\xi)| e^{\pi(\|x\|^2+\|\xi\|^2)/2}\, dx\, d\xi$$
$$\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_K \|G_\varphi f(x,k,\xi,\omega)\|_{\mathrm{HS}}\; e^{\pi(\|x\|^2+\|\xi\|^2)/2}\, dx\, d\xi\, dk < \infty.$$

Then by the Beurling theorem for the Gabor transform on $\mathbb{R}^n$ (see [14]) or Theorem 5.9 above, we conclude that $g = 0$ a.e. Since $\omega \in \widehat{K}$ is arbitrary, we get $f = 0$ a.e. □

Remark 5.11. Using Theorem 5.2, the above theorem can be proved for the group $G \times K$, where G is an exponential solvable Lie group with a nontrivial center and K is a compact group in the following setting:
Let $f \in L^2(G \times K)$ and $\psi \in C_c(G \times K)$ such that

$$\int_G\int_K\int_{\mathcal{W}} \sum_{\gamma\in\widehat{K}} \|K_\xi G_\psi f(x,k,\pi_\xi,\gamma)\|^2_{\mathrm{HS}}\; e^{\pi(\|x\|^2+\|\xi\|^2)}\, dx\, dk\, d\xi \; < \infty.$$

Then either $f = 0$ a.e. or $\psi = 0$ a.e.

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