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AN ALGORITHM FOR DOUBLY UNITARY LAURENT POLYNOMIALS

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Abstract. We propose two algorithms that for any ring $\mathbf{R}$, given a doubly unitary Laurent polynomial $g \in \mathbf{R}[X, X^{-1}]$, compute $h \in \mathbf{R}[X, X^{-1}]$ such that $gh \in \mathbf{R}[X^{-1} + X]$ and gh is monic. The first algorithm is directly extracted from the classical proof. The second algorithm is more direct and simpler. It relies on a symmetrization technique.

1. Introduction and preliminaries

In [2, Proposition 9], it was shown that for any ring $\mathbf{R}$, any doubly unitary Laurent polynomial in $\mathbf{R}[X, X^{-1}]$ divides a monic polynomial at $X^{-1} + X$. As a consequence of this result, we know that for any ring $\mathbf{R}$, $\mathbf{R}\langle X, X^{-1}\rangle$ (the localization of the ring $\mathbf{R}[X, X^{-1}]$ at the monoid of doubly monic polynomials) is a finitely-generated free $R\langle X^{-1} + X\rangle$-module of rank 2, where for a ring $\mathbf{A}$, $\mathbf{A}\langle X\rangle$ denotes the localization of $\mathbf{A}[X]$ at the monoid $U(X)$ of monic polynomials at X. This also gives a process that systematically translates results related to projective modules over $\mathbf{R}[X_1, \ldots, X_n]$ to projective modules over $\mathbf{R}[X_1^{\pm}, \ldots, X_n^{\pm}]$; see [2, 4]. It is also worth pointing out that doubly unitary Laurent polynomials play an important role in the conception of algorithms for completion of unimodular vectors with entries in a multivariate Laurent polynomial ring $\mathbf{K}[X_1^{\pm}, \ldots, X_n^{\pm}]$, where $\mathbf{K}$ is an infinite field [1, 4].

In this paper, we propose two algorithms realizing the above-mentioned result. The first algorithm is directly extracted from the classical proof. The second algorithm is more direct and simple. It relies on a symmetrization technique.

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All the considered rings are commutative and unitary. The undefined terminology is standard as in [3].

2. An algorithm extracted from the classical proof

Definition 2.1. Let $\mathbf{R}$ be a ring.

(1) For $f = a_mX^m + a_{m+1}X^{m+1} + \cdots + a_{m+n}X^{m+n} \in \mathbf{R}[X, X^{-1}]$, with $a_m, a_{m+n} \in \mathbf{R} \setminus \{0\}$, $n \in \mathbb{N}$, and $m \in \mathbb{Z}$, the nonnegative integer n will be called the *degree* of f and denoted by $\deg(f)$. We convene that $\deg(0) = -1$.

Also, $\mathfrak{h}(f) := a_{m+n}$ is called the *head coefficient* of f, and $\mathfrak{t}(f) := a_m$ is called the *tail coefficient* of f.

(2) A Laurent polynomial $f(X) \in \mathbf{R}[X, X^{-1}]$ is said to be *doubly monic* (resp., *doubly unitary*) if both $\mathfrak{h}(f)$ and $\mathfrak{t}(f)$ are equal to 1 (resp., are invertible). Note that if the basic ring $\mathbf{R}$ is trivial, so is the ring $\mathbf{R}[X, X^{-1}]$ of Laurent polynomials, and 0 is doubly monic.

Recall that an element b of a ring $\mathbf{B}$ is said to be *integral* over a subring $\mathbf{A}$ of $\mathbf{B}$, if there are $n \geq 1$ and $a_j \in \mathbf{A}$ such that $b^n + a_{n-1}b^{n-1} + \cdots + a_1b + a_0 = 0$. That is to say, b is a root of a monic polynomial over $\mathbf{A}$. If every element of $\mathbf{B}$ is integral over $\mathbf{A}$, then it is said that $\mathbf{B}$ is integral over $\mathbf{A}$, or also, $\mathbf{B}$ is an integral extension of $\mathbf{A}$. Recall also that the integral closure of $\mathbf{A}$ in $\mathbf{B}$ is the set of elements in $\mathbf{B}$ that are integral over $\mathbf{A}$. It is a subring of $\mathbf{B}$ containing $\mathbf{A}$.

