Khayyam J. Math. 8 (2022), no. 2, 195-203
DOI: 10.22034/KJM.2022.327963.2476

Khayyam Journal of Mathematics
emis.de/journals/KJM
kjm-math.org

MATRIX SUMMABILITY OF SEQUENCES OF SETS

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Communicated by A.M. Peralta

Abstract. The concept of strong Cesàro summability of sequences of closed sets with respect to a modulus is extended to a concept of strong T-summability with respect to a modulus when T is a nonnegative regular matrix summability method. Also, we show that if a sequence of closed sets is strongly T-summable with respect to an arbitrary modulus, then it is T-statistically convergent and that T-statistical convergence and strong T-summability with respect to a modulus are equivalent on the bounded sequences of closed sets.

1. Introduction and preliminaries

The natural density of a set K of positive integers is defined by

$$\delta(K) := \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : \ k \in K\}|,$$

where $|\{k \leq n : \ k \in K\}|$ denotes the number of elements of K not exceeding n.

Statistical convergence of number sequences was introduced by Fast [9]. Schoenberg [19] established some basic properties of statistical convergence and studied the concept as a summability method.

A sequence $x = (x_k)$ is said to be statistically convergent to the number ℓ if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : \ |x_k - \ell| \geq \varepsilon\}| = 0.$$

In this case, we write $st - \lim x_k = \ell$. Moreover,$\lim x_k = \ell$ implies $st - \lim x_k = \ell$, so statistical convergence may be considered as a regular summability method. This was observed by Schoenberg [19] along with the fact that the statistical limit is a linear functional on some sequence spaces.

Date: Received: 3 February 2022; Accepted: 23 June 2022.
2010 *Mathematics Subject Classification.* Primary 40D25; Secondary 40A05, 40C05.
Key words and phrases. Sequence of sets, Wijsman convergence, Matrix transformation, Regular matrix.

If $x = (x_k)$ is a sequence such that x_k satisfies property P for all k except a set of natural density zero, then we say that x_k satisfies P for almost all k, and we abbreviate this by "a.a. k.". Fridy [11] proved that if $x = (x_k)$ is a statistically convergent sequence, then there is a convergent sequence $y = (y_k)$ such that $x_k = y_k$ a.a. k.

The concepts of strong summability and statistical convergence were separately introduced and, until recently, followed independent lines of development. Strong summability first appeared in the paper by Hardy and Littlewood [12], who improved Fejer's theorem on the Cesàro convergence of a Fourier series. The strong summability of Fourier series continues to be an active area of research. Connor's papers [7, 8], on matrix summability and statistical convergence, are essential articles that contribute significantly to the subject.

Recently, many authors published articles on the Wijsman convergence of set sequences. The reader can refer to the papers [1–6, 10, 13, 14, 20, 21] for more detailed information about Wijsman convergence of set sequences.

Let (X, ρ) be a metric space. For any point $x \in X$ and any nonempty subset A of X, we define the distance from x to A as

$$d(x, A) = \inf_{a \in A} \rho(x, a).$$

Let (X, ρ) be a metric space. For any nonempty closed subsets $B, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman convergent to A if

$$\lim_{k \to \infty} d(x, A_k) = d(x, B)$$

for each $x \in X$. In this case, we write $W - \lim A_k = B$ (see [22,23]). All Wijsman convergent sequences of sets will be denoted by W.

Example 1.1. Consider the following sequence of circles in the (x, y)-plane: $\{A_k\} = \{(x, y) : \ x^2 + y^2 - 2ky = 0\}$. As $k \to \infty$, the sequence is Wijsman convergent to the x-axis $B = \{(x, y) : \ y = 0\}$.

