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WEAKLY MID-$(p_1,\dots,p_m)$-SUMMING MULTILINEAR OPERATORS

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Abstract. We introduce a new ideal of the weakly mid-$(p_1,\dots,p_m)$-summing multilinear operators as multilinear version of weakly mid-p- summing linear operators. Using the space of mid-p-summable sequences, we present a characterization given by summability property. Also, we give an analogue of the Pietsch domination theorem.

1. Introduction

Using the notion of operator p-summability, Karn and Sinha [9] introduced the class of weakly mid-p-summing operators. There are many attempts available in the literature to study the ideal of weakly mid-p-summing operators in different textures; see [3, 4, 8, 9]. Recently, Baweja and Philip [1] studied some aspects of the operator ideal of weakly mid-p-summing operators using the theory of tensor products. This concept led them to a new space of summability between p-summable and weakly p-summable sequences. A bounded linear operator $T: X \to Y$ between Banach spaces is weakly mid-p-summing if it sends weakly p-summable sequences to mid p-summable sequences. Equivalently, by [9, Theorem 4.4], $S\circ T : X \to \ell_p$ is p-summing for all bounded linear operators S from Y to ℓ_p. The aim of this paper is to study the multilinear version of this class by considering the $(p_1,\dots,p_m)$-dominated multilinear operators $S\circ T$.

This paper is divided into three sections. In the first section, we recall some basic definitions and notations concerning the linear and multilinear summing operators, some facts on sequence spaces. We introduce in the second section, a

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multilinear version of weakly mid-p-summing operators introduced in [9] for which the resulting vector space is a Banach ideal of multilinear mappings. Finally, in last section we prove a natural analogue of Pietsch domination theorem of this new class.

2. Notations and Backgrounds

The notation used in the paper is in the general standard. The letters X and Y shall denote Banach spaces over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). The closed unit ball of X is denoted by B_X, and its topological dual is denoted by X^*. Let m be in $\mathbb{N}$ and let $X_1, \ldots, X_m, Y$ be Banach spaces over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). We will denote by $\mathcal{L}(X_1, \ldots, X_m; Y)$ the Banach space of all bounded multilinear operators from $X_1 \times \cdots \times X_m$ into Y equipped with the operator norm. If $m = 1$, then $\mathcal{L}(X_1, \ldots, X_m; Y) = \mathcal{L}(X; Y)$ is the Banach space of all bounded linear operators $T : X \to Y$ endowed with the usual sup norm.

Let X be a Banach space and let $1 \leq p < \infty$. Let us recall some important Banach sequence spaces, which will be used in what follows.
The Banach space of absolutely p-summable sequences is given by

$$\ell_p(X) := \left\{ (x_j)_{j=1}^{\infty} \subset X \colon \parallel (x_j)_{j=1}^{\infty} \parallel_p^p := \sum_{j=1}^{\infty} \parallel x_j \parallel^p < \infty \right\}.$$

The Banach space of weakly p-summable sequences is given by

$$\ell_p^w(X) := \left\{ (x_j)_{j=1}^{\infty} \subset X \colon \parallel (x_j)_{j=1}^{\infty} \parallel_{w,p}^p := \sup_{x^* \in B_{X^*}} \sum_{j=1}^{\infty} |x^*(x_j)|^p < \infty \right\}.$$

The Banach space of unconditionally p-summable sequences is given by

$$\ell_p^u(X) := \left\{ (x_j)_{j=1}^{\infty} \in \ell_p^w(X) : lim_k \parallel (x_j)_{j=k}^{\infty} \parallel_{w,p} = 0 \right\}.$$

For more details on these spaces, we refer the reader to [6].

Definition 2.1 ([3,9])**.** A sequence $(x_j)_{j=1}^{\infty}$ in a Banach space X is said to be *mid-p-summable*, $1 \leq p < \infty$, if $((x_n^*(x_j))_{j=1}^{\infty})_{n=1}^{\infty} \in \ell_p(\ell_p)$ whenever $(x_n^*)_{n=1}^{\infty} \in \ell_p^w(X^*)$. The space of all such sequences shall be denoted by $\ell_p^{mid}(X)$, which is a Banach space under the norm

$$\parallel (x_j)_{j=1}^{\infty} \parallel_{mid,p} := \sup_{(x_n^*)_{n=1}^{\infty} \in B_{\ell_p^w(X^*)}} \left(\sum_{j=1}^{\infty} \sum_{n=1}^{\infty} |x_n^*(x_j)|^p \right)^{\frac{1}{p}}.$$

