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DESCRIPTION OF J-SETS AND C-SETS BY MATRICES

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Abstract. We redefine the notion of J-sets in a commutative semigroup S with the help of matrices whose entries are functions from the natural numbers into S. We show that our definition of J-sets is equivalent to the standard definition of J-sets. We also introduce a new notion of C-set using matrices whose entries are functions from the natural numbers into S.

1. Introduction

Let $(S,+)$ be an infinite semigroup. The collection of all ultrafilters on S is denoted by βS. When provided with the topology generated by

$$\{\overline{A}=\{p\in\beta S : A\in p\} : A\subseteq S\},$$

βS is called the Stone-Čech compactification of the discrete space S. There exists a unique extension of the operation to βS such that $(\beta S,+)$ is a compact right topological semigroup with S contained in its topological center. That is, $\rho_q:\beta S\to\beta S$ defined by $\rho_q(p)=p+q$ is continuous for each $q\in\beta S$, and $\lambda_x:\beta S\to\beta S$ defined by $\lambda_x(p)=x+p$ is continuous for each $x\in S$. Also for $p,q\in\beta S$ and $A\subseteq S$, $A\in p+q$ if and only if $\{x\in S:\lambda_x^{-1}(A)\in q\}\in p$. For more details, see [3].

If $(S,+)$ is a compact Hausdorff right topological semigroup, then $K(S)$ is the smallest two-sided ideal of S. Also an idempotent x in $K(S)$ is called a minimal idempotent. Moreover, L is called a left ideal if and only if $S+L\subseteq L$, where $S+L=\{x+y:x\in S,y\in L\}$, and L is called a minimal left ideal if L is a minimal set of the collection of all left ideals with respect to inclusion.

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Let $(S,+)$ be a commutative semigroup and let

$$\mathcal{V}=\left\{\left\langle\langle y_{i,t}\}_{t=1}^{\infty}\right\rangle_{i=1}^{\infty}: \text{ for all } i,t\in\mathbb{N}\,,\, y_{i,t}\in S\right\}.$$

If $Y=\left\langle\langle y_{i,t}\}_{t=1}^{\infty}\right\rangle_{i=1}^{\infty}\in\mathcal{V}$, then a set $A\subseteq S$ is called a J_Y-set if and only if for each $n\in\mathbb{N}$, there exist $a\in S$ and $H\in\mathcal{P}_f(\mathbb{N})$ with $\min H>n$ such that $a+\sum_{t\in H}y_{i,t}\in A$ for each $i\in\{1,2,\ldots,n\}$. This definition was stated by Hindman, Maleki, and Strauss in 1996. They showed that J_Y-sets are partition regular sets and that $J_Y=\{p\in\beta S:$ for all $A\in p, A$ is a J_Y-set $\}$ is a closed two-sided ideal of βS; see [2, Definition 2.4 and Theorem 2.6].

In 2008, the concept of J-set has been derived from the concept of J_Y-set by De, Hindman, and Strauss; see [1, Definition 3.3]. Later, in 2010, they restated the concept of J-sets. In their definition, ${}^{\mathbb{N}}S$ is the set of sequences in S. Also $\mathcal{P}_f(X)$ denotes the set of finite nonempty subsets of X.

Definition 1.1. Let $(S,+)$ be a commutative semigroup and let $A\subseteq S$. Then A is a J-set if and only if whenever $F\in\mathcal{P}_f({}^{\mathbb{N}}S)$, there exist $a\in S$ and $H\in\mathcal{P}_f(\mathbb{N})$ such that $a+\sum_{t\in H}f(t)\in A$ for all $f\in F$.

In 2015, Johnson stated a simpler version of the central sets theorem with a simpler definition of J-sets for a noncommutative semigroup; see [4, Definition 2.1].

Let $(S,+)$ be a commutative semigroup and let $\mathcal{J}$ be the collection of all J-sets. Then $\mathcal{J}$ is partition regular and $J(S)=\{p\in\beta S:$ for all $A\in p, A$ is a J-set$\}$ is a closed ideal of βS; see [3, Theorem 14.14.4].

