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WHEN NILPOTENCE IMPLIES THE ZERONESS OF LINEAR OPERATORS

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Abstract. We give conditions forcing nilpotent operators (everywhere bounded or closed) to be null. More precisely, it is mainly shown that any closed or everywhere defined bounded nilpotent operator with a positive (self-adjoint) real part is automatically null. Some other interesting examples and results accompany our results.

1. Introduction

First, we assume that readers have some familiarity with the standard notions and results in matrix and operator theories (see, e.g., [3,16]), as well as unbounded operators (see [25] for the needed notions, cf. [19]).

Let H be a Hilbert space and let $B(H)$ be the algebra of all bounded linear operators defined from H into H. Recall that $T \in B(H)$ is said to be positive, symbolically $T \geq 0$, if $\langle Tx, x\rangle \geq 0$ for all $x \in H$. Recall also that any T may always be expressed as $T = A + iB$ with $A, B \in B(H)$ being both self-adjoint and $i = \sqrt{-1}$. Necessarily, $A = (T + T^*)/2$, which will be denoted by $\operatorname{Re} T$, and it is called the real part of T. Also, $B = (T - T^*)/2i$ is the imaginary part of T, written $\operatorname{Im} T$.

As is well known, the nilpotence plays an important role in matrix theory, and in operator theory in general. The following result was shown in [18].

Proposition 1.1. *If $T \in B(H)$ is such that $\operatorname{Re} T \geq 0$ and $T^2 = 0$, then $T = 0$.*

In this paper, we carry on this investigation and deal with the general case.

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We recall a few well-established facts. For example, if $T \in B(H)$ is normal, then

$$\|T^n\| = \|T\|^n \quad \text{for all } n \in \mathbb{N}.$$

It seems noteworthy to emphasize that thanks to the previous equality, if T is nilpotent, then "$T = 0 \Leftrightarrow T$ is normal". Therefore, when we furthermore assume that $\operatorname{Re} T \geq 0$ and prove Theorem 2.1 below, then this will become yet another characterization to be added to the 89 conditions equivalent to the normality of a matrix already obtained in [9] and [11]. A somehow related paper is [10].

The second main topic of the paper deals with (unbounded) closed operators. Hence let us recall briefly some notions about non-necessarily bounded operators.

If S and T are two linear operators with domains $D(S)$ and $D(T)$, respectively, then T is said to be an extension of S, written as $S \subset T$, whenever $D(S) \subset D(T)$ and S and T coincide on $D(S)$.

The product ST and the sum $S + T$ of two operators S and T are defined in the usual fashion on the natural domains:

$$D(ST) = \{x \in D(T) : \ Tx \in D(S)\}$$

and

$$D(S + T) = D(S) \cap D(T).$$

When $\overline{D(T)} = H$, we say that T is densely defined. In such case, the adjoint T^* exists and is unique. If $S \subset T$ and S is densely defined, then T too is densely defined and $T^* \subset S^*$.

An operator T is called closed if its graph is closed in $H \oplus H$. If T is densely defined, then we say that T is self-adjoint when $T = T^*$; symmetric if $T \subset T^*$; normal if T is *closed* and $TT^* = T^*T$. A symmetric operator T is called positive if

$$\langle Tx, x\rangle \geq 0 \quad \text{for all } x \in D(T).$$

Note that unlike positive operators in $B(H)$, an unbounded positive operator need not be self-adjoint.

In the event of the density of all of $D(S)$, $D(T)$, and $D(ST)$, then

$$T^*S^* \subset (ST)^*,$$

with equality occurring when $S \in B(H)$. Also, when S, T, and $S+T$ are densely defined, then

$$S^* + T^* \subset (S + T)^*,$$

and the equality holds if $S \in B(H)$.