It can be easily seen that

$$\ell_p(X) \subsetneq \ell_p^{mid}(X) \subsetneq \ell_p^w(X).$$

Consider $1 \leq p, p_1, \ldots, p_m < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. A multilinear operator T from $X_1 \times \cdots \times X_m$ into Y is called $(p_1, \ldots, p_m)$-dominated (see [10–12]) if $(T(x_j^1, \ldots, x_j^m))_{j=1}^{\infty} \in \ell_p(Y)$ whenever $(x_j^i)_{j=1}^{\infty} \in \ell_{p_i}^w(X_i)$. This is equivalent to say that, there is a constant $C>0$ such that for any $(x_j^i)_{j=1}^\infty \in \ell_{p_i}^w(X_i), (1\le i\le m)$, we have

$$\left\|(T(x_j^1,\dots,x_j^m))_{j=1}^\infty\right\|_p \le C\prod_{i=1}^m \left\|(x_j^i)_{j=1}^\infty\right\|_{w,p_i}. \tag{2.1}$$

The class of all $(p_1,\dots,p_n)$-dominated multilinear operators from $X_1\times\cdots\times X_m$ into Y is denoted by $\Pi_{p_1,\dots,p_m}(X_1,\dots,X_m;Y)$, which is a Banach space equipped with the norm $\pi_{p_1,\dots,p_m}(T)$ that is the smallest constant C such that the inequality (2.1) holds.
This class satisfies a Pietsch's domination theorem, so T is $(p_1,\dots,p_m)$-dominated if and only if there are a constant $C>0$ and regular Borel probability measures μ_i on $B_{X_i^*}, 1\le i\le m$ (with the weak star topology), so that for all $(x^1,\dots,x^m)\in X_1\times\cdots\times X_m$, the inequality

$$\left\|T\left(x^1,\dots,x^m\right)\right\| \le C\prod_{i=1}^m\left(\int_{B_{X_i^*}}\left|\left\langle x^i,x_i^*\right\rangle\right|^{p_i}d\mu_i\left(x_i^*\right)\right)^{\frac{1}{p_i}} \tag{2.2}$$

is valid. In this case, $\pi_{p_1,\dots,p_m}(T)$ is the least of all $C>0$ such that (2.2) holds.

3. Weakly mid-$(p_1,\dots,p_m)$-summing multilinear mappings

The following definition was given by Karn and Sinha [9, Definition 4.1] and [3, Definition 2.1].

Definition 3.1. Let T be in $\mathcal{L}(X;Y)$. For $1\le p<\infty$, we say that T is weakly *mid-p*-summing, if $(T(x_j))_{j=1}^\infty\in\ell_p^{mid}(Y)$ whenever $(x_j)_{j=1}^\infty\in\ell_p^w(X)$. The set of all weakly *mid-p*-summing operators from X into Y will be denoted by $Lt_p(X,Y)$. Moreover, $lt_p(T)=\sup\{\pi_p(ST): S\in\mathcal{L}(Y,\ell_p), \|S\|\le 1\}$ and $(Lt_p(X,Y), lt_p(\cdot))$ is a Banach operator ideal (see [9, Proposition 4.7]).

The weakly mid-$(p_1,\dots,p_m)$-summing operators are characterized by the following theorem (see [9, Theorem 4.4] and [3, Theorem 2.3]).

Theorem 3.2. *Let T be in $\mathcal{L}(X;Y)$. For $1\le p<\infty$, the following properties are equivalent:*

(1) *T is weakly mid-$(p_1,\dots,p_m)$-summing.*

(2) *$S\circ T\in\Pi_p(X;\ell_p)$ for every $S\in\mathcal{L}(Y;\ell_p)$.*

(3) *$T\circ U\in\Pi_p^d(\ell_{p^*};Y)$ for every $U\in\mathcal{L}(\ell_{p^*};X)$, $1/p+1/p^*=1$ (where Π_p^d is the dual ideal of Π_p).*

(4) *The map $\widehat{T}:\ell_p^w(X)\to\ell_p^{mid}(Y)$ given by $\widehat{T}((x_j)_{j=1}^\infty)=(T(x_j))_{j=1}^\infty$ is well-defined and continuous, and $\|\widehat{T}\|=lt_p(T)$.*

Now, we extend the preceding definition to the case of multilinear operators.