Proposition 2.2. *Let $\mathbf{R}$ be a ring. Then, for any doubly unitary Laurent polynomial $g \in \mathbf{R}[X, X^{-1}]$, there exists $h \in \mathbf{R}[X, X^{-1}]$ such that gh is a monic polynomial at $X^{-1} + X$.*

In other words, for any $g(X) = a_0X^m + a_1X^{m+1} + \cdots + a_nX^{m+n}$ in $\mathbb{Z}[a_0, a_1, \ldots, a_{n-1}, a_n][X, X^{-1}]$, there exists $h \in \mathbb{Z}[a_0^{\pm}, a_1, \ldots, a_{n-1}, a_n^{\pm}][X, X^{-1}]$ such gh is a monic polynomial at $X^{-1}+X$ with coefficients in $\mathbb{Z}[a_0^{\pm}, a_1, \ldots, a_{n-1}, a_n^{\pm}]$.

Classical proof ([2]). If $g = X^n$ for some $n \in \mathbb{Z}$, then $X^n(X^{-n-1} + X^{-n+1}) = X^{-1} + X \in U(X + X^{-1})$. So, we can suppose that $g \in U(X)$ and $g(0) \in \mathbf{R}^{\times}$. We have the inclusions

$$\begin{aligned}\mathbf{R} &\subseteq \mathbf{R}[X^{-1} + X]/(g\mathbf{R}[X, X^{-1}] \cap \mathbf{R}[X^{-1} + X]) \\ &\subseteq \mathbf{R}[X, X^{-1}]/g\mathbf{R}[X, X^{-1}] = S^{-1}\mathbf{R}[X]/S^{-1}g\mathbf{R}[X] \\ &\cong \overline{S}^{-1}(\mathbf{R}[X]/g\mathbf{R}[X]) \cong \mathbf{R}[\theta, \theta^{-1}],\end{aligned}$$

where $\overline{S}$ is the multiplicative set generated by the class $\theta = \overline{X}$ of X modulo $g\mathbf{R}[X]$. Since g is a doubly unitary polynomial, both θ and θ^{-1} are integral over $\mathbf{R}$, and thus, $\mathbf{R}[\theta, \theta^{-1}]$ is integral over $\mathbf{R}$. It follows that $\mathbf{R}[X^{-1} + X]/(g\mathbf{R}[X, X^{-1}] \cap \mathbf{R}[X^{-1} + X])$ is integral over $\mathbf{R}$, that is, $g\mathbf{R}[X, X^{-1}] \cap \mathbf{R}[X^{-1} + X]$ contains a monic polynomial ($\in U(X^{-1} + X)$), as desired.

Roughly speaking, the proof above says that in the ring $\mathbf{R}[X, X^{-1}]$ modulo g, as both X^{-1} and X are integral over $\mathbf{R}$, $X^{-1} + X$ is integral over $\mathbf{R}$ as well.

The computation hidden in the classical proof.

The proof above is good, but not enough. Imagine that we pick a polynomial in $g = \mathbf{R}[X, X^{-1}]$, say $g = X^{-2} + 2X^{-2} + 3 - X$, and want to explicitly find $h \in \mathbf{R}[X, X^{-1}]$ such that gh is a monic polynomial at $X^{-1} + X$. How can we find h?
The solution is (as often) to find the algorithm behind the classical proof. In fact, in our situation, it is just a polynomial identity ensuing from equality to zero modulo g in the ring $\mathbf{R}[X, X^{-1}]$. This latter equality follows from "gluing" two integral dependencies over $\mathbf{R}$ (namely, those of X^{-1} and X modulo g). In more details, consider a Laurent polynomial $g(X) = a_0X^m + a_1X^{m+1} + \cdots + a_nX^{m+n} = X^m(a_0 + a_1X + \cdots + a_{n-1}X^{n-1} + a_nX^n) = X^m\tilde{g}$ of degree less than or equal to n, where $m \in \mathbb{Z}$. Set

