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A NONMONOTONIC EXPLICIT PROXIMAL-LIKE METHOD FOR SOLVING EQUILIBRIUM PROGRAMMING WITH CONVEX CONSTRAINTS

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Abstract. We propose a new proximal-type method to solve equilibrium problems in a real Hilbert space. The new method is analogous to the famous two-step extragradient method that is used to solve variational inequalities in the Hilbert spaces. The proposed iterative scheme uses a new non-monotone step size rule based on local bifunction information instead of any line search method. A strong convergence theorem for the proposed method is well-established by taking mild conditions on a bifunction. The applications of the main results to solve fixed point problems and variational inequalities are presented. Finally, we examine two test problems for computational experiments and demonstrate the validity and effectiveness of the proposed method.

1. Introduction

Let $\mathcal{C}$ be a nonempty, closed and convex subset of a real Hilbert space $\mathcal{H}$. Let $f : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be a bifunction with $f(y, y) = 0$ for each $y \in \mathcal{C}$ and a *equilibrium problem* (EP) for bifunction f on $\mathcal{C}$ is defined in the following manner: Find $u^* \in \mathcal{C}$ such that

$$f(u^*, y) \geq 0, \quad \text{for all } y \in \mathcal{C}. \tag{EP}$$

In this study, the equilibrium problem is studied based on the following conditions.

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(**C1**) *pseudomonotone* on a set $\mathcal{C}$, that is, the bifunction f satisfying (see for more details [4,5])

$$f(y_1, y_2) \geq 0 \Longrightarrow f(y_2, y_1) \leq 0, \quad \text{for all } \ y_1, y_2 \in \mathcal{C};$$

(**C2**) *Lipschitz-type continuous* on $\mathcal{C}$ (see [15]), that is, for the bifunction $f : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$, there exist two constants $k_1, k_2 > 0$ such that

$$\begin{aligned} &f(y_1, y_3) \\ &\leq f(y_1, y_2) + f(y_2, y_3) + k_1\|y_1 - y_2\|^2 + k_2\|y_2 - y_3\|^2, \quad \text{for all } \ y_1, y_2, y_3 \in \mathcal{C}; \end{aligned}$$

(**C3**) For any weakly convergent $\{y_n\} \subset \mathcal{C}$ ($y_n \rightharpoonup y^*$), the following inequality holds:

$$\limsup_{n \to +\infty} f(y_n, y) \leq f(y^*, y), \quad \text{for all } \ y \in \mathcal{C};$$

(**C4**) $f(y, \cdot)$ is convex and subdifferentiable on $\mathcal{H}$ for each $y \in \mathcal{H}$.

The general format of an equilibrium problem greatly interests researchers because it includes a variety of mathematical problems such as fixed-point problems, scalar and vector minimization problems, complementarity problems, variational inequalities problems, Nash equilibrium problems in non-cooperative games, saddle point problems, and inverse minimization problems [5, 11, 18] and iterative methods in [2, 16, 17, 21–24, 26, 27]. The equilibrium problem has applications in economics [7] or the dynamics of offer and demand [1] and continues to exploit the theoretical structure of non-cooperative games and Nash's equilibrium idea [19, 20]. To the best of our knowledge, the term "equilibrium problem" was first used in 1992 by Muu and Oettli [18] and has since been thoroughly researched by Blum and Oettli [5].

Flam and Antipin [8] and Tran, Dung, and Nguyen [29] extended the Korpelevich extragradient method and proposed the extragradient method for solving equilibrium problems involving monotone and Lipschitz-type bifunctions. Choose a random starting point $u_0 \in \mathcal{C}$; based on the given iterate u_n, choose the next iteration using the following scheme:

$$\begin{cases} y_n = \arg\min\limits_{y \in \mathcal{C}} \{\chi f(u_n, y) + \frac{1}{2}\|u_n - y\|^2\}, \\ u_{n+1} = \arg\min\limits_{y \in \mathcal{C}} \{\chi f(y_n, y) + \frac{1}{2}\|u_n - y\|^2\}, \end{cases} \tag{1.1}$$

where $0 < \chi < \min\left\{\frac{1}{2k_1}, \frac{1}{2k_2}\right\}$ and k_1 and k_2 are two Lipschitz-type constants of a bifunction. It is important to note that the above well-established method carries two serious drawbacks. The first is the constant step size that involves the knowledge or approximation of the Lipschitz constant of the related bifunction, and it only converges weakly in Hilbert spaces. From the computational point of view, it might be problematic to use a fixed step size, and hence the convergence rate and appropriateness of the method could be affected.

As a result, a natural question arises:

Is it possible to develop a new strongly convergent extragradient method with a non-monotone step size rule to determine the numerical solution of problem (EP) *involving a pseudomonotone bifunction?*

In this paper, we provide a positive answer to this question by demonstrating that the gradient method is still valid in the case of nonmonotonic step size rules for solving equilibrium problems accompanied by pseudomonotone bifunctions and maintains strong convergence. Motivated by the works in [14, 29], we design a new extragradient-type method to solve problem (EP) in the context of an infinite-dimensional real Hilbert space. (i) We introduce a self-adaptive subgradient extragradient method for solving equilibrium problems that employ a non-monotone step size rule, and we demonstrate that the generated sequence is strongly convergent. These results are regarded as the modification of the method (1.1). (ii) We numerically investigate Algorithm 1 with [9, Algorithm 2] and [25, Algorithm 1]. The numerical results show that the proposed method is appropriate and outperforms the existing ones.

2. Preliminaries

Let $\mathcal{C}$ be a nonempty, closed, and convex subset of a real Hilbert space $\mathcal{H}$. The *metric projection* $P_{\mathcal{C}}(u)$ of $u \in \mathcal{H}$ onto a closed and convex subset $\mathcal{C}$ of $\mathcal{H}$ is defined by $P_{\mathcal{C}}(u) = \arg\min_{y \in \mathcal{C}} \|y - u\|$. Next, some useful properties of the metric projection are given.

Lemma 2.1 ([12])**.** *Let $P_{\mathcal{C}} : \mathcal{H} \to \mathcal{C}$ be a metric projection. Then it satisfies the following properties:*

(i) $\|y_1 - P_{\mathcal{C}}(y_2)\|^2 + \|P_{\mathcal{C}}(y_2) - y_2\|^2 \leq \|y_1 - y_2\|^2,\ y_1 \in \mathcal{C}, y_2 \in \mathcal{H}$;
(ii) $y_3 = P_{\mathcal{C}}(y_1)$ *if and only if* $\langle y_1 - y_3, y_2 - y_3\rangle \leq 0,$ *for all* $y_2 \in \mathcal{C}$;
(iii) $\|y_1 - P_{\mathcal{C}}(y_1)\| \leq \|y_1 - y_2\|,\ y_2 \in \mathcal{C}, y_1 \in \mathcal{H}$.

Definition 2.2. Let $\mathcal{C}$ be a subset of a real Hilbert space $\mathcal{H}$ and let $\varkappa : \mathcal{C} \to \mathbb{R}$ be a given convex function.