Example 1.2. Let $X = l^2 = \{(x_n) : \ \sum_{n=1}^{\infty} |x_n|^2 < \infty\}$ and let $\{e_n\}$ be its standard orthonormal basis. Set $A_n = \{x \in X : \ x = \lambda e_1 + (1-\lambda)e_n, \ 0 \le \lambda \le 1\}$, $n \ge 1$, and $B = \{e_1\}$. Since

$$d(0, A_n) = \frac{1}{2}\|e_1 + e_2\| = \frac{\sqrt{2}}{2}, \qquad d(0, B) = \|e_1\| = 1,$$

$\{A_n\}$ is not Wijsman convergent to B.

Wijsman convergence was introduced by Wijsman in the seminal paper [23], where it was shown to be preserved under Fenchel conjugation in finite-dimensional spaces.

Let (X, ρ) be a metric space. For any nonempty closed subsets $A_k \subseteq X$, we say that the sequence $\{A_k\}$ is bounded if $\sup_k d(x, A_k) < \infty$ for each $x \in X$. In addition, W_∞ denotes the space of all bounded sequences of sets.

Following [16, Definitions 1.3, 1.6, 1.8, 1.9 and Theorem 1.10], we state the following definition.

Definition 1.3. Let (X,ρ) be a metric space. For any nonempty closed subsets $B, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman statistically convergent to B if $\{d(x,A_k)\}$ is statistically convergent to $d(x,B)$; that is, for each $\epsilon > 0$ and for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n}|\{k \le n : \ |d(x,A_k) - d(x,B)| \ge \epsilon\}| = 0;$$

in other words,

$$|d(x,A_k) - d(x,B)| < \epsilon \ a.a.k. \tag{1.1}$$

In this case, we write $st - \lim_W A_k = B$.

It is clear that if the inequality in (1.1) holds for all but finitely many k, then $W - \lim A_k = B$. It follows that $W - \lim A_k = B$ implies $st - \lim_W A_k = B$.

Example 1.4. Let $X = \mathbb{R}$ and let $\{A_k\}$ be the following sequence:

$$A_k := \begin{cases} \{x \in \mathbb{R} : \ 2 \le x \le k\} & \text{if } k \ge 2 \text{ and k is a square integer,} \\ \{1\} & \text{otherwise.} \end{cases}$$

This sequence is not Wijsman convergent. However, since

$$\frac{1}{n}|\,\{k \le n : \ |d(x,A_k) - d(x,\{1\})| \ge \varepsilon\}\,| \le \frac{\sqrt{n}}{n},$$

this sequence is Wijsman statistically convergent to set $B = \{1\}$.

Example 1.5. Let $X = \mathbb{R}^2$ and let $\{A_k\}$ be the following sequence:

$$A_k := \begin{cases} \{(x,y) \in \mathbb{R}^2 : \ (x-1)^2 + y^2 = \frac{1}{k}\} & \text{if } k \text{ is a square integer,} \\ \{(0,0)\} & \text{otherwise.} \end{cases}$$

This sequence is also Wijsman statistically convergent to the set $B = \{(0,0)\}$ but it is not Wijsman convergent.

Definition 1.6. Let (X,ρ) be a metric space. For any nonempty closed subsets $B, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman Cesàro summable to B if $\{d(x,A_k)\}$ is Cesàro summable to $d(x,B)$; that is, for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} d(x,A_k) = d(x,B).$$

Example 1.7. In the extended real numbers $\overline{\mathbb{R}}$, let $\{A_k\}$ be the following sequence: $A_1 = [-\infty, 0]$, $A_2 = [2+\frac{1}{2}, \infty]$, $A_3 = [-\infty, 1-\frac{1}{3}]$, $A_4 = [2+\frac{1}{4}, \infty]$, $\ldots$. Since

$$\lim_{n} \frac{1}{n}\sum_{k=1}^{n} d(x,A_k) = d\left(x, [-\infty,1] \cup [2,\infty]\right),$$

this sequence is Wijsman Cesàro summable, but since

$$\lim_{k} d(x,A_k) \ne d\left(x, [-\infty,1] \cup [2,\infty]\right)$$

this sequence is not Wijsman convergent.