Definition 3.3. Let $1\le p,p_1,\dots,p_m<\infty$ with $1/p=1/p_1+\cdots+1/p_m$. A multilinear operator $T:X_1\times\cdots\times X_m\to Y$ is weakly *mid*-$(p_1,\dots,p_m)$-summing if $(T(x_j^1,\dots,x_j^m))_{j=1}^\infty\in\ell_p^{mid}(Y)$ whenever $(x_j^i)_{j=1}^\infty\in\ell_{p_i}^w(X_i)(1\le i\le m)$. The set of all weakly *mid*-$(p_1,\ldots,p_m)$-summing operators from $X_1\times\cdots\times X_m$ into Y will be denoted by $Lt_{p_1,\ldots,p_m}(X_1,\ldots,X_m;Y)$.

Replacing Y in [8, Definition 2.1] by ℓ_p, we have the following lemma.

Lemma 3.4. *Let $(x_j)_{j=1}^\infty\in\ell_p^w(X)$. Then the following properties are equivalent.*

(1) $(x_j)_{j=1}^\infty\in\ell_p^{mid}(X)$.

(2) *$(S(x_j))_{j=1}^\infty\in\ell_p(\ell_p)$ for all $S\in\mathcal{L}(X;\ell_p)$. Moreover*

$$\|(x_j)_{j=1}^\infty\|_{mid,p}=\sup\{\|(S((x_j))_{j=1}^\infty\|:\ S\in\mathcal{L}(X;\ell_p),\|S\|\leq 1\}.$$

The following proposition characterizes the weakly *mid*-$(p_1,\ldots,p_m)$-summing multilinear operators.

Proposition 3.5. *For $T\in\mathcal{L}(X_1,\ldots,X_m;Y)$, the following statements are equivalent:*

(1) *T is weakly mid-$(p_1,\ldots,p_m)$-summing.*
(2) *$S\circ T\in\Pi_{p_1,\ldots,p_m}(X_1,\ldots,X_m;\ell_p)$ for every $S\in\mathcal{L}(Y;\ell_p)$.*

Proof. We take $T\in Lt_{p_1,\ldots,p_m}(X_1,\ldots,X_m;Y)$. Then by the definition of weakly *mid*-$(p_1,\ldots,p_m)$-summing operators, we have $(T(x_j^1,\ldots,x_j^m))_{j=1}^\infty\in\ell_p^{mid}(Y)$ for every $(x_j^i)_{j=1}^\infty\in\ell_{p_i}^w(X_i)$. Lemma 3.4 gives us $(S\circ T(x_j^1,\ldots,x_j^m))_{j=1}^\infty\in\ell_p(\ell_p)$ for every $S\in\mathcal{L}(Y;\ell_p)$ and $(x_j^i)_{j=1}^\infty\in\ell_{p_i}^w(X_i)$. Hence $(S\circ T(x_j^1,\ldots,x_j^m))_{j=1}^\infty\in\ell_p(\ell_p)$ for every $(x_j^i)_{j=1}^\infty\in\ell_{p_i}^w(X_i)$ and for every $S\in\mathcal{L}(Y;\ell_p)$, which yields $S\circ T\in\Pi_{p_1,\ldots,p_m}(X_1,\ldots,X_m,\ell_p)$ for every $S\in\mathcal{L}(Y;\ell_p)$. □

Using [3, Proposition 1.9] and the abstract approach of [2], we can see that the next theorem is immediate consequences of [2, Proposition 2.4], with the equivalences involving due to [2, Corollary 2.6].

Theorem 3.6. *For $T\in\mathcal{L}(X_1,\ldots,X_m;Y)$, the following statements are equivalent:*

(1) $T\in Lt_{p_1,\ldots,p_m}(X_1,\ldots,X_m;Y)$.
(2) *The induced map $\widetilde{T}:\ell_{p_1}^w(X_1)\times\cdots\times\ell_{p_m}^w(X_m)\to\ell_p^{mid}(Y)$ given by*

$$\widetilde{T}((x_j^1)_{j=1}^\infty,\ldots,(x_j^m)_{j=1}^\infty)=(T(x_j^1,\ldots,x_j^m))_{j=1}^\infty$$

is a well-defined continuous multilinear operator.
(3) *The induced map $\widehat{T}:\ell_{p_1}^u(X_1)\times\cdots\times\ell_{p_m}^u(X_m)\to\ell_p^{mid}(Y)$ given by*

$$\widehat{T}((x_j^1)_{j=1}^\infty,\ldots,(x_j^m)_{j=1}^\infty)=(T(x_j^1,\ldots,x_j^m))_{j=1}^\infty$$

is a well-defined and continuous multilinear operator.