(1) The *subdifferential of set* $\varkappa$ at $u \in \mathcal{C}$ is defined by
$$\partial\varkappa(u) = \{z \in \mathcal{H} : \varkappa(y) - \varkappa(u) \geq \langle z, y - u\rangle, \text{ for all } y \in \mathcal{C}\}.$$
(2) The *normal cone* at $u \in \mathcal{C}$ is defined by
$$N_{\mathcal{C}}(u) = \{z \in \mathcal{H} : \langle z, y - u\rangle \leq 0, \text{ for all } y \in \mathcal{C}\}.$$

Lemma 2.3 ([28])**.** *Suppose that $\varkappa : \mathcal{C} \to \mathbb{R}$ is a convex, subdifferentiable, and lower semi-continuous function on $\mathcal{C}$. An element $u \in \mathcal{C}$ is a minimizer of a function $\varkappa$ if and only if $0 \in \partial\varkappa(u) + N_{\mathcal{C}}(u)$, where $\partial\varkappa(u)$ stands for the subdifferential of $\varkappa$ at $u \in \mathcal{C}$ and $N_{\mathcal{C}}(u)$ is the normal cone of $\mathcal{C}$ at u.*

Lemma 2.4 ([30])**.** *Suppose that $\{a_n\} \subset (0, +\infty)$ is a sequence satisfying*
$$a_{n+1} \leq (1 - b_n)a_n + b_n\eta_n, \text{ for all } n \in \mathbb{N}.$$
Moreover, $\{b_n\} \subset (0,1)$ and $\{\eta_n\} \subset \mathbb{R}$ are sequences such that $\lim_{n\to+\infty} b_n = 0$, $\sum_{n=1}^{+\infty} b_n = +\infty$ and $\limsup_{n\to+\infty} \eta_n \leq 0$. Then, $\lim_{n\to+\infty} a_n = 0$.

Lemma 2.5 ([13])**.** *Let $\{a_n\} \subset \mathbb{R}$ be a sequence, and let there exist a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_{i+1}}$ for all $i \in \mathbb{N}$. Then, there exists a nondecreasing* *sequence* $m_k \subset \mathbb{N}$ *such that* $m_k \to +\infty$ *as* $k \to +\infty$, *and the subsequent conditions are fulfilled by all (sufficiently large) numbers* $k \in \mathbb{N}$:

$$a_{m_k} \leq a_{m_{k+1}} \quad \text{and} \quad a_k \leq a_{m_{k+1}}.$$

Indeed, $m_k = \max\{j \leq k : a_j \leq a_{j+1}\}$.

Lemma 2.6 ([3])**.** *For all* $y_1, y_2 \in \mathcal{H}$ *and* $\Im \in \mathbb{R}$, *the following subsequent relationships hold:*

(i) $\|\Im y_1 + (1-\Im)y_2\|^2 = \Im\|y_1\|^2 + (1-\Im)\|y_2\|^2 - \Im(1-\Im)\|y_1 - y_2\|^2$;
(ii) $\|y_1 + y_2\|^2 \leq \|y_1\|^2 + 2\langle y_2, y_1 + y_2\rangle$.

3. Main results

Next, we introduce a variant of Algorithm (1.1) in which the constant step size χ is chosen adaptively and thus yields a sequence χ_n that improves the performance of the method.

Algorithm 1

Step 0: Choose $u_0 \in \mathcal{C}$, $\mu \in (0,1)$, $\chi_0 > 0$, $0 < \sigma < \min\left\{1, \frac{1}{2k_1}, \frac{1}{2k_2}\right\}$, $\{\phi_n\} \subset (a,b) \subset (0, 1-\varphi_n)$, and $\{\varphi_n\} \subset (0,1)$ satisfies the conditions

$$\lim_{n\to+\infty} \varphi_n = 0 \text{ and } \sum_{n=1}^{+\infty} \varphi_n = +\infty.$$

Step 1: Compute

$$y_n = \arg\min_{y\in\mathcal{C}}\{\chi_n f(u_n, y) + \frac{1}{2}\|u_n - y\|^2\}.$$

If $u_n = y_n$, then STOP. Otherwise go to **Step 2**.

Step 2: Firstly choose $\omega_n \in \partial_2 f(u_n, y_n)$ satisfying $u_n - \chi_n\omega_n - y_n \in N_{\mathcal{C}}(y_n)$ and then create a half-space $\mathcal{H}_n = \{z \in \mathcal{H} : \langle u_n - \chi_n\omega_n - y_n, z - y_n\rangle \leq 0\}$ and compute

$$z_n = \arg\min_{y\in\mathcal{H}_n}\{\chi_n f(y_n, y) + \frac{1}{2}\|u_n - y\|^2\}.$$

Step 3: Compute

$$u_{n+1} = (1 - \phi_n - \varphi_n)u_n + \phi_n z_n.$$

Step 4: Next, the step size rule χ_{n+1} is evaluated as follows:

$$\chi_{n+1} = \begin{cases} \min\left\{\sigma, \frac{\mu f(y_n,z_n)}{f(u_n,z_n)-f(u_n,y_n)-k_1\|u_n-y_n\|^2-k_2\|z_n-y_n\|^2+1}\right\}, \\ \qquad \text{if } \frac{\mu f(y_n,z_n)}{f(u_n,z_n)-f(u_n,y_n)-k_1\|u_n-y_n\|^2-k_2\|z_n-y_n\|^2+1} > 0, \\ \sigma \qquad \textit{otherwise}. \end{cases} \tag{3.1}$$

Put $n := n+1$ and move back to **Step 1**.

Remark 3.1. By the use of χ_{n+1} in (3.1), we obtain

$$\chi_{n+1}\left[f(u_n, z_n) - f(u_n, y_n) - k_1\|u_n - y_n\|^2 - k_2\|y_n - z_n\|^2\right] \leq \mu f(y_n, z_n).$$

Theorem 3.2. *Assume that the conditions* (**C1**)–(**C4**) are satisfied. Then, the sequence $\{u_n\}$ generated by Algorithm 1 strongly converges to an element $u^* = P_{EP(f,\mathcal{C})}(0)$.

Proof. First, start to prove the boundedness of the sequence $\{u_n\}$. By the use of the definition of z_n, we have

$$0 \in \partial_2 \Big\{ \chi_n f(y_n, y) + \frac{1}{2}\|u_n - y\|^2 \Big\}(z_n) + N_{\mathcal{C}}(z_n).$$

Therefore, there exist $\omega_n \in \partial_2 f(y_n, z_n)$ and $\overline{\omega}_n \in N_{\mathcal{C}}(z_n)$ such that

$$\chi_n \omega_n + z_n - u_n + \overline{\omega}_n = 0.$$

The above relationship implies that

$$\langle u_n - z_n, y - z_n \rangle = \chi_n \langle \omega_n, y - z_n \rangle + \langle \overline{\omega}_n, y - z_n \rangle, \quad \text{for all } \ y \in \mathcal{C}.$$

Then $\overline{\omega}_n \in N_{\mathcal{C}}(z_n)$ implies that $\langle \overline{\omega}_n, y - z_n \rangle \le 0$, for every $y \in \mathcal{C}$. Thus, we have

$$\chi_n \langle \omega_n, y - z_n \rangle \ge \langle u_n - z_n, y - z_n \rangle, \quad \text{for all } \ y \in \mathcal{C}. \tag{3.2}$$

Given $\omega_n \in \partial_2 f(y_n, z_n)$, we obtain

$$f(y_n, y) - f(y_n, z_n) \ge \langle \omega_n, y - z_n \rangle, \quad \text{for all } \ y \in \mathcal{C}. \tag{3.3}$$