Definition 1.8. Let (X, ρ) be a metric space. For any nonempty closed subsets $B, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman strongly Cesàro summable to B if $\{d(x, A_k)\}$ strongly summable to $d(x, B)$; that is, for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |d(x, A_k) - d(x, B)| = 0.$$

Definition 1.9. Let (X, ρ) be a metric space. For any nonempty closed subsets $B, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman strongly p-Cesàro summable to B if $\{d(x, A_k)\}$ strongly p-summable to $d(x, B)$; that is, for each p positive real number and for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |d(x, A_k) - d(x, B)|^p = 0.$$

Theorem 1.10. *Let (X, ρ) be a metric space and let p be a positive real number. Then for any nonempty closed subsets $B, A_k \subseteq X$*

(a) *$\{A_k\}$ is bounded and Wijsman statistically convergent to B if it is Wijsman strongly p-Cesàro summable to B;*
(b) *if $\{A_k\}$ is strongly p-Cesàro summable to B, then it is Wijsman statistically convergent to B.*

2. Matrix summability

There have been many studies in the literature on matrix transformation of sequences of numbers. When it comes to sequences of sets, these questions arise: Can we define a matrix transformation on the space of set sequences, and if so, how is defined. In this section, we will answer these questions.

Definition 2.1. Let E and F be any two nonempty subset of the space of all set sequences and let $T = (t_{nk})$ be a nonnegative infinite matrix of real numbers. We write $TA = (T_n(A))$ if

$$T_n(A) = \sum_{k=1}^{\infty} t_{nk} d(x, A_k)$$

converges for every $n \in \mathbb{N}$ and $x \in X$. If $(A_k) \in E$ implies that $TA = (T_n(A)) \in F$, then we say that T defines a matrix transformations from E into F, and we denote it by $T : E \to F$. The sequence TA is called the T transform of A. By (E, F) we mean the class of matrices T such that $T : E \to F$.

A matrix method T is called regular if all convergent sequences $x = (x_k)$ are T-summable and $\lim_n T_n x = \lim x_k$. This is denoted by $T \in (c, c, P)$. It is known that T is regular if and only if

(R1) $\lim_n t_{nk} = 0, \qquad (k \in \mathbb{N})$,
(R2) $\lim \sum_k t_{nk} = 1$,
(R3) $\sup_n \sum_k |t_{nk}| < \infty$.

If $\lim_k d(x, A_k) = d(x, B)$, $a_k = d(x, A_k)$, and $b = d(x, B)$, then for the real number sequence (a_k), we have $\lim_k a_k = b$. Thus $T \in (W, W, P)$ if and only if (R1), (R2), and (R3) hold.

Definition 2.2. If $T = (t_{nk})$ is a nonnegative regular matrix summability method, then we let

$$W_{\emptyset}(T) = \left\{ \{A_k\} : \lim_{n\to\infty} \sum_{k=1}^{\infty} t_{nk} d(x, A_k) = 0 \right\},$$

$$W(T) = \left\{ \{A_k\} : \lim_{n\to\infty} \sum_{k=1}^{\infty} t_{nk} |d(x, A_k) - d(x, B)| = 0, \text{ for every } x \in X \right\}.$$

The collection $W(T)$ is referred to as the collection of strongly T-summable sequences of closed sets. If $\{A_k\} \in W(T)$, then we say that $\{A_k\}$ is strongly T-summable to B. In the case where $T = (C, 1)$, we get the strongly Cesàro summable sequences of closed sets.