(4) *There is a constant $C>0$ such that for any $(x_j^i)_{j=1}^\infty\in\ell_{p_i}^w(X_i)(1\leq i\leq m)$, we have*

$$\left\|\left(T\left(x_j^1,\ldots,x_j^m\right)\right)_{j=1}^\infty\right\|_{mid,p}\leq C\prod_{i=1}^m\left\|\left(x_j^i\right)_{j=1}^\infty\right\|_{w,p_i},\tag{3.1}$$

with $lt_{p_1,\dots,p_m}(T) = \|\widetilde{T}\| = \|\widehat{T}\| = inf(C)$ *such that the inequality* (3.1) *holds.*

Proof. Clearly, the implication (2) $\Longrightarrow$ (1) is obvious. Supposing (1), it is easy to prove that $\widetilde{T}$ is well-defined and multilinear. Let us show that it is continuous in the case $m = 2$ (identically, we can show that the general case). To do this, let $(x_j^1)_{j=1}^\infty \in \ell_{p_1}^w(X_1)$ and $(x_j^2)_{j=1}^\infty \in \ell_{p_2}^w(X_2)$ be sequences such that $(x_j^1, x_j^2) \mapsto (x, y)$ in $\ell_{p_1}^w(X_1) \times \ell_{p_2}^w(X_2)$ and $\widetilde{T} : (x_j, y_j) \longrightarrow z$ in $\ell_p^{mid}(Y)$. Then $x_j \longrightarrow x$ in $\ell_{p_1}^w(X_1)$ and $y_j \longrightarrow y$ in $\ell_{p_1}^w(X_1)$. Set

$$x_j = (\xi_{j,m})_{m=1}^\infty\,,\ \ y_j = (\eta_{j,m})_{m=1}^\infty\,,\ \ x = (\xi_m)_{m=1}^\infty\,,\ \ y = (\eta_m)_{m=1}^\infty\,,\ \ z = (w_m)_{m=1}^\infty\,.$$

By the condition $\ell_p^w(\cdot) \overset{1}{\hookrightarrow} \ell_\infty(\cdot)$, the convergence in the sequence spaces is guaranteed. We are working with coordinatewise convergence, so $\xi_{j,m} \overset{j}{\longrightarrow} \xi_m$ in X_1 and $\eta_{j,m} \overset{j}{\longrightarrow} \eta_m$ in X_2 for every m. The continuity of T gives $T\left(\xi_{j,m}, \eta_{j,m}\right) \overset{j}{\longrightarrow} T\left(\xi_m, \eta_m\right)$ in Y for every m. From

$$\left(T\left(\xi_{j,m}, \eta_{j,m}\right)\right)_{m=1}^\infty = \widetilde{T}\left(x_j, y_j\right) \overset{j}{\longrightarrow} z = (w_m)_{m=1}^\infty \ \text{ in } \ell_p^{mid}(Y),$$

it follows that $T\left(\xi_{j,m}, \eta_{j,m}\right) \overset{j}{\longrightarrow} w_m$ in Y for any m. Therefore $T\left(\xi_m, \eta_m\right) = w_m$ for every m. Finally,

$$\widetilde{T}(x, y) = \widetilde{T}\left((\xi_m)_{m=1}^\infty, (\eta_m)_{m=1}^\infty\right) = \left(T\left(\xi_m, \eta_m\right)\right)_{m=1}^\infty = (w_m)_{m=1}^\infty = z,$$

showing that the graph of $\widetilde{T}$ is closed. It follows from the closed graph theorem for multilinear operators (see, [7]) that the continuity of $\widetilde{T}$ is obvious.
The equivalence (2) $\Leftrightarrow$ (3) is obvious by replacing in [2, Corollary 2.6] X_i with $\ell_{p_i}^u$ and Z_i by $\ell_{p_i}^w$ for all $1 \leq i \leq m$.
For (3) $\Longleftrightarrow$ (4), it is clear that to see the equivalence (2) $\Longleftrightarrow$ (4) and we have (2) $\Longleftrightarrow$ (3), and then (3) $\Longleftrightarrow$ (4). $\square$