From expressions (3.2) and (3.3), we have

$$\chi_n f(y_n, y) - \chi_n f(y_n, z_n) \ge \langle u_n - z_n, y - z_n \rangle, \quad \text{for all } \ y \in \mathcal{C}. \tag{3.4}$$

In a similar way, y_n gives that

$$\chi_n \big\{ f(u_n, y) - f(u_n, y_n) \big\} \ge \langle u_n - y_n, y - y_n \rangle, \quad \text{for all } \ y \in \mathcal{C}. \tag{3.5}$$

By the use of $y = z_n$ into (3.5), we have

$$\chi_n \big\{ f(u_n, z_n) - f(u_n, y_n) \big\} \ge \langle u_n - y_n, z_n - y_n \rangle. \tag{3.6}$$

By the use of $y = u^*$ into (3.4), we obtain

$$\chi_n f(y_n, u^*) - \chi_n f(y_n, z_n) \ge \langle u_n - z_n, u^* - z_n \rangle. \tag{3.7}$$

Since $u^* \in S_{EP}$, so $f(u^*, y_n) \ge 0$ and the pseudomonotonicity of a bifunction f provides $f(y_n, u^*) \le 0$. Thus, expression (3.7) implies that

$$\langle u_n - z_n, z_n - u^* \rangle \ge \chi_n f(y_n, z_n). \tag{3.8}$$

From expression (3.1), we have

$$f(y_n, z_n) \ge \chi_{n+1} \Big[f(u_n, z_n) - f(u_n, y_n) - k_1 \|u_n - y_n\|^2 - k_2 \|y_n - z_n\|^2 \Big]. \tag{3.9}$$

Combining (3.8) and (3.9) provides that

$$\begin{aligned} \langle u_n - z_n, z_n - u^* \rangle \ge \chi_{n+1} \Big[& \chi_n \big\{ f(u_n, z_n) - f(u_n, y_n) \big\} \\ & - k_1 \chi_n \|u_n - y_n\|^2 - k_2 \chi_n \|z_n - y_n\|^2 \Big]. \end{aligned} \tag{3.10}$$

From expressions (3.6) and (3.10), we obtain

$$2\langle u_n - z_n, z_n - u^*\rangle \geq \chi_{n+1}\Big[2\langle u_n - y_n, z_n - y_n\rangle - 2k_1\chi_n\|u_n - y_n\|^2 - 2k_2\chi_n\|z_n - y_n\|^2\Big].$$

By the use of following formulas:

$$2\langle u_n - z_n, z_n - u^*\rangle = \|u_n - u^*\|^2 - \|z_n - u_n\|^2 - \|z_n - u^*\|^2,$$

$$2\langle u_n - y_n, z_n - y_n\rangle = \|u_n - y_n\|^2 + \|z_n - y_n\|^2 - \|u_n - z_n\|^2,$$

we have

$$\|z_n - u^*\|^2 \leq \|u_n - u^*\|^2 - (1 - \chi_{n+1})\|z_n - u_n\|^2 - \chi_{n+1}(1 - 2k_1\chi_n)\|u_n - y_n\|^2 - \chi_{n+1}(1 - 2k_2\chi_n)\|z_n - y_n\|^2. \tag{3.11}$$

Since $0 < \chi_n < \min\left\{1, \frac{1}{2k_1}, \frac{1}{2k_2}\right\}$ for all $n \geq 1$. Thus, we have

$$(1 - \chi_{n+1}) > 0,\ (1 - 2k_1\chi_n) > 0 \text{ and } (1 - 2k_2\chi_n) > 0, \quad \text{for all } n \geq 1.$$

Thus, expression (3.11) implies that

$$\|z_n - u^*\|^2 \leq \|u_n - u^*\|^2, \quad \text{for all } n \geq 1. \tag{3.12}$$

Given that $u^* \in EP(f, \mathcal{C})$, we obtain

$$\begin{aligned}\big\|u_{n+1} - u^*\big\| &= \big\|(1 - \phi_n - \varphi_n)u_n + \phi_n z_n - u^*\big\| \\ &= \big\|(1 - \phi_n - \varphi_n)(u_n - u^*) + \phi_n(z_n - u^*) - \varphi_n u^*\big\| \\ &\leq \big\|(1 - \phi_n - \varphi_n)(u_n - u^*) + \phi_n(z_n - u^*)\big\| + \varphi_n\big\|u^*\big\|.\end{aligned} \tag{3.13}$$

Next, we estimate the following:

$$\begin{aligned}&\big\|(1 - \phi_n - \varphi_n)(u_n - u^*) + \phi_n(z_n - u^*)\big\|^2 \\ &= (1 - \phi_n - \varphi_n)^2\big\|u_n - u^*\big\|^2 + \phi_n^2\big\|z_n - u^*\big\|^2 + 2\big\langle(1 - \phi_n - \varphi_n)(u_n - u^*), \phi_n(z_n - u^*)\big\rangle \\ &\leq (1 - \phi_n - \varphi_n)^2\big\|u_n - u^*\big\|^2 + \phi_n^2\big\|z_n - u^*\big\|^2 + 2\phi_n(1 - \phi_n - \varphi_n)\big\|u_n - u^*\big\|\big\|z_n - u^*\big\| \\ &\leq (1 - \phi_n - \varphi_n)^2\big\|u_n - u^*\big\|^2 + \phi_n^2\big\|z_n - u^*\big\|^2 \\ &\quad + \phi_n(1 - \phi_n - \varphi_n)\big\|u_n - u^*\big\|^2 + \phi_n(1 - \phi_n - \varphi_n)\big\|z_n - u^*\big\|^2 \\ &\leq (1 - \phi_n - \varphi_n)(1 - \varphi_n)\big\|u_n - u^*\big\|^2 + \phi_n(1 - \varphi_n)\big\|z_n - u^*\big\|^2.\end{aligned} \tag{3.14}$$

Substituting (3.12) into (3.14), we obtain

$$\begin{aligned}&\big\|(1 - \phi_n - \varphi_n)(u_n - u^*) + \phi_n(z_n - u^*)\big\|^2 \\ &\leq (1 - \phi_n - \varphi_n)(1 - \varphi_n)\big\|u_n - u^*\big\|^2 + \phi_n(1 - \varphi_n)\big\|u_n - u^*\big\|^2 \\ &= (1 - \varphi_n)^2\big\|u_n - u^*\big\|^2.\end{aligned} \tag{3.15}$$

Therefore we have

$$\big\|(1 - \phi_n - \varphi_n)(u_n - u^*) + \phi_n(z_n - u^*)\big\| \leq (1 - \varphi_n)\big\|u_n - u^*\big\|. \tag{3.16}$$

Combining (3.13) and (3.16), we get

$$\begin{aligned}\|u_{n+1}-u^*\| &\le (1-\varphi_n)\|u_n-u^*\|+\varphi_n\|u^*\|\\ &\le \max\Big\{\|u_n-u^*\|,\|u^*\|\Big\}\\ &\le \max\Big\{\|u_1-u^*\|,\|u^*\|\Big\}.\end{aligned}$$

Thus, the above expression implies that $\{u_n\}$ is bounded sequence.
Next, our aim is to prove that the sequence $\{u_n\}$ is strongly convergent. Indeed, by the use of the definition of $\{u_{n+1}\}$, we have

$$\begin{aligned}\|u_{n+1}-u^*\|^2 &= \|(1-\phi_n-\varphi_n)u_n+\phi_n z_n-u^*\|^2\\ &= \|(1-\phi_n-\varphi_n)(u_n-u^*)+\phi_n(z_n-u^*)-\varphi_n u^*\|^2\\ &= \|(1-\phi_n-\varphi_n)(u_n-u^*)+\phi_n(z_n-u^*)\|^2+\varphi_n^2\|u^*\|^2\\ &\quad -2\big\langle(1-\phi_n-\varphi_n)(u_n-u^*)+\phi_n(z_n-u^*),\,\varphi_n u^*\big\rangle. \qquad (3.17)\end{aligned}$$