Definition 2.3. Let T be a nonnegative regular summability method and let $\{A_k\}$ be a sequence of sets. Then $\{A_k\}$ is said to be T-statistically convergent to B if $\chi_{S(A_k,B;\varepsilon)}$ is contained in $W_{\emptyset}(T)$ for every $\varepsilon > 0$, where

$$S(A_k, B; \varepsilon) = \{k \in \mathbb{N} : |d(x, A_k) - d(x, B)| \geq \varepsilon\}.$$

Theorem 2.4. *Let $T = (t_{nk})$ be a nonnegative regular matrix summability method.*

(i) *If $\{A_k\}$ is bounded and T-statistically convergent to B, then it is strongly T-summable to B.*

(ii) *If $\{A_k\}$ is strongly T-summable to B, then it is T-statistically convergent to B.*

Proof. (i) Let $\{A_k\}$ is bounded and T-statistically convergent to B. Then for every $x \in X$,

$$\begin{aligned} &\sum_k t_{nk} |d(x, A_k) - d(x, B)| \\ &\quad \leq \sum_{k \notin S(A_k,B;\varepsilon)} t_{nk} |d(x, A_k) - d(x, B)| + \sum_{k \in S(A_k,B;\varepsilon)} t_{nk} |d(x, A_k) - d(x, B)| \\ &\quad \leq \varepsilon \sum_{k \notin S(A_k,B;\varepsilon)} t_{nk} + \sup_k |d(x, A_k) - d(x, B)| \sum_{k \in S(A_k,B;\varepsilon)} t_{nk}. \end{aligned}$$

By using the definition of T-statistical convergence and the regularity of T, since $\varepsilon > 0$ arbitrary, we get

$$\lim_n \sum_k t_{nk} |d(x, A_k) - d(x, B)| = 0.$$

Thus $\{A_k\}$ is strongly T-summable to B.

(ii) Let $\{A_k\}$ be strongly T-summable to B. Since

$$\sum_{k \notin S(A_k,B;\varepsilon)} t_{nk} |d(x, A_k) - d(x, B)| \leq \sum_k t_{nk} |d(x, A_k) - d(x, B)| \to 0, \quad \text{as } n \to \infty,$$

for every $x \in X$, $\{A_k\}$ is T-statistically convergent to B. □

A function $f : [0, \infty) \to [0, \infty)$ is called a modulus, if the following conditions hold:

(a) $f(x) = 0$ if and only if $x = 0$,
(b) $f(x + y) \leq f(x) + f(y)$,
(c) f is increasing and
(d) f is continuous from right at 0.

By using the idea of modulus function f, Ruckle [17] introduced the sequence space

$$L(f) = \left\{ x = (x_k) : \sum_{k=1}^{\infty} f(|x_k|) < \infty \right\}.$$

Later, Maddox [15] introduced the sequence spaces

$$w_0(f) = \left\{ x = (x_k) : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(|x_k|) = 0 \right\},$$

$$w(f) = \left\{ x = (x_k) : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(|x_k - \ell|) = 0 \text{ for some number } \ell \right\},$$

and

$$w_\infty(f) = \left\{ x = (x_k) : \sup_n \frac{1}{n} \sum_{k=1}^{n} f(|x_k|) < \infty \right\},$$

which generalize the classical spaces of strongly summable sequences.

By using a modulus function f and a nonnegative regular matrix $A = (a_{nk})$, Connor [8] defined the following sequence spaces:

$$w_0(A, f) = \left\{ x = (x_k) : \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} f(|x_k|) = 0 \right\},$$

$$w(A, f) = \left\{ x = (x_k) : \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} f(|x_k - \ell|) = 0 \text{ for some number } \ell \right\},$$

which generalize the strong A-summability.

Now, we introduce and examine some properties of two sequence spaces of sets that are defined by using a modulus function, which generalize the sequence spaces $W_\emptyset(T)$ and $W(T)$.

Definition 2.5. Let f be a modulus function and let $T = (t_{nk})$ be a nonnegative regular matrix summability method. Then we let

$$W_\emptyset(T, f) = \left\{ \{A_k\} : \lim_{n\to\infty} \sum_{k=1}^{\infty} t_{nk} f(d(x, A_k)) = 0 \right\},$$

$$W(T, f) = \left\{ \{A_k\} : \lim_{n\to\infty} \sum_{k=1}^{\infty} t_{nk} f(|d(x, A_k) - d(x, B)|) = 0 \right\}.$$

If $\{A_k\} \in W(T, f)$, then we say that $\{A_k\}$ is strongly T-summable to B with respect to the modulus f.