The real and imaginary parts of a densely defined operator T are defined, respectively, by

$$\operatorname{Re} T = \frac{T + T^*}{2} \text{ and } \operatorname{Im} T = \frac{T - T^*}{2i}.$$

Clearly, if T is closed, then $\operatorname{Re} T$ is symmetric, but it is not always self-adjoint (it may even fail to be closed).

Definition 1.2 ([21])**.** Let T be a densely defined operator with domain $D(T) \subset H$. If there exist densely defined symmetric operators A and B with domains $D(A)$ and $D(B)$, respectively, and such that

$$T = A + iB \text{ with } D(A) = D(B),$$

then T is said to have a Cartesian decomposition.

Remark 1.3. A densely defined operator T admits a Cartesian decomposition if and only if $D(T) \subset D(T^*)$. In this case, $T = A + iB$, where

$$A = \operatorname{Re} T \text{ and } B = \operatorname{Im} T.$$

2. Bounded case

The first result tells us that a (nonzero) operator T with a positive (or negative) real or imaginary part is never nilpotent. It may be known to some readers especially when $\dim H < \infty$. The proof when $\dim H = \infty$ here relies on the finite-dimensional case. In the next section, we generalize this result to closed operators.

Theorem 2.1. *Let $T = A + iB \in B(H)$ and let $n \geq 2$. If $T^n = 0$ and $A \geq 0$ (or $B \geq 0$), then $T = 0$.*

Proof. The proof is carried out in two steps.

(1) Let $\dim H < \infty$. The proof uses a trace argument. First, assume that $A \geq 0$. Clearly, the nilpotence of T does yield $\operatorname{tr} T = 0$. Hence

$$0 = \operatorname{tr}(A + iB) = \operatorname{tr} A + i \operatorname{tr} B.$$

Since A and B are self-adjoint, we know that $\operatorname{tr} A, \operatorname{tr} B \in \mathbb{R}$. By the above equation, this forces $\operatorname{tr} B = 0$ and $\operatorname{tr} A = 0$. The positiveness of A now intervenes to make $A = 0$. Therefore, $T = iB$ and so T is normal. Thus, as alluded above,

$$0 = \|T^n\| = \|T\|^n,$$

thereby, $T = 0$.

In the event $B \geq 0$, the reason as above obtains $T = A$ and so $T = 0$, as wished.

(2) Let $\dim H = \infty$. The condition $\operatorname{Re} T \geq 0$ is being equivalent to $\operatorname{Re}\langle Tx, x\rangle \geq 0$ for all $x \in H$. Hence if E is a closed invariant subspace of T, then the previous condition also holds for $T|E : E \to E$.

Now, we proceed to show that $T = 0$, that is, we must show that $Tx = 0$ for all $x \in H$. Therefore, let $x \in H$ and let E be the span of $x, Tx, \dots, T^{n-1}x$ (that is, the orbit of x under the action of T). Hence E is a finite-dimensional subspace of H (and so it is equally a Hilbert space). By the nilpotence assumption, we have

$$T^n x = 0,$$

from which it follows that E is invariant for T. Hence, by the first part of the proof (the finite-dimensional case), we know that $T = 0$ on E whereby $Tx = 0$. As this holds for any x, it follows that $T = 0$ on H, as needed. ∎

Remark 2.2. For example, the condition $A \geq 0$ may not just be dropped. Indeed, if $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then $T^2 = 0$, but $T \neq 0$. Observe finally that

$$A = \operatorname{Re} T = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is neither positive nor negative for $\sigma(A) = \{-1/2, 1/2\}$.

Remark 2.3. As mentioned above, the power of Theorem 2.1 lies in the fact that it easily allows us to test the non-nilpotence of a given operator. For example, let V be the Volterra's operator defined on $L^2(0,1)$, that is,

$$Vf(x) = \int_0^x f(t)dt, \quad f \in L^2(0,1).$$

Then, it is well known that V is not nilpotent. Let us corroborate this fact using Theorem 2.1. Since $\operatorname{Re} V \geq 0$ (see, e.g., [16, Exercise 9.3.21]), assuming the nilpotence of V would make $V = 0$, and this is impossible. Thus, V is not nilpotent.