By the use of (3.14), we have

$$\begin{aligned}&\|(1-\phi_n-\varphi_n)(u_n-u^*)+\phi_n(z_n-u^*)\|^2\\ &\le (1-\phi_n-\varphi_n)(1-\varphi_n)\|u_n-u^*\|^2+\phi_n(1-\varphi_n)\|z_n-u^*\|^2. \qquad (3.18)\end{aligned}$$

Combining (3.17) and (3.18) (for some $K_2>0$), we get

$$\begin{aligned}&\|u_{n+1}-u^*\|^2\\ &\le (1-\phi_n-\varphi_n)(1-\varphi_n)\|u_n-u^*\|^2+\phi_n(1-\varphi_n)\|z_n-u^*\|^2+\varphi_n K_2\\ &\le (1-\phi_n-\varphi_n)(1-\varphi_n)\|u_n-u^*\|^2+\varphi_n K_2\\ &\quad +\phi_n(1-\varphi_n)\Big[\|u_n-u^*\|^2-(1-\chi_{n+1})\|z_n-u_n\|^2\\ &\quad -\chi_{n+1}(1-2k_1\chi_n)\|u_n-y_n\|^2-\chi_{n+1}(1-2k_2\chi_n)\|z_n-y_n\|^2\Big]\\ &= (1-\varphi_n)^2\|u_n-u^*\|^2+\varphi_n K_2\\ &\quad -\phi_n(1-\varphi_n)\Big[(1-\chi_{n+1})\|z_n-u_n\|^2\\ &\quad +\chi_{n+1}(1-2k_1\chi_n)\|u_n-y_n\|^2+\chi_{n+1}(1-2k_2\chi_n)\|z_n-y_n\|^2\Big]\\ &\le \|u_n-u^*\|^2+\varphi_n K_2\\ &\quad -\phi_n(1-\varphi_n)\Big[(1-\chi_{n+1})\|z_n-u_n\|^2\\ &\quad +\chi_{n+1}(1-2k_1\chi_n)\|u_n-y_n\|^2+\chi_{n+1}(1-2k_2\chi_n)\|z_n-y_n\|^2\Big]. \qquad (3.19)\end{aligned}$$

By following the conditions (**C1**) and (**C2**), the solution set $EP(f,\mathcal{C})$ is a closed and convex set; see, for example, [29]. Given that $u^*=P_{EP(f,\mathcal{C})}(0)$, and by Lemma 2.1 (ii), we have

$$\langle 0-u^*, y-u^*\rangle \le 0, \quad \text{for all } y\in EP(f,\mathcal{C}).$$

Now we divide the rest of the proof into the following two parts:

Case 1: Suppose that there is a fixed number $n_1 \in \mathbb{N}$ such that

$$\|u_{n+1} - u^*\| \leq \|u_n - u^*\|, \quad \text{for all } n \geq n_1.$$

Then $\lim_{n\to+\infty} \|u_n - u^*\|$ exists. From (3.19), we have

$$\begin{aligned}
&\phi_n(1-\varphi_n)\Big[(1-\chi_{n+1})\|z_n - u_n\|^2 \\
&\quad + \chi_{n+1}(1-2k_1\chi_n)\|u_n - y_n\|^2 + \chi_{n+1}(1-2k_2\chi_n)\|z_n - y_n\|^2\Big] \\
&\qquad \leq \|u_n - u^*\|^2 + \varphi_n K_2 - \|u_{n+1} - u^*\|^2.
\end{aligned} \tag{3.20}$$

The existence of $\lim_{n\to+\infty} \|u_n - u^*\|$ and $\varphi_n \to 0$, we infer that

$$\lim_{n\to+\infty} \|u_n - y_n\| = \lim_{n\to+\infty} \|z_n - y_n\| = 0. \tag{3.21}$$

It follows that

$$\lim_{n\to+\infty} \|u_n - z_n\| = 0. \tag{3.22}$$

It follows from (3.22) and $\varphi_n \to 0$, that

$$\begin{aligned}
\big\|u_{n+1} - u_n\big\| &= \big\|(1-\phi_n-\varphi_n)u_n + \phi_n z_n - u_n\big\| \\
&= \big\|u_n - \varphi_n u_n + \phi_n z_n - \phi_n u_n - u_n\big\| \\
&\leq \phi_n\big\|z_n - u_n\big\| + \varphi_n\big\|u_n\big\|,
\end{aligned}$$

which gives that

$$\|u_{n+1} - u_n\| \to 0 \quad \text{as} \quad n \to +\infty.$$

We can also deduce that $\{y_n\}$ and $\{z_n\}$ are bounded. The reflexivity of $\mathcal{H}$ and the boundedness of $\{u_n\}$ guarantee that there is a subsequence $\{u_{n_k}\}$ such that $\{u_{n_k}\} \rightharpoonup \hat{u} \in \mathcal{H}$ as $k \to +\infty$. Next, we need to show that $\hat{u} \in EP(f, \mathcal{C})$. By the use of expression (3.4), the Lipschitz-type continuous of f, and (3.9), we get

$$\begin{aligned}
&\chi_{n_k} f(y_{n_k}, y) \\
&\geq \chi_{n_k} f(y_{n_k}, z_{n_k}) + \langle u_{n_k} - z_{n_k}, y - z_{n_k}\rangle \\
&\geq \chi_{n_k}\chi_{n_k+1} f(u_{n_k}, z_{n_k}) - \chi_{n_k}\chi_{n_k+1} f(u_{n_k}, y_{n_k}) - k_1\chi_{n_k}\chi_{n_k+1}\|u_{n_k} - y_{n_k}\|^2 \\
&\quad - k_2\chi_{n_k}\chi_{n_k+1}\|y_{n_k} - z_{n_k}\|^2 + \langle u_{n_k} - z_{n_k}, y - z_{n_k}\rangle \\
&\geq \chi_{n_k+1}\langle u_{n_k} - y_{n_k}, z_{n_k} - y_{n_k}\rangle - k_1\chi_{n_k}\chi_{n_k+1}\|u_{n_k} - y_{n_k}\|^2 \\
&\quad - k_2\chi_{n_k}\chi_{n_k+1}\|y_{n_k} - z_{n_k}\|^2 + \langle u_{n_k} - z_{n_k}, y - z_{n_k}\rangle,
\end{aligned}$$

where y is an arbitrary point in $\mathcal{H}_n$. Using the boundedness of $\{u_n\}$ and from (3.21) and (3.22), the right-hand side converges to zero. Since $\chi_{n_k} > 0$, using the condition (**C3**) and $y_{n_k} \rightharpoonup \hat{u}$, we have

$$0 \leq \limsup_{k\to+\infty} f(y_{n_k}, y) \leq f(\hat{u}, y), \quad \text{for all } y \in \mathcal{H}_n.$$