Theorem 2.6. *If f is a modulus and $\{A_k\}$ is strongly T-summable to B, then $\{A_k\}$ is strongly T-summable to B with respect to the modulus f.*

Proof. Let $\{A_k\}$ be strongly T-summable to B. Then

$$s_n = \sum_k t_{nk}|d(x, A_k) - d(x, B)| \to 0, \qquad n \to \infty.$$

Let $\varepsilon > 0$, and choose δ with $0 < \delta < 1$ such that $f(u) < \varepsilon$ for $0 \leq u \leq \delta$. Write $v_k = |d(x, A_k) - d(x, B)|$, and consider

$$\sum_k t_{nk} f(v_k) = \sum_{\{k:\ v_n \leq \delta\}} t_{nk} f(v_k) + \sum_{\{k:\ v_k > \delta\}} t_{nk} f(v_k).$$

Then

$$\sum_{\{k:\ v_k \leq \delta\}} t_{nk} f(v_k) \leq \varepsilon \sum_{\{k:\ v_k \leq \delta\}} t_{nk},$$

and for $v_k > \delta$, we use the fact that

$$v_k < \frac{v_k}{\delta} < 1 + [\frac{v_k}{\delta}],$$

where $[a]$ denotes the integer part of a. Since f is a modulus function, we have for $v_k > \delta$,

$$f(v_k) \leq 1 + [\frac{v_k}{\delta}] f(1) \leq 2f(1)\frac{v_k}{\delta}.$$

Hence

$$\sum_{\{k:\ v_k > \delta\}} t_{nk} f(v_k) \leq 2\delta^{-1} f(1) \sum_{\{k:\ v_k > \delta\}} t_{nk}.$$

Thus

$$\sum_k t_{nk} f(v_k) \leq \varepsilon \sum_{\{k:\ \ v_k \leq \delta\}} t_{nk} + 2\delta^{-1} f(1) \sum_{\{k:\ v_k > \delta\}} t_{nk}.$$

Since $\varepsilon > 0$ is arbitrary and T is regular, we get the result. □

Lemma 2.7 ([18]). *Let f be a modulus and let $\alpha > 0$ be a given constant. Then there exists a constant $c > 0$ such that $f(x) > cx$ $(0 < x < \alpha)$.*

The proofs of the following theorems follows from Theorem 2.6 and Lemma 2.7.

Theorem 2.8. *Let $\{A_k\}$ be a bounded sequence of sets, let f be a modulus, and let T be a nonnegative regular matrix summability method. Then $\{A_k\}$ is strongly T-summable to B with respect to the modulus f if and only if $\{A_k\}$ is strongly T-summable to B.*

Theorem 2.9. *Let T be a nonnegative regular matrix summability method and let f be a modulus.*

(i) *If $\{A_k\} \in W$ is strongly T-summable to B with respect to f, then $\{A_k\}$ is T-statistically convergent to B.*

(ii) *If* $\{A_k\} \in W$ *is bounded and* T*-statistically convergent to* B*, then* $\{A_k\}$ *is strongly* T*-summable to* B *with respect to the modulus* f*.*

It is easy to check that if a sequence of sets is T-statistically convergent to B, then it must have a subsequence that is convergent to B. The above theorem now yields the following corollary.