The previous example also tells us that the assumption may not be weakened to quasinilpotence (recall that quasinilpotence means that spectrum is reduced to the singleton $\{0\}$).

Here is an alternative reformulation of Theorem 2.1 over finite-dimensional spaces.

Corollary 2.4. *Let $T \in M_n(\mathbb{C})$ be nilpotent (with $T \neq 0$). Then $(T + T^*)/2$ (or $(T - T^*)/2i$) has at least two eigenvalues of opposite signs.*

In many results in operator theory, the asymmetric condition $\sigma(A) \cap \sigma(-A) \subseteq \{0\}$ yields similar conclusions as when assuming the positivity of A (for instance, it was used in [1] to define the square root of A^2, where A is self-adjoint). It is also known that this asymmetric condition is weaker than positiveness (and negativeness) of A.

Nonetheless, we have the following result.

Theorem 2.5. *Let H be a Hilbert space of dimension k, where $k = 2$ or $k = 3$. Let $T = A + iB \in B(H)$ be nilpotent. If $\sigma(A) \cap \sigma(-A) = \{0\}$ or $\sigma(B) \cap \sigma(-B) = \{0\}$, then $T = 0$.*

Proof. (1) Let $k = 2$. As above, we may obtain that $\operatorname{tr} A = 0$. Since A is self-adjoint, it follows that A is similar to $\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}$, where $\alpha \in \mathbb{R}$. Hence, if $\alpha \neq 0$, then $\sigma(A) \cap \sigma(-A) = \{0\}$ will be violated. Thence, $\alpha = 0$, that is, $A = 0$. Consequently, we obtain $T = 0$ as above. The corresponding case for B can be dealt with similarly.

(2) Assume now that $k = 3$. If $\sigma(A) \cap \sigma(-A) = \{0\}$, then in view of the self-adjointness of A, we know that A is similar to $\begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & -\alpha \end{pmatrix}$ (where $\alpha \in \mathbb{R}$)

given that $\operatorname{tr} A = 0$ and $0 \in \sigma(A)$. As before, we must necessarily have $\alpha = 0$ and so $A = 0$. The nilpotence of $T = iB$ then gives $T = 0$. $\square$

When $\dim H = 4$, a similar idea is just not applicable. Let us therefore give a counterexample.

Example 2.6. Take

$$T = \begin{pmatrix} 2 & 2 & -2 & 0 \\ 5 & 1 & -3 & 0 \\ 1 & 5 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and so } A = \begin{pmatrix} 2 & 7/2 & -1/2 & 0 \\ 7/2 & 1 & 1 & 0 \\ -1/2 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence (approximatively)

$$\sigma(A) = \{0, -3.71, -1.33, 5.04\}$$

and so $\sigma(A) \cap \sigma(-A) = \{0\}$ is trivially satisfied. Observe finally that $T \neq 0$ whereas $T^3 = 0$, that is, T is nilpotent.

We may easily prove the following result.

Proposition 2.7. *Let H be a four-dimensional Hilbert space. Let $T = A + iB \in B(H)$ be nilpotent. If $\sigma(A) \cap \sigma(-A) = \{0\}$ with 0 being an eigenvalue of multiplicity 2, then $T = 0$.*

Proof. Just write

$$A \sim \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & -\alpha \end{pmatrix}$$

and obtain $A = 0$. Hence $T = iB$ and so $T = 0$, as above. $\square$

We also have the following related result.