Thus, the above equation implies that $f(\hat{u}, y) \geq 0$, for all $y \in \mathcal{C}$, and hence $\hat{u} \in EP(f, \mathcal{C})$. Thus

$$\begin{aligned}&\limsup_{n\to+\infty}\langle u^*, u^* - u_n\rangle \\ &= \limsup_{k\to+\infty}\langle u^*, u^* - u_{n_k}\rangle = \langle u^*, u^* - \hat{u}\rangle \leq 0.\end{aligned}$$

By the use of $\lim_{n\to+\infty}\|u_{n+1} - u_n\| = 0$, we might conclude that

$$\begin{aligned}&\limsup_{n\to+\infty}\langle u^*, u^* - u_{n+1}\rangle \\ &\leq \limsup_{n\to+\infty}\langle u^*, u^* - u_n\rangle + \limsup_{n\to+\infty}\langle u^*, u_n - u_{n+1}\rangle \leq 0. \end{aligned} \tag{3.23}$$

Next, assume that $t_n = (1 - \phi_n)u_n + \phi_n z_n$. Thus, we obtain

$$u_{n+1} = t_n - \varphi_n u_n = (1 - \varphi_n)t_n - \varphi_n(u_n - t_n) = (1 - \varphi_n)t_n - \varphi_n\phi_n(u_n - z_n),$$

where $u_n - t_n = u_n - (1 - \phi_n)u_n - \phi_n z_n = \phi_n(u_n - z_n)$. Thus, we have

$$\begin{aligned}&\|u_{n+1} - u^*\|^2 \\ &= \|(1 - \varphi_n)t_n + \phi_n\varphi_n(z_n - u_n) - u^*\|^2 \\ &= \|(1 - \varphi_n)(t_n - u^*) + [\phi_n\varphi_n(z_n - u_n) - \varphi_n u^*]\|^2 \\ &\leq (1 - \varphi_n)^2\|t_n - u^*\|^2 + 2\langle\phi_n\varphi_n(z_n - u_n) - \varphi_n u^*,\, (1 - \varphi_n)(t_n - u^*) + \phi_n\varphi_n(z_n - u_n) - \varphi_n u^*\rangle \\ &= (1 - \varphi_n)^2\|t_n - u^*\|^2 + 2\langle\phi_n\varphi_n(z_n - u_n) - \varphi_n u^*,\, t_n - \varphi_n t_n - \varphi_n(u_n - t_n) - u^*\rangle \\ &= (1 - \varphi_n)\|t_n - u^*\|^2 + 2\phi_n\varphi_n\langle z_n - u_n,\, u_{n+1} - u^*\rangle + 2\varphi_n\langle u^*,\, u^* - u_{n+1}\rangle \\ &\leq (1 - \varphi_n)\|t_n - u^*\|^2 + 2\phi_n\varphi_n\|z_n - u_n\|\|u_{n+1} - u^*\| + 2\varphi_n\langle u^*,\, u^* - u_{n+1}\rangle. \end{aligned} \tag{3.24}$$

Next, we need to evaluate

$$\begin{aligned}&\|t_n - u^*\|^2 \\ &= \|(1 - \phi_n)u_n + \phi_n z_n - u^*\|^2 \\ &= \|(1 - \phi_n)(u_n - u^*) + \phi_n(z_n - u^*)\|^2 \\ &= (1 - \phi_n)^2\|u_n - u^*\|^2 + \phi_n^2\|z_n - u^*\|^2 + 2\langle(1 - \phi_n)(u_n - u^*), \phi_n(z_n - u^*)\rangle \\ &\leq (1 - \phi_n)^2\|u_n - u^*\|^2 + \phi_n^2\|z_n - u^*\|^2 + 2\phi_n(1 - \phi_n)\|u_n - u^*\|\|z_n - u^*\| \\ &\leq (1 - \phi_n)^2\|u_n - u^*\|^2 + \phi_n^2\|z_n - u^*\|^2 + \phi_n(1 - \phi_n)\|u_n - u^*\|^2 + \phi_n(1 - \phi_n)\|z_n - u^*\|^2 \\ &= (1 - \phi_n)\|u_n - u^*\|^2 + \phi_n\|z_n - u^*\|^2 \\ &\leq (1 - \phi_n)\|u_n - u^*\|^2 + \phi_n\|u_n - u^*\|^2 \\ &= \|u_n - u^*\|^2. \end{aligned} \tag{3.25}$$

Combining expressions (3.24) and (3.25) gives that

$$\begin{aligned}&\|u_{n+1} - u^*\|^2 \\ &\leq (1 - \varphi_n)\|u_n - u^*\|^2 + \varphi_n\Big[2\phi_n\|z_n - u_n\|\|u_{n+1} - u^*\| + 2\varphi_n\langle u^*,\, u^* - u_{n+1}\rangle\Big]. \end{aligned} \tag{3.26}$$

By the use of expressions (3.23) and (3.26) and Lemma 2.4, we can derive that $\|u_n - u^*\| \to 0$ as $n \to +\infty$.

Case 2: Assume that there is a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\|u_{n_i} - u^*\| \leq \|u_{n_{i+1}} - u^*\|, \quad \text{for all } i \in \mathbb{N}.$$

By the use of Lemma 2.5, there exists a sequence $\{m_k\} \subset \mathbb{N}$ ($\{m_k\} \to +\infty$), such that

$$\|u_{m_k} - u^*\| \leq \|u_{m_{k+1}} - u^*\| \quad \text{and} \quad \|u_k - u^*\| \leq \|u_{m_{k+1}} - u^*\|, \quad \text{for all } k \in \mathbb{N}. \tag{3.27}$$

By the use of expression (3.20), we have

$$\begin{aligned}
&\phi_{m_k}(1-\varphi_{m_k})\Big[(1-\chi_{m_k+1})\|z_{m_k} - u_{m_k}\|^2 \\
&\quad + \chi_{m_k+1}(1-2k_1\chi_{m_k})\|u_{m_k} - y_{m_k}\|^2 + \chi_{m_k+1}(1-2k_2\chi_{m_k})\|z_{m_k} - y_{m_k}\|^2\Big] \\
&\qquad \leq \|u_{m_k} - u^*\|^2 + \varphi_{m_k}K_2 - \|u_{m_k+1} - u^*\|^2.
\end{aligned}$$

Due to $\varphi_{m_k} \to 0$, we can deduce that

$$\lim_{n\to+\infty} \|z_{m_k} - u_{m_k}\| = \lim_{n\to+\infty} \|u_{m_k} - y_{m_k}\| = \lim_{n\to+\infty} \|z_{m_k} - y_{m_k}\| = 0.$$

It continues from that

$$\begin{aligned}
\big\|u_{m_k+1} - u_{m_k}\big\| &= \big\|(1-\phi_{m_k} - \varphi_{m_k})u_{m_k} + \phi_{m_k} z_{m_k} - u_{m_k}\big\| \\
&= \big\|u_{m_k} - \varphi_{m_k}u_{m_k} + \phi_{m_k}z_{m_k} - \phi_{m_k}u_{m_k} - u_{m_k}\big\| \\
&\leq \phi_{m_k}\big\|z_{m_k} - u_{m_k}\big\| + \varphi_{m_k}\big\|u_{m_k}\big\| \longrightarrow 0.
\end{aligned}$$