Corollary 2.10. *If* $\{A_k\}$ *is strongly* T*-summable to* B *with respect to the modulus* f*, then* $\{A_k\}$ *has a subsequence that is Wijsman convergent to* B*.*

Proposition 2.11. *Let* $\{A_n\} \in W$ *such that* $d(x, A_n) = O(\sqrt{n})$ *and let*

$$\lim_n \frac{1}{\sqrt{n}} |\{k \leq n : \ |d(x, A_k) - d(x, B)| \geq \varepsilon\}| = 0$$

for all $\varepsilon > 0$*. Then* $\{A_n\}$ *is strongly summable to* B*.*

Proof. Note that $|d(x, A_n) - d(x, B)| \leq M\sqrt{n}$ for all n. Then

$$\frac{1}{n}\sum_{k=1}^{n} |d(x, A_n) - d(x, B)| \leq \varepsilon + M\frac{1}{\sqrt{n}} |\{k \leq n : \ |d(x, A_k) - d(x, B)| \geq \varepsilon\}|$$

for all $\varepsilon > 0$. □

This result is unsatisfying since it requires the sequence to be more than statistically convergent. Theorem 2.9(ii) is the best partial converse we have been able to find, which only requires the sequence be statistically convergent.

References

1. J.P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhauser, Boston, 1990.
2. M. Baronti and P. Papini, *Convergence of sequences of sets,* in: Methods of Functional analysis in Approximation Theory, pp. 133–155, ISNM 76, Birkhäuser, Basel, 1986.
3. G. Beer, *On convergence of closed sets in a metric space and distance functions*, Bull. Austral. Math. Soc. **31** (1985) 421–432.
4. G. Beer, *Convergence of continuous linear functionals and their level sets*, Arch. Math. **52** (1989) 482–491.
5. G. Beer, *On the compactness theorem for sequences of closed sets*, Math. Balkanica **16** (2002) 327–338.
6. J.M. Borwein and J. Vanderwerk, *Dual Kadec-Klee norms and the relationships between Wijsman, slice and Mosco convergence*, Michigan Math. J. **41** (1994) 371–387.
7. J. Connor, *The statistical and strong p-Cesàro convergence of sequences*, Analysis **8** (1988) 46–63.
8. J. Connor, *On strong matrix summability with respect to a modulus and statistical convergence*, Canad. Math. Bull. **32** (2) (1989) 194–198.
9. H. Fast, *Sur la convergence statistique*, Colloq. Math. **2** (1951) 241–244.
10. J. Ferrera, *Convergence of polinomial level sets*, Trans. Amer. Math. Soc. **350** (12) (1988) 4757–4773.
11. J.A. Fridy, *On statistical convergence*, Analysis **5** (1985) 301–313.
12. G.H. Hardy and J.E. Littlewood, *Sur la série d'une fonction à carré sommable*, C. R. Acad. Sci. Paris **153** (1913) 1307–-1309.
13. F. Hausdorff, *Grundzüge der Mengenlehre,* Verlag Von Veit & Company, Leipzig, 1914. Reprinted by Chelsea Publishing Company, New York, 1949.
14. C. Kuratowski, *Topology*, Vol. I, Academic Press, New York, 1966.

15. I.J. Maddox, *Sequence spaces defined by a modulus function*, Math. Proc. Cambridge Phil. Soc. **100** (1986) 161–166.
16. F. Nuray and B. E. Rhodes, *Statistical convergence of sequences of sets*, Fasc. Math. **49** (2012) 87–99.
17. W.H. Ruckle. *FK spaces in which the sequence of coordinate vectors is bounded*, Canad. J. Math. **25** (1973) 973–978.
18. E. Savaş, *On strong almost A-summability with respect to a modulus and statistical convergence*, Indian J. Pure and Appl. Math., **23** (1992) 217–222.
19. I. J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly **66** (1959) 361–375.
20. Y. Sonntag and C. Zalinescu, *Set Convergences. An Attempt of Classification*, Trans. Amer. Math. Soc. **340** (1) (1993) 199–226.
21. Y. Sonntag and C. Zalinescu, *Convergences for sequences of sets and linear mappings*, J. Math. Anal. Appl. **188** (1994) 616–640.
22. R.A. Wijsman, *Convergence of sequences of convex sets, cones and functions*, Bull. Amer. Math. Soc. **70** (1964) 186–188.
23. R.A. Wijsman, *Convergence of sequences of convex sets*, Cones and Functions II, Trans. Amer. Math. Soc. **123** (1) (1966) 32–45.

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