Definition 1.2. Let $(S,+)$ be a commutative semigroup and let $A\subseteq S$. Then A is a C-set if and only if there exist functions $\alpha:\mathcal{P}_f({}^{\mathbb{N}}S)\to S$ and $H:\mathcal{P}_f({}^{\mathbb{N}}S)\to\mathcal{P}_f(\mathbb{N})$ such that

(a) whenever $F,G\in\mathcal{P}_f({}^{\mathbb{N}}S)$ and $F\subsetneq G$, then $\max H(F)<\min H(G)$ and
(b) whenever $r\in\mathbb{N}$ and $G_1,G_2,\ldots,G_r\in\mathcal{P}_f({}^{\mathbb{N}}S)$ such that $G_1\subsetneq G_2\subsetneq\cdots\subsetneq G_r$ and $f_i\in G_i$, for each $i\in\{1,\ldots,r\}$, one has $\sum_{i=1}^{r}\left(\alpha(G_i)+\sum_{t\in H(G_i)}f_i(t)\right)\in A$.

If e is an idempotent in $J(S)$, then every element of e is a C-set; see [3, Theorem 14.14.9].

In this article, $M_{n\times m}({}^{\mathbb{N}}S)$ denotes the collection of all $m\times n$ matrices with entries from ${}^{\mathbb{N}}S$ for $n,m\in\mathbb{N}$, and we define $\overline{a}=(a,\ldots,a)\in S^n$ for $a\in S$.

By the definition of J-set, $A\subseteq S$ is J-set if for each $F=\{f_1,\ldots,f_k\}\in\mathcal{P}_f({}^{\mathbb{N}}S)$, there exist $H=\{n_1,\ldots,n_m\}$ and $a\in S$ such that $a+\sum_{t\in H}f(t)\in A$ for each $f\in F$. Therefore, for $F=\{f_1,\ldots,f_k\}$, $a\in S$, and $H=\{n_1,\ldots,n_m\}$, we have

$$\begin{bmatrix} a\\ a\\ \vdots\\ a\end{bmatrix}_{k\times 1}+\begin{bmatrix} f_1 & f_1 & \ldots & f_1\\ f_2 & f_2 & \ldots & f_2\\ \vdots & \vdots & \ddots & \vdots\\ f_k & f_k & \ldots & f_k\end{bmatrix}_{k\times m}\begin{bmatrix} n_1\\ n_2\\ \vdots\\ n_m\end{bmatrix}_{m\times 1}=\begin{bmatrix} a+\sum_{t\in H} & f_1(t)\\ a+\sum_{t\in H} & f_2(t)\\ & \vdots\\ a+\sum_{t\in H} & f_k(t)\end{bmatrix}_{k\times 1}\in A^k. \tag{1}$$

Therefore, we have a matrix description for the concept of J-set, and the above way leads to a new version of the definition of J-sets.

Definition 1.3. Let $(S,+)$ be a commutative semigroup and let $m \in \mathbb{N}$. We say that $A \subseteq S$ is a J_m-set if and only if whenever $F \in \mathcal{P}_f\big(M_{1\times m}({}^{\mathbb{N}}S)\big)$, there exist $a \in S$ and $H \in \mathcal{P}_f(\mathbb{N}^m)$ such that $a + \sum_{\mathbf{t}\in H} M\mathbf{t} \in A$ for each $M \in F$.

In the above definition, $M\mathbf{t} = \sum_{i=1}^{m} f_i(t_i)$ for each $M = [f_1 \ \dots \ f_m] \in F$ and $\mathbf{t} = (t_1, \dots, t_m) \in H$. In this paper, we show that A is a J-set if and only if A is a J_m-set. Also, we define C_m-set and prove that C-sets are C_m-sets.

2. Matrix description of J-sets

We now describe the relationship between J-set and J_m-set.