By using a similar argument as in Case 1, we get

$$\limsup_{k\to+\infty} \langle u^*, u_{m_k+1} - u^*\rangle \leq 0. \tag{3.28}$$

By the use of expressions (3.26) and (3.27), we have

$$\begin{aligned}
&\big\|u_{m_k+1} - u^*\big\|^2 \\
&\leq (1-\varphi_{m_k})\big\|u_{m_k} - u^*\big\|^2 \\
&\quad + \varphi_{m_k}\Big[2\phi_{m_k}\big\|z_{m_k} - u_{m_k}\big\|\big\|u_{m_k+1} - u^*\big\| + 2\varphi_{m_k}\big\langle u^*,\, u^* - u_{m_k+1}\big\rangle\Big] \\
&\leq (1-\varphi_{m_k})\big\|u_{m_{k+1}} - u^*\big\|^2 \\
&\quad + \varphi_{m_k}\Big[2\phi_{m_k}\big\|z_{m_k} - u_{m_k}\big\|\big\|u_{m_k+1} - u^*\big\| + 2\varphi_{m_k}\big\langle u^*,\, u^* - u_{m_k+1}\big\rangle\Big].
\end{aligned}$$

It follows that

$$\big\|u_{m_k+1} - u^*\big\|^2 \leq 2\phi_{m_k}\big\|z_{m_k} - u_{m_k}\big\|\big\|u_{m_k+1} - u^*\big\| + 2\varphi_{m_k}\big\langle u^*,\, u^* - u_{m_k+1}\big\rangle. \tag{3.29}$$

Since $\varphi_{m_k} \to 0$ and $\big\|u_{m_k} - u^*\big\|$ is bounded, (3.28) and (3.29) yield

$$\|u_{m_k+1} - u^*\|^2 \to 0, \text{ as } k \to +\infty.$$

The above equation implies that

$$\lim_{n\to+\infty} \|u_k - u^*\|^2 \leq \lim_{n\to+\infty} \|u_{m_k+1} - u^*\|^2 \leq 0.$$

As a result, $u_n \to u^*$ and the desired result will be obtained.

□

4. Applications

In this section, we derive the results from our main proposed results to solve variational inequalities. The variational inequality problem for an operator $\mathcal{A} : \mathcal{H} \to \mathcal{H}$ is defined as follows:

$$\text{Find } u^* \in \mathcal{C} \text{ such that } \left\langle \mathcal{A}(u^*), y - u^* \right\rangle \geq 0, \quad \text{for all } y \in \mathcal{C}. \tag{VIP}$$

To study variational inequalities, we consider the following conditions:

($\mathcal{A}1$) A solution set of problem (VIP) is denoted by $VI(\mathcal{A}, \mathcal{C})$ and it is nonempty;

($\mathcal{A}2$) $\mathcal{A} : \mathcal{H} \to \mathcal{H}$ is pseudomonotone, that is,
$$\left\langle \mathcal{A}(u), y - u \right\rangle \geq 0 \Longrightarrow \left\langle \mathcal{A}(y), u - y \right\rangle \leq 0, \quad \text{for all } u, y \in \mathcal{C};$$

($\mathcal{A}3$) $\mathcal{A} : \mathcal{H} \to \mathcal{H}$ is *Lipschitz continuous*, that is, there exists a constant $L > 0$ such that
$$\|\mathcal{A}(u) - \mathcal{A}(y)\| \leq L\|u - y\|, \quad \text{for all } u, y \in \mathcal{C};$$

($\mathcal{A}4$) $\mathcal{A} : \mathcal{H} \to \mathcal{H}$ is *sequentially weakly continuous*, that is, $\{\mathcal{A}(u_n)\}$ weakly converges to $\mathcal{A}(u)$ for every sequence $\{u_n\}$ converges weakly to u.

On the other hand, the fixed point problem for an operator $\mathcal{B} : \mathcal{H} \to \mathcal{H}$ is defined as follows:

$$\text{Find } u^* \in \mathcal{C} \text{ such that } \mathcal{B}(u^*) = u^*. \tag{FPP}$$

The following conditions are taken to study fixed point theorems.

($\mathcal{B}1$) Solution set of a problem (FPP), denoted by $Fix(\mathcal{B}, \mathcal{C})$, is nonempty;

($\mathcal{B}2$) $\mathcal{B} : \mathcal{C} \to \mathcal{C}$ is κ*-strict pseudocontraction* [6] on $\mathcal{C}$, that is,
$$\|\mathcal{B}u - \mathcal{B}y\|^2 \leq \|u - y\|^2 + \kappa\|(u - \mathcal{B}u) - (y - \mathcal{B}y)\|^2, \quad \text{for all } u, y \in \mathcal{C};$$

($\mathcal{B}3$) $\mathcal{B} : \mathcal{H} \to \mathcal{H}$ is *weakly sequentially continuous*, that is, $\{\mathcal{B}(u_n)\}$ weakly converges to $\mathcal{B}(u)$ for every sequence $\{u_n\}$ converges weakly to u.

Corollary 4.1. *Assume that an operator $\mathcal{A} : \mathcal{C} \to \mathcal{H}$ satisfies the conditions ($\mathcal{A}1$)–($\mathcal{A}4$). Choose $u_0 \in \mathcal{C}$, $\mu \in (0, 1)$, $\chi_0 > 0$, $0 < \sigma < \min\left\{1, \frac{1}{L}\right\}$, $\{\phi_n\} \subset (a, b) \subset (0, 1 - \varphi_n)$, and $\{\varphi_n\} \subset (0, 1)$ satisfying the conditions $\lim_{n\to+\infty} \varphi_n = 0$ and $\sum_{n=1}^{+\infty} \varphi_n = +\infty$. Consider the iterative sequence as follows:*

$$\begin{cases} y_n = P_{\mathcal{C}}(u_n - \chi_n \mathcal{A}(u_n)), \\ z_n = P_{\mathcal{H}_n}(u_n - \chi_n \mathcal{A}(y_n)), \\ u_{n+1} = (1 - \phi_n - \varphi_n)u_n + \phi_n z_n, \end{cases}$$

where $\mathcal{H}_n = \{z \in \mathcal{H} : \langle u_n - \chi_n \mathcal{A}(u_n) - y_n, z - y_n \rangle \leq 0\}$. Compute

$$\chi_{n+1} = \begin{cases} \min\left\{\sigma, \frac{\mu\langle \mathcal{A}y_n, z_n - y_n\rangle}{\langle \mathcal{A}u_n, z_n - y_n\rangle - \frac{L}{2}\|u_n - y_n\|^2 - \frac{L}{2}\|z_n - y_n\|^2 + 1}\right\}, \\ \qquad \text{if} \quad \frac{\mu\langle \mathcal{A}y_n, z_n - y_n\rangle}{\langle \mathcal{A}u_n, z_n - y_n\rangle - \frac{L}{2}\|u_n - y_n\|^2 - \frac{L}{2}\|z_n - y_n\|^2 + 1} > 0, \\ \sigma \qquad \text{otherwise}. \end{cases}$$

Then, $\{u_n\}$ strongly converges to $u^* \in VI(\mathcal{A}, \mathcal{C})$.