Proposition 2.8. *If A is a self-adjoint 2×2 matrix such that $\sigma(A) \cap \sigma(-A) = \varnothing$, then $T = A + iB$ is never nilpotent.*

Proof. If T were nilpotent, then $\operatorname{tr} A = 0$. This would necessarily make A look like $\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}$ (with $\alpha \in \mathbb{R}$). This condition is, however, not consistent with $\sigma(A) \cap \sigma(-A) = \varnothing$. Thus, T cannot be nilpotent. $\square$

Remark 2.9. Finally, note that there are nilpotent matrices $T = A + iB$ of higher order such that $\sigma(A) \cap \sigma(-A) = \varnothing$. We may just consider the nonzero block matrix from Example 2.6.

3. Unbounded case

We confine our attention now to the case of unbounded nilpotent operators. We choose to use Ôta's definition in [20] of nilpotence (note that Ôta gave the definition in the case $n = 2$).

Definition 3.1. Let T be a non-necessarily bounded operator with a dense domain $D(T)$. We say that T is nilpotent if T^n is well defined and

$$T^n = 0 \text{ on } D(T)$$

for some $n \in \mathbb{N}$ (hence $D(T^n) = D(T^{n-1}) = \cdots = D(T)$).

Thanks to the following lemma, there are not any unbounded self-adjoint *nilpotent* operators!

Lemma 3.2 ([24])**.** *If H and K are two Hilbert spaces and if $T : D(T) \subset H \to K$ is a densely defined closed operator, then*

$$D(T) = D(T^*T) \Longleftrightarrow T \in B(H, K).$$

Since for a normal T, $D(T^*T) = D(T^2)$, we see that there are not any unbounded normal *nilpotent* operators either. Hence, it is natural to ask is whether there are unbounded closed symmetric unbounded operators? The answer is still negative! In fact, any densely defined closed nilpotent operator T with $D(T) \subset D(T^*)$ is everywhere bounded.

Proposition 3.3. *Let T be a densely defined closed nilpotent operator with domain $D(T)$ such that $D(T) \subset D(T^*) \subset H$. Then $T \in B(H)$.*

In particular, if T is a closed densely defined nilpotent symmetric or hyponormal operator, then $T = 0$ everywhere on H.

Proof. Let T be a densely defined closed operator with domain $D(T) \subset H$ such that $T^n = 0$ on $D(T)$ for some n and $D(T) \subset D(T^*)$. It is seen that

$$D(T) = D(T^2) \subset D(T^*T) \subset D(T)$$

whereby $D(T^*T) = D(T)$. Since T is closed, Lemma 3.2 yields $T \in B(H)$.

The last statement of the proposition follows from the general theory. Indeed, when $T \in B(H)$, then T is self-adjoint if and only if it is symmetric. Accordingly, $T = 0$ since $T^n = 0$ everywhere on H. The case of hyponormality is also known to readers. $\square$

Remark 3.4. The previous result may be reformulated as follows: Any closed densely defined nilpotent operator having a Cartesian decomposition is necessarily everywhere bounded.

As alluded above, Theorem 2.1 remains valid in the context of closed operators.

Theorem 3.5. *Let $T = A + iB$, where either A or B is positive with $D(T) \subset D(T^*)$. If T is nilpotent, then $T \in B(H)$ is normal thereby $T = 0$ everywhere on H.*

Proof. Since $T^n = 0$ on $D(T)$ for some natural integer n, we have $D(T^2) = D(T)$. The reason as in Proposition 3.3 obtains $T \in B(H)$. We have thus gone back to the setting of Theorem 2.1, that is, we obtain $T = 0$, as wished. $\square$

Before stating and proving the last result in this paper, we give some auxiliary results, which are also interesting in their own. Note that they might well be known to specialists, however, they are not documented (to the best of our knowledge).

It is worth noting in passing that there are unbounded self-adjoint operators A and B such that $A+iB \subset 0$ (where 0 designates the zero operator on all of H), yet $A \not\subset 0$ and $B \not\subset 0$. For example, let A and B be unbounded self-adjoint operators such that $D(A) \cap D(B) = \{0_H\}$ (see, e.g., [14]). Assuming $D(A) = D(B)$ makes the whole difference.