Theorem 2.1. *Let $(S,+)$ be a commutative semigroup, let $A \subseteq S$, and let $m \in \mathbb{N}$. Then A is a J-set if and only if A is a J_m-set.*

Proof. Necessity. Let $F = \{M_1, M_2, \dots, M_k\} \in \mathcal{P}_f\big(M_{1\times m}({}^{\mathbb{N}}S)\big)$, where each $M_i = \begin{pmatrix} f_{i,1} & f_{i,2} & \dots & f_{i,m} \end{pmatrix}$. For $i \in \{1, 2, \dots, k\}$, let $g_i = \sum_{j=1}^{m} f_{i,j}$. Then $\{g_1, g_2, \dots, g_k\} \in \mathcal{P}_f({}^{\mathbb{N}}S)$, so pick $a \in S$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that $a + \sum_{t\in H} g_i(t) \in A$ for $i \in \{1, 2, \dots, k\}$. Given $t \in H$, let $\mathbf{s}_t = \begin{pmatrix} t & t & \dots & t \end{pmatrix} \in \mathbb{N}^m$. Then $\{\mathbf{s}_t : t \in H\} \in \mathcal{P}_f(\mathbb{N}^m)$ and $a + \sum_{t\in H} M_i\mathbf{s}_t = a + \sum_{t\in H} g_i(t) \in A$ for $i \in \{1, 2, \dots, k\}$.

Sufficiency. Pick $b \in S$, and define $\bar{b} \in {}^{\mathbb{N}}S$ by $\bar{b}(t) = b$ for each $t \in \mathbb{N}$. Let $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$. For $f \in F$, let $M_f = \begin{pmatrix} f & \bar{b} & \dots & \bar{b} \end{pmatrix} \in M_{1\times m}({}^{\mathbb{N}}S)$. Then $\{M_f : f \in F\} \in \mathcal{P}_f\big(M_{1\times m}({}^{\mathbb{N}}S)\big)$. Pick $a \in S$ and $H \in \mathcal{P}_f(\mathbb{N}^m)$ such that $a + \sum_{\mathbf{t}\in H} M_f\mathbf{t} \in A$ for each $f \in F$. Given $\mathbf{t} \in H$, we have$M_f\mathbf{t} = f(t_1) + (m-1)b$. Hence let $K = \{t_1 : \mathbf{t} \in H\}$ and let $c = a + (m-1)b$. Then for $f \in F$, $c + \sum_{t\in K} f(t) \in A$. □

For $H \in \mathcal{P}_f(\mathbb{N}^m)$, we let $\max H = \max\{\max \mathbf{r} : \mathbf{r} \in H\}$ and $\min H = \min\{\min \mathbf{r} : \mathbf{r} \in H\}$.

Definition 2.2. Let $(S,+)$ be a commutative semigroup and let $B \subseteq S$. Then B is a C_m-set if and only if there exist functions $\alpha : \mathcal{P}_f\big(M_{1\times m}({}^{\mathbb{N}}S)\big) \to S$ and $H : \mathcal{P}_f\big(M_{1\times m}({}^{\mathbb{N}}S)\big) \to \mathcal{P}_f(\mathbb{N}^m)$ such that

(a) if $F, G \in \mathcal{P}_f\big(M_{1\times m}({}^{\mathbb{N}}S)\big)$ and $F \subsetneq G$, then $\max H(F) < \min H(G)$, and

(b) whenever $r \in \mathbb{N}$, $G_1, G_2, \dots, G_r \in \mathcal{P}_f\big(M_{1\times m}({}^{\mathbb{N}}S)\big)$ such that $G_1 \subsetneq G_2 \subsetneq \dots \subsetneq G_r$ and $A_i \in G_i$ for each $i \in \{1, \dots, r\}$, one has $\sum_{i=1}^{r}\big(\alpha(G_i) + \sum_{\mathbf{t}\in H(G_i)} A_i\mathbf{t}\big) \in B$.