Corollary 4.2. *Assume that $\mathcal{B} : \mathcal{C} \to \mathcal{C}$ is a mapping satisfying the conditions* $(\mathcal{B}1)$–$(\mathcal{B}3)$ *and* $Fix(\mathcal{B}, \mathcal{C}) \neq \emptyset$. *Choose* $u_0 \in \mathcal{C}$, $\mu \in (0, 1)$, $\chi_0 > 0$, $0 < \sigma < \min\left\{1, \frac{1-\kappa}{3-2\kappa}\right\}$, $\{\phi_n\} \subset (a, b) \subset (0, 1 - \varphi_n)$ *and* $\{\varphi_n\} \subset (0, 1)$ *satisfying the conditions* $\lim_{n\to+\infty} \varphi_n = 0$ *and* $\sum_{n=1}^{+\infty} \varphi_n = +\infty$. *Consider the iterative sequence update as follows:*

$$\begin{cases} y_n = P_{\mathcal{C}}\big[u_n - \chi_n(u_n - \mathcal{B}(u_n))\big], \\ z_n = P_{\mathcal{H}_n}\big[u_n - \chi_n(y_n - \mathcal{B}(y_n))\big], \\ u_{n+1} = (1 - \phi_n - \varphi_n)u_n + \phi_n z_n, \end{cases}$$

where $\mathcal{H}_n = \{z \in \mathcal{H} : \langle (1 - \chi_n)u_n + \chi_n \mathcal{B}(u_n) - y_n, z - y_n\rangle \leq 0\}$. *Compute*

$$\chi_{n+1} = \begin{cases} \min\left\{\sigma, \frac{\mu\langle y_n - \mathcal{B}y_n, z_n - y_n\rangle}{\langle u_n - \mathcal{B}(u_n), z_n - y_n\rangle - \left(\frac{3-2\kappa}{2-2\kappa}\right)\|u_n - y_n\|^2 - \left(\frac{3-2\kappa}{2-2\kappa}\right)\|z_n - y_n\|^2 + 1}\right\}, \\ \qquad \textit{if} \quad \frac{\mu\langle y_n - \mathcal{B}y_n, z_n - y_n\rangle}{\langle u_n - \mathcal{B}(u_n), z_n - y_n\rangle - \left(\frac{3-2\kappa}{2-2\kappa}\right)\|u_n - y_n\|^2 - \left(\frac{3-2\kappa}{2-2\kappa}\right)\|z_n - y_n\|^2 + 1} > 0, \\ \sigma \qquad \textit{otherwise.} \end{cases}$$

Then, $\{u_n\}$ *converges strongly to* $Fix(\mathcal{B}, \mathcal{C})$.

5. Numerical illustrations

In this section, we give two numerical test problems and explain the numerical behavior of the designed method in comparison to some related works in the literature. The control parameters are taken in the following way: (1) $\chi = \frac{1}{5k_1}$, $\phi_n = \frac{1}{100(n+2)}$, and $D_n = \|u_n - y_n\|$ for Algorithm 3.2 (**Halp-EGM**) in [9]; (2) $\chi = \frac{1}{5k_1}$, $\phi_n = \frac{1}{100(n+2)}$, $f(u) = \frac{u}{2}$, and $D_n = \|u_n - y_n\|$ for Algorithm 1 (**Visc-EGM**) in [25]; (3) $\chi_0 = \frac{1}{3k_1}$, $\mu = 0.33$, $\sigma = \frac{1}{2.2k_1}$, $\varphi_n = \frac{1}{50(n+2)}$, $\phi_n = \frac{2}{10}(1 - \varphi_n)$, and $D_n = \|u_n - y_n\|$ for Algorithm 1 (**Mann-EGM**).

Example 5.1. Let the set $\mathcal{C}$ be defined by $\mathcal{C} := \{u \in \mathbb{R}^m : -10 \leq u_i \leq 10\}$ and let $f : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ be defined as follows:

$$f(u, y) = \langle Mu + Ny + r, y - u\rangle, \quad \text{for all } \ u, y \in \mathcal{C},$$

where $r \in \mathbb{R}^m$, M and N are matrices of order m, and $k_1 = k_2 = \frac{1}{2}\|M - N\|$ (see [29] for details). Two matrices M and N are taken as follows:

$$M = \begin{pmatrix} 3.1 & 2 & 0 & 0 & 0 \\ 2 & 3.6 & 0 & 0 & 0 \\ 0 & 0 & 3.5 & 2 & 0 \\ 0 & 0 & 2 & 3.3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}, \quad N = \begin{pmatrix} 1.6 & 1 & 0 & 0 & 0 \\ 1 & 1.6 & 0 & 0 & 0 \\ 0 & 0 & 1.5 & 1 & 0 \\ 0 & 0 & 1 & 1.5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \quad r = \begin{pmatrix} 1 \\ -2 \\ -1 \\ 2 \\ -1 \end{pmatrix}.$$

Numerical results are shown in Figures 1-4 by letting different values of starting points u_0 and $TOL = 10^{-4}$.

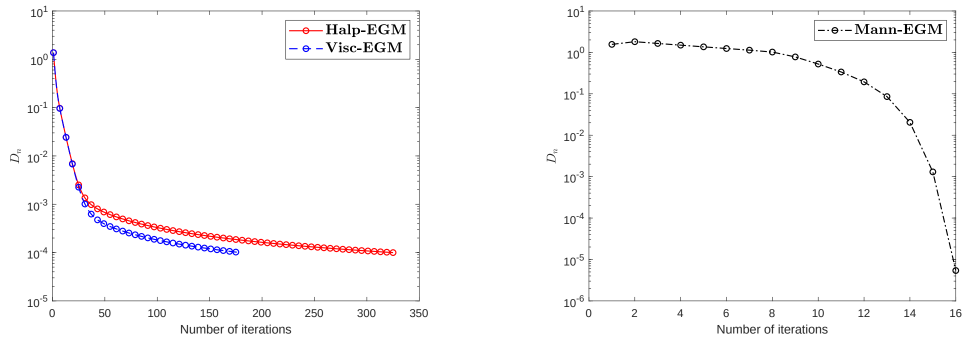

FIGURE 1. Numerical comparison of Algorithm 1 with [9, Algorithm 3.2] and [25, Algorithm 1] while $u_0 = (1, 1, 1, 1, 1)^T$.

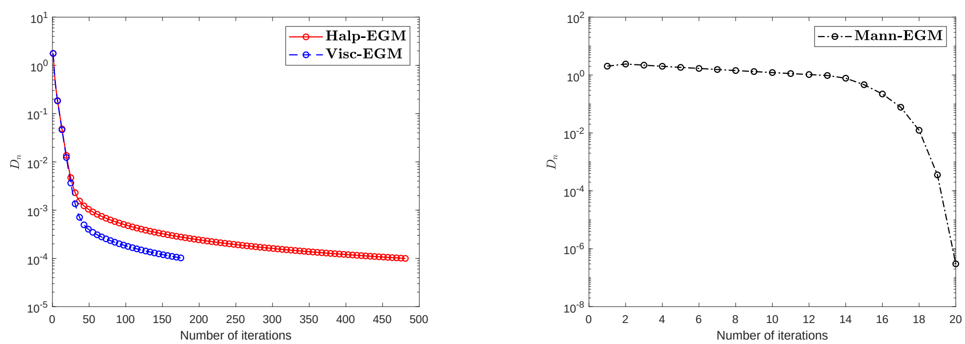

FIGURE 2. Numerical comparison of Algorithm 1 with [9, Algorithm 3.2] and [25, Algorithm 1] while $u_0 = (2, 1, 0, 2, 1)^T$.