Proposition 3.6. *Let A and B be two densely defined symmetric operators with domains $D(A), D(B) \subset H$, respectively. Assume that $D(A) = D(B)$. If $A+iB \subset 0$, then $A \subset 0$ and $B \subset 0$. If A (or B) is furthermore taken to be closed, then $A = B = 0$ everywhere on H.*

Proof. By the assumption, $A + iB \subset 0$. Since $D(A) = D(B)$, it ensures that $A = -iB$. Indeed A and B are both symmetric, and so the only possible outcome is $A \subset 0$ and $B \subset 0$.

Since $A \subset 0$, it follows that $A^* = 0$ everywhere on H. By the closedness of A, we obtain $A = 0$. A similar reasoning applies to B because $A = -iB$ makes B closed and so $B = 0$ as well. □

It is known that the pointwise commutativity of unbounded (self-adjoint or normal) operators does not always mean their strong commutativity (witness Nelson's or Schmüdgen's counterexamples). Recall here that the pointwise commutativity of two unbounded (self-adjoint) operators A and B means that AB coincides with BA on some common dense domain. The strong commutativity of A and B signifies the commutativity of their spectral measures. Hence, the next result on (strong) commutativity might be unknown to some readers.

Proposition 3.7 (cf. [15])**.** *Let A and B be two unbounded self-adjoint operators with domains $D(A)$ and $D(B)$, respectively. Assume that A is also positive and that $D(A) = D(B)$. If $BA \subset AB$, then A commutes strongly with B.*

The proof is based on the following auxiliary result.

Lemma 3.8 (see [23, Proposition 5.27])**.** *Let A and B be self-adjoint unbounded operators. Then A commutes strongly with B if and only if $(A - \lambda I)^{-1}B \subset B(A - \lambda I)^{-1}$ for any $\lambda \in \rho(A)$ ($\rho(A)$ being the resolvent set of A).*

Now, we prove Proposition 3.7.

Proof of Proposition 3.7. By the hypothesis, $BA \subset AB$. Hence $B(A + I) \subset (A + I)B$ because

$$D[B(A + I)] \subset D[(A + I)B].$$

Since A is self-adjoint and positive, it results that $A+I$ is boundedly invertible. Left and right multiplying by $(A + I)^{-1}$ yield

$$(A + I)^{-1}B \subset B(A + I)^{-1}.$$

By Lemma 3.8, this means that A commutes strongly with B, which completes the proof. □

As readers are aware, the condition $D(T^2) = D(T)$ is strong. Why not call a densely defined operator T nilpotent when $T^n \subset 0$ for a certain n? The main issue would be that it is quite conceivable to have T^n defined only at 0; see, for example, [6, 7, 17, 19, 22] (cf. [2, 5]). A recent somewhat related paper [8] might be of some interest to readers.

Let us treat this case anyway.

Theorem 3.9. *Let $T = A + iB$, where A and B are self-adjoint (one of them is also positive), with $D(A) = D(B)$ and $D(BA) \subset D(AB)$. If $T^2 \subset 0$, then $T \in B(H)$ is normal, and so $T = 0$ everywhere on H.*

Proof. Assume that A is positive (the proof in the case of the positiveness of B is similar). Let $T = A + iB$. Clearly,

$$A^2 - B^2 + i(AB + BA) \subset (A + iB)A + i(A + iB)B = T^2 \subset 0.$$

Since $D(A) = D(B)$, it follows that

$$D(A^2) = \{x \in D(A) : Ax \in D(A)\} = \{x \in D(A) : Ax \in D(B)\} = D(BA).$$