Theorem 2.3. *Let $(S,+)$ be a commutative semigroup and let $m \in \mathbb{N}$. Then every C-set is a C_m-set.*

Proof. The conclusion is trivial if $m = 1$, so assume that $m \geq 2$, and pick functions $\alpha : \mathcal{P}_f({}^{\mathbb{N}}S) \to S$ and $H : \mathcal{P}_f({}^{\mathbb{N}}S) \to \mathcal{P}_f(\mathbb{N})$ such that

(a) whenever $F, G \in \mathcal{P}_f({}^{\mathbb{N}}S)$ and $F \subsetneq G$, then $\max H(F) < \min H(G)$ and

(b) whenever $r \in \mathbb{N}$ and $G_1, G_2, \ldots, G_r \in \mathcal{P}_f({}^{\mathbb{N}}S)$ such that $G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_r$ and $f_i \in G_i$ for each $i \in \{1, \ldots, r\}$, one has $\sum_{i=1}^{r} \big(\alpha(G_i) + \sum_{t \in H(G_i)} f_i(t)\big) \in A$.

Given $M = \begin{pmatrix} f_1 & \ldots & f_m \end{pmatrix} \in G$ such that $G \in \mathcal{P}_f\big(M_{1\times m}({}^{\mathbb{N}}S)\big)$, let $g_M := \sum_{j=1}^{m} f_i$. Now we define $\tau : \mathcal{P}_f\big(M_{1\times m}({}^{\mathbb{N}}S)\big) \to \mathcal{P}_f({}^{\mathbb{N}}S)$ by induction on $|G|$ for $G \in \mathcal{P}_f\big(M_{1\times m}({}^{\mathbb{N}}S)\big)$. If $G = \{M\}$, then set $\tau(G) = \{g_M\}$. Now assume that $G \in \mathcal{P}_f\big(M_{1\times m}({}^{\mathbb{N}}S)\big)$, that $|G| > 1$, and that $\tau(F)$ is defined whenever $\varnothing \neq F \subsetneq G$. Pick $f \in {}^{\mathbb{N}}S \setminus \bigcup\{\tau(F) : \varnothing \neq F \subsetneq G\}$ and let

$$\tau(G) = \bigcup\{\tau(F) : \varnothing \neq F \subsetneq G\} \cup \{g_M : M \in G\} \cup \{f\}\,.$$

Then whenever $F, G \in \mathcal{P}_f\big(M_{1\times m}({}^{\mathbb{N}}S)\big)$ and $F \subsetneq G$, one has $\tau(F) \subsetneq \tau(G)$. For $G \in \mathcal{P}_f\big(M_{1\times m}({}^{\mathbb{N}}S)\big)$, let $\alpha'(G) = \alpha\big(\tau(G)\big)$ and $H'(G) = \{\bar{t} : t \in H(\tau(G))\}$.

Now assume that $F, G \in \mathcal{P}_f\big(M_{1\times m}({}^{\mathbb{N}}S)\big)$ and that $F \subsetneq G$. Then $\tau(F) \subsetneq \tau(G)$, so $\max H'(F) = \max H\big(\tau(F)\big) < \min H\big(\tau(G)\big) = \min H'(F)$.

Next assume that $r \in \mathbb{N}$, that $G_1, \ldots, G_r \in \mathcal{P}_f\big(M_{1\times m}({}^{\mathbb{N}}S)\big)$, that $G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_r$, and that $M_i \in G_i$ for $i \in \{1, 2, \ldots, r\}$. Then $\tau(G_1) \subsetneq \tau(G_2) \subsetneq \cdots \subsetneq \tau(G_r)$, and for $i \in \{1, 2, \ldots, r\}, g_{M_i} \in \tau(G_i)$, so

$$\sum_{i=1}^{r}(\alpha'(G_i) + \sum_{\mathbf{t} \in H'(G_i)} M_i\mathbf{t}) = \sum_{i=1}^{r}(\alpha(\tau(G_i)) + \sum_{t \in H(\tau(G_i))} g_{M_i}(t)) \in A.$$

□

Question: Let $m \in \mathbb{N} \setminus \{1\}$. Is every C_m-set a C-set?

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