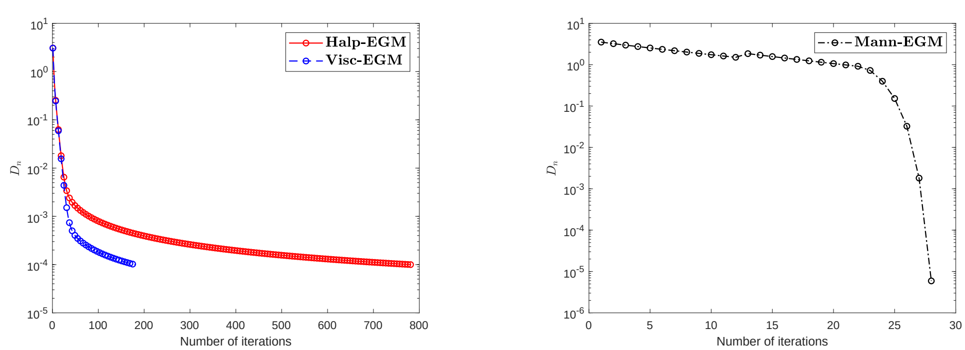

FIGURE 3. Numerical comparison of Algorithm 1 with [9, Algorithm 3.2] and [25, Algorithm 1] while $u_0 = (2, 3, -1, 3, 4)^T$.

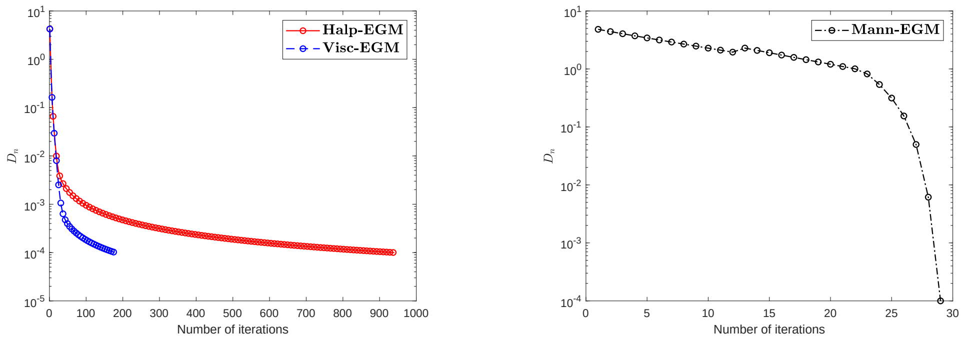

FIGURE 4. Numerical comparison of Algorithm 1 with [9, Algorithm 3.2] and [25, Algorithm 1] while $u_0 = (-2, -1, 3, 4, -5)^T$.

Example 5.2. Let $f : \mathcal{C}\times\mathcal{C} \to \mathbb{R}$ be defined by $f(u, y) = \sum_{i=2}^{5}(y_i - u_i)\|u\|$, for all $u, y \in \mathbb{R}^5$, where $\mathcal{C} \subset \mathbb{R}^5$ is taken as follows:

$$\mathcal{C} = \Big\{(u_1, \ldots, u_5) : u_1 \geq -1, u_i \geq 1, i = 2, \ldots, 5\Big\}.$$

Then, f is Lipschitz-like continuous with $k_1 = k_2 = 2$, and satisfies the conditions (C1)–(C4). All numerical results are reported in Table 1-3 by letting different initial points and $TOL = 10^{-3}$.

TABLE 1. Example 5.2: Numerical results of [9, Algorithm 3.2] while $u_0 = (5, 2, 1, -3, 4)^T$.

Iter (n)	u_1	u_2	u_3	u_4	u_5
1	4.999999	1.4967643	1.0000000	2.49676432	3.4967643
2	4.999999	1.0666826	1.0000007	2.04283026	3.0428300
3	4.999999	1.0500000	1.0000000	1.61835734	2.6183571
4	4.999999	1.0400009	1.0000008	1.21606861	2.2160654
5	5.000000	1.0333342	1.0000008	1.06666832	1.8269685
6	4.999999	1.0285714	1.0000000	1.05714290	1.4467949
7	5.000000	1.0250009	1.0000008	1.05000099	1.0753428
8	4.999999	1.0222222	1.0000000	1.04444449	1.0666667
⋮	⋮	⋮	⋮	⋮	⋮
63	4.999998	1.0027778	1.0000000	1.00555560	1.0083333
64	4.999998	1.0027397	1.0000000	1.00547949	1.0082192
65	4.999998	1.0027027	1.0000000	1.00540545	1.0081081
66	4.999998	1.0026667	1.0000000	1.00533337	1.0080000
67	4.999998	1.0026316	1.0000000	1.00526320	1.0078947
CPU time is seconds	2.506290				

Table 2. Example 5.2: Numerical results of [10, Algorithm 4.1] while $u_0 = (5, 2, 1, -3, 4)^T$.

Iter (n)	u_1	u_2	u_3	u_4	u_5
1	4.9999999	1.4688068	1.0000000	2.468806	3.468806
2	4.9999999	1.0357217	1.0000007	1.985704	2.985704
3	4.9999999	1.0277778	1.0000000	1.533361	2.533361
4	4.9999999	1.0227281	1.0000008	1.104510	2.104495
5	4.9999999	1.0192308	1.0000000	1.038461	1.688463
6	4.9999999	1.0166673	1.0000008	1.033334	1.281918
7	5.0000000	1.0147068	1.0000008	1.029412	1.044120
8	4.9999999	1.0131579	1.0000000	1.026315	1.039473
⋮	⋮	⋮	⋮	⋮	⋮
31	4.9999988	1.0028090	1.0000000	1.005618	1.008427
32	4.9999988	1.0027472	1.0000000	1.005494	1.008241
33	4.9999988	1.0026882	1.0000000	1.005376	1.008064
34	4.9999987	1.0026316	1.0000000	1.005263	1.007894
35	4.9999987	1.0025773	1.0000000	1.005154	1.007732
CPU time is seconds	0.9549447599				

Table 3. Example 5.2: Numerical results of Algorithm 1 while $u_0 = (5, 2, 1, -3, 4)^T$.

Iter (n)	u_1	u_2	u_3	u_4	u_5
1	4.9750000	1.194000	0.99500010	1.393000	1.6223834
2	4.9584169	1.035337	0.99567056	1.075005	1.1207291
3	4.9460211	1.004550	0.99663682	1.012464	1.0215860
4	4.9361293	0.998908	0.99732925	1.000488	1.0023091
5	4.9279026	0.998116	0.99780062	0.998431	0.99879493
6	4.9208630	0.998195	0.99813272	0.998251	0.99833130
7	4.9147122	0.998390	0.99837756	0.998405	0.99841722
8	4.9092516	0.998567	0.99856531	0.998570	0.99857323
⋮	⋮	⋮	⋮	⋮	⋮
14	4.8649957	0.999450	0.999450950	0.999450	0.99945092
15	4.8629689	0.999474	0.999474123	0.999474	0.99947412
16	4.8610240	0.999495	0.999495421	0.999495	0.99949542
17	4.8591546	0.999515	0.999515062	0.999515	0.99951506
18	4.8573551	0.999533	0.999533233	0.999533	0.99953323
CPU time is seconds	0.44989097144				

Conclusions

We developed an explicit extragradient-type method to find a numerical solution to the problem of pseudomonotone equilibrium in a real Hilbert space. This method is a variant of a two-step gradient method. The proposed algorithm produced a strongly convergent result. Numerical conclusions have been drawn to demonstrate the numerical potency of our algorithms in comparison to existing methods. Such computational results showed that the variable step size rule tends to improve the usefulness of the iterative sequence in this context.

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