Recall that, an ideal of multilinear operators (or multi-ideal) is a subclass $\mathcal{M}$ of all continuous multilinear operators between Banach spaces such that for all $m \in \mathbb{N}$ and Banach spaces $X_1, \dots, X_m$ and Y, the components

$$\mathcal{M}\left(X_1, \dots, X_m; Y\right) := \mathcal{L}\left(X_1, \dots, X_m; Y\right) \cap \mathcal{M}$$

satisfy the following conditions:

(1) $\mathcal{M}\left(X_1, \dots, X_m; Y\right)$ is a linear subspace of $\mathcal{L}\left(X_1, \dots, X_m; Y\right)$ that contains the m-linear mappings of finite type.

(2) The ideal property: If $T \in \mathcal{M}\left(G_1, \dots, G_m; F\right), u_j \in \mathcal{L}\left(X_j, G_j\right)$ for $j = 1, \dots, m$ and $v \in \mathcal{L}(F, Y)$, then $v \circ T \circ (u_1, \dots, u_m)$ is in $\mathcal{M}\left(X_1, \dots, X_m; Y\right)$.

If $\|\cdot\|_{\mathcal{M}} : \mathcal{M} \to \mathbb{R}^+$ satisfies

(a) $\left(\mathcal{M}\left(X_1, \dots, X_m; Y\right), \|\cdot\|_{\mathcal{M}}\right)$ is a normed (Banach) space for all Banach spaces $X_1, \dots, X_m, Y$ and all m;
(b) The m-linear form $T^m : \mathbb{K}^m \to \mathbb{K}$ given by $T^m\left(x^1, \dots, x^m\right) = x^1 \dots x^m$ satisfies $\|T^m\|_{\mathcal{M}} = 1$ for all m;

(c) If $T \in \mathcal{M}(G_1, \ldots, G_m; F), u_j \in \mathcal{L}(X_j, G_j)$ for $j = 1, \ldots, m$ and $v \in \mathcal{L}(F, Y)$. then

$$\|v \circ T \circ (u_1, \ldots, u_m)\|_{\mathcal{M}} \leq \|v\| \|T\|_{\mathcal{M}} \|u_1\| \ldots \|u_m\|.$$

Then we say that $(\mathcal{M}, \|\cdot\|_{\mathcal{M}})$ is a normed (Banach) multi-ideal. For more details, see [12].

Theorem 3.7. *The class $(Lt_{p_1,\ldots,p_m}, lt_{p_1,\ldots,p_m}(\cdot))$ is a Banach ideal of multilinear operators.*

Proof. Using the abstract framework, notation, and language in [2], we find that a multilinear operator T is weakly mid-$(p_1, \ldots, p_m)$-summing multilinear mappings if and only if T is $(\ell_{p_1}^w(\cdot), \ldots, \ell_{p_m}^w(\cdot); \ell_p^{mid}(\cdot))$-summing. Since $1/p = 1/p_1 + \cdots + 1/p_m$, we obtain

$$\ell_{p_1}^w(\mathbb{K}) \cdots \ell_{p_m}^w(\mathbb{K}) = \ell_{p_1}(\mathbb{K}) \cdots \ell_{p_m}(\mathbb{K}) \overset{1}{\hookrightarrow} \ell_p^{mid}(\mathbb{K}) = \ell_p(\mathbb{K}).$$

In addition, all the sequence classes involved are linearly stable (see [3]). So, from [2, Theorem 3.6] it follows that $(Lt_{p_1,\ldots,p_m}, lt_{p_1,\ldots,p_m}(\cdot))$ is a Banach ideal of multilinear operators. □

Proposition 3.8. *For $T \in Lt_{p_1,\ldots,p_m}(X_1, \ldots, X_m; Y)$, we have $lt_{p_1,\ldots,p_m}(T) = w_{p_1,\ldots,p_m}(T)$, where $lt_{p_1,\ldots,p_m}(T) = sup\{\pi_{p_1,\ldots,p_m}(ST) : S \in \mathcal{L}(Y; \ell_p), \|S\| \leq 1\}$.*