In a similar manner, it is seen that $D(B^2) = D(AB)$. Thus,

$$D(A^2 - B^2) = D(AB + BA).$$

We also have $D(BA) \subset D(AB)$. Accordingly,

$$D(A^2 - B^2) = D(AB + BA) = D(BA) = D(A^2).$$

Since A is self-adjoint, so is A^2 and in particular A^2 is necessarily densely defined. Thus, $A^2 - B^2$ and $AB + BA$ are both densely defined. Now, by the symmetricity (only) of both A and B, we have that both $AB + BA$ and $A^2 - B^2$ are symmetric. By Proposition 3.6, we get $AB + BA \subset 0$. Hence $BA \subset -AB$ (for $D(BA) \subset D(AB)$) and so

$$BA^2 \subset -ABA \subset A^2B.$$

As A is positive, we obtain $BA \subset AB$ by say [4]. By Proposition 3.7, we have that A commutes strongly with B. Whence T is normal. Hence T^2 too is normal, and so by using maximality, $T^2 \subset 0$ becomes $T^2 = 0$ everywhere on H. Consequently, $D(T^2) = H$ and hence $D(T) = H$. Since T is closed, it follows by the closed graph theorem that $T \in B(H)$. Finally, $T = 0$ follows by the normality of T, as needed. □

Remark 3.10. Another way of obtaining $A = B = 0$ in the result above (and without using Proposition 3.7) reads: Since B is self-adjoint, $B = U|B| = |B|U$, where U is unitary and self-adjoint, that is, $U^2 = I$ and $U^* = U$ (see, e.g., [12] or [13]).

As above, we may obtain $BA \subset -AB$ and $A^2 - B^2 \subset 0$. Since $D(BA) = D(A^2)$ and $D(AB) = D(B^2)$, we get $A^2 \subset B^2$. Since A^2 and B^2 are both self-adjoint, a maximality argument yields $A^2 = B^2$ which, upon passing to the unique positive square root, implies that $A = |B|$ as A is positive. Hence $B = UA = AU$. Therefore

$$UA^2 = U|B|A = BA \subset -AB = -UA^2.$$

Hence

$$U^2A^2 = A^2 \subset -U^2A^2 = -A^2.$$

Thus, $A^2 = -A^2$ and so $A = 0$, thereby $B = 0$ as well.

A variant of Theorem 3.9 reads as follows.

Theorem 3.11. *Let A and B be two self-adjoint operators (one of them is also positive) such that $D(A^2) = D(B^2)$. If $T = A + iB$ and $T^2 \subset 0$, then $T = 0$ everywhere on H.*

Proof. We need only go back to the assumptions of Theorem 3.9. Without loss of generality, assume that A is positive. Since B is self-adjoint, B^2 is self-adjoint and positive. Then, by [25, Theorem 9.4], we obtain

$$D(A) = D(\sqrt{A^2}) = D(\sqrt{B^2}) = D(|B|) = D(B)$$

by invoking the closedness of B and the positiveness of A. Hence

$$D(BA) = D(A^2) = D(B^2) = D(AB).$$

Therefore, we have recovered all of the assumptions of Theorem 3.9. The remaining parts of the proof stay unchanged. □

We finish with an example showing the importance of the self-adjointness of A and B.

Example 3.12. There is a densely defined closed symmetric positive operator A such that $T := A + iA$ obeys $T^2 \subset 0$ yet $T \not\subset 0$.

To obtain such an example, recall that Chernoff [6] obtained a densely defined closed, symmetric and positive operator A such that $D(A^2) = \{0\}$. Now, let $B = A$ and set $T = A + iA = (1 + i)A$. Then

$$D(T^2) = D(A^2) = \{0\}$$

and so $T^2 \subset 0$ trivially. Observe in the end that $T \not\subset 0$, that is, T does not vanish on $D(T)$. Note in the end that neither A nor B were self-adjoint.

Note that the idea of using the polar decomposition in the remark below the proof of Theorem 3.9, comes from one of the readers of the paper. This was suggested when the assumption $D(BA) = D(AB)$ was made instead of $D(BA) \subset D(AB)$ in an earlier version of that theorem. Then, we saw the improvement using the assumption $D(BA) \subset D(AB)$.

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