Proof. Let $T \in Lt_{p_1,\ldots,p_m}(X_1, \ldots, X_m; Y)$ and $S \in \mathcal{L}(Y; \ell_p)$ with $\|S\| \leq 1$. Here we use that the spaces $\ell_p^w(Y^*)$ and $\mathcal{L}(Y; \ell_p)$ are canonically isometrically isomorphic via the correspondence $S : \ell_p^w(Y^*) \longrightarrow \mathcal{L}(Y; \ell_p)$ defined by $S(y^*)(y) = (y_n^*)_{n=1}^{\infty}$ for every $y^* = (y_n^*)_{n=1}^{\infty} \in \ell_p^w(Y^*)$ and $y \in Y$ (see [5, Proposition 8.2(2)]). So there exists $(y_n^*)_{n=1}^{\infty} \in B_{\ell_p^w(Y^*)}$ such that $S(y) = (y_n^*(y))_{n=1}^{\infty}$ for every $y \in Y$. Thus

$$\begin{aligned}\left(\sum_{j=1}^{\infty} \left\|ST(x_j^1, \ldots, x_j^m)\right\|_p^p\right)^{1/p} &= \left(\sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \left|\langle y_n^*, T(x_j^1, \ldots, x_j^m)\rangle\right|^p\right)^{1/p} \\ &\leq lt_{p_1,\ldots,p_m}(T) \prod_{i=1}^{m} \left\|(x_j^i)_{j=1}^{\infty}\right\|_{w,p_i}\end{aligned}$$

for every $(x_j^i)_{j=1}^{\infty} \in \ell_{p_i}^w(X_i)$. Therefore $ST \in \Pi_{p_1,\ldots,p_m}(X_1, \ldots, X_m; \ell_p)$ and $w_{p_1,\ldots,p_m}(T) \leq lt_{p_1,\ldots,p_m}(T)$. From

$$\begin{aligned}\left(\sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \left|\langle y_n^*, T(x_j^1, \ldots, x_j^m)\rangle\right|^p\right)^{1/p} &= \left(\sum_{j=1}^{\infty} \left\|ST(x_j^1, \ldots, x_j^m)\right\|_p^p\right)^{1/p} \\ &\leq \pi_{p_1,\ldots,p_m}(S \circ T) \prod_{i=1}^{m} \left\|(x_j^i)_{j=1}^{\infty}\right\|_{w,p_i}.\end{aligned}$$

we obtain $lt_{p_1,\ldots,p_m}(T) \leq w_{p_1,\ldots,p_m}^{mid}(T)$, proving that $lt_{p_1,\ldots,p_m}(T) = w_{p_1,\ldots,p_m}^{mid}(T)$. □

4. Pietsch domination theorem

Using the domination theorem for the class of $\Pi_{p_1,\ldots,p_m}$, we give the domination theorem the class of weakly *mid*-$(p_1,\ldots,p_m)$-summing multilinear operators.

Theorem 4.1. *Let $T \in \mathcal{L}(X_1,\ldots,X_m;Y)$. The following statements are equivalent:*

(a) *T is weakly mid-$(p_1,\ldots,p_m)$-summing*

(b) *There are a constant $C>0$ and a regular Borel probability measures μ_i on $B_{X_i^*}$ such that*

$$\left(\sum_{n=1}^{\infty}|\langle y_n^*, T(x^1,\ldots,x^m)\rangle|^p\right)^{1/p} \le C\prod_{i=1}^{m}\left(\int_{B_{X_i^*}}\left|x^i\left(x_i^*\right)\right|^{p_i} d\mu(x_i^*)\right)^{1/p_i} \tag{4.1}$$

for all $(y_n^*)_{n=1}^{\infty} \in \ell_p^w(Y^*), x^i \in X_i (1\le i\le m)$.

Proof. (a)⇒(b) If $T \in Lt_{p_1,\ldots,p_m}(X_1,\ldots,X_m;Y)$, then $S\circ T : X_1\times\cdots\times X_m \to \ell_p$ is $(p_1,\ldots,p_m)$-dominated. Therefore, there are a constant $C>0$ and a regular Borel probability measures μ_i on $B_{X_i^*}$ such that

$$\|S\circ T(x^1,\ldots,x^m)\| \le C\prod_{i=1}^{m}\left(\int_{B_{X_i^*}}\left|x^i\left(x_i^*\right)\right|^{p_i} d\mu(x_i^*)\right)^{1/p_i}$$

for all $x^i\in X_i$ and for all $S\in\mathcal{L}(Y;\ell_p)$. For $S\in\mathcal{L}(Y;\ell_p)$, there exists $(y_n^*)_{n=1}^{\infty}\in \ell_p^w(Y^*)$ such that $S(y) = (y_n^*(y))_{n=1}^{\infty}$ for every $y\in Y$. Thus

$$\|S\circ T(x^1,\ldots,x^m)\| = \left(\sum_{n=1}^{\infty}|\langle y_n^*, T(x^1,\ldots,x^m)\rangle|^p\right)^{1/p}$$

for all $x_i\in X_i$, that is,

$$\left(\sum_{n=1}^{\infty}|\langle y_n^*, T(x^1,\ldots,x^m)\rangle|^p\right)^{1/p} \le C\prod_{i=1}^{m}\left(\int_{B_{X_i^*}}\left|x^i\left(x_i^*\right)\right|^{p_i} d\mu(x_i^*)\right)^{1/p_i}$$

for all $(y_n^*)_{n=1}^{\infty}\in\ell_p^w(Y^*), x^i\in X_i$.

(b)⇒(a) Let $(x_j^1,\ldots,x_j^m)\in(X_1\times\cdots\times X_m)$ $(1\le j\le k)$. Form (4.1), we get

$$\sum_{n=1}^{\infty}|\langle y_n^*, T(x_j^1,\ldots,x_j^m)\rangle|^p \le C^p\prod_{i=1}^{m}\left(\int_{B_{X_i^*}}\left|x^i\left(x_i^*\right)\right|^{p_i} d\mu(x_i^*)\right)^{p/p_i},$$

and by the Hölder's inequality $((\sum_{j=1}^{k}\prod_{i=1}^{m}(\lambda_j^i)^p)^{1/p} \leq \prod_{i=1}^{m}(\sum_{j=1}^{k}(\lambda_j^i)^{p_i})^{1/p_i},\ \lambda_j^i \geq 0)$, we obtain

$$\begin{aligned}
\left(\sum_{j=1}^{k}\sum_{n=1}^{\infty}|\langle y_n^*, T(x_j^1,\dots,x_j^m)\rangle|^p\right)^{1/p} &\leq C\left(\sum_{j=1}^{k}\prod_{i=1}^{m}\left(\int_{B_{X_i^*}}\left|x_j^i\left(x_i^*\right)\right|^{p_i}d\mu(x_i^*)\right)^{p/p_i}\right)^{1/p}\\
&= C\left(\sum_{j=1}^{k}\left[\prod_{i=1}^{m}\left(\int_{B_{X_i^*}}\left|x_j^i\left(x_i^*\right)\right|^{p_i}d\mu(x_i^*)\right)^{1/p_i}\right]^p\right)^{1/p}\\
&\leq C\prod_{i=1}^{m}\left(\sum_{j=1}^{k}\int_{B_{X_i^*}}\left|x_j^i\left(x_i^*\right)\right|^{p_i}d\mu(x_i^*)\right)^{1/p_i}\\
&\leq C\prod_{i=1}^{m}\left(\int_{B_{X_i^*}}\sum_{j=1}^{k}\left|x_j^i\left(x_i^*\right)\right|^{p_i}d\mu(x_i^*)\right)^{1/p_i}\\
&\leq C\prod_{i=1}^{m}\|(x_j^i)_{j=1}^k\|_{w,p_i}.
\end{aligned}$$

We deduce $(((\langle T(x_j^1,\dots,x_j^m), y_n^*\rangle)_{n=1}^{\infty})_{j=1}^{k} \in \ell_p(\ell_p)$ whenever $(y_n^*)_{n=1}^{\infty} \in \ell_p^w(Y^*)$, $(x_j^i)_{j=1}^k \in \ell_{p_i}^w(X_i)$, that is, $(T(x_j^1,\dots,x_j^m))_{j=1}^k \in \ell_p^{mid}(Y)$ whenever $(x_j^i)_{j=1}^k \in \ell_{p_i}^w(X_i)$. $\square$

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