$$\begin{aligned}
\mathbf{B} &= ((X^{-1})^{n-1}, (X^{-1})^{n-2}, \ldots, (X^{-1})^2, X^{-1}, 1, X, X^2, \ldots, X^{n-2}, X^{n-1}),\\
&= (u_1, \ldots, u_{2n-1}),\\
L_1 &= (X^{-1} + X) \cdot (X^{-1})^{n-1} - a_0^{-1}\tilde{g}(X)X^{-n}\\
&= (-a_0^{-1}a_1, 1 - a_0^{-1}a_2, -a_0^{-1}a_3, \ldots, -a_0^{-1}a_{n-1}, -a_0^{-1}a_n, 0, \ldots, 0)_{\mathbf{B}},\\
L_2 &= (X^{-1} + X) \cdot (X^{-1})^{n-2} = (1, 0, 1, \ldots, 0, \ldots, 0)_{\mathbf{B}},\\
&\ \vdots\\
L_{n-1} &= (X^{-1} + X) \cdot (X^{-1}) = (\overbrace{0, \ldots, 0}^{n-3}, 1, 0, 1, \overbrace{0, \ldots, 0}^{n-1})_{\mathbf{B}},\\
L_n &= (X^{-1} + X) \cdot 1 = (\overbrace{0, \ldots, 0}^{n-2}, 1, 0, 1, \overbrace{0, \ldots, 0}^{n-2})_{\mathbf{B}},\\
L_{n+1} &= (X^{-1} + X) \cdot X = (\overbrace{0, \ldots, 0}^{n-1}, 1, 0, 1, \overbrace{0, \ldots, 0}^{n-3})_{\mathbf{B}},\\
&\ \vdots\\
L_{2n-2} &= (X^{-1} + X) \cdot X^{n-2} = (0, \ldots, 0, 1, 0, 1)_{\mathbf{B}},\\
L_{2n-1} &= (X^{-1} + X) \cdot X^{n-1} - a_n^{-1}\tilde{g}(X)\\
&= (0, \ldots, 0, -a_n^{-1}a_0, -a_n^{-1}a_1, \ldots, -a_n^{-1}a_{n-3}, 1 - a_n^{-1}a_{n-2}, -a_n^{-1}a_{n-1})_{\mathbf{B}}.
\end{aligned}$$

Thus, for $1 \leq i \leq 2n-1$, denoting by $L_i = (b_{i,1}, \ldots, b_{i,2n-1})_{\mathbf{B}}$, and setting

$$B = (b_{i,j})_{1 \leq i,j \leq 2n-1}$$
$$= \begin{pmatrix}
-a_0^{-1}a_1 & 1 - a_0^{-1}a_2 & a_0^{-1}a_3 & \cdots & -a_0^{-1}a_n & 0 & \cdots & 0\\
1 & 0 & 1 & & & & & \\
& \ddots & \ddots & \ddots & & & & \\
& & 1 & 0 & 1 & & & \\
& & & & \ddots & \ddots & \ddots & \\
& & & & & 1 & 0 & 1\\
0 & \cdots & 0 & -a_n^{-1}a_0 & \cdots & -a_n^{-1}a_{n-3} & 1 - a_n^{-1}a_{n-2} & -a_n^{-1}a_{n-1}
\end{pmatrix},$$

and $A = (X^{-1} + X)\mathbf{I}_{2n-1} - B$, we have

$$B\ {}^{\mathrm{t}}(u_1,\dots,u_{n-1},1,u_{n+1},\dots,u_{2n-1}) = {}^{\mathrm{t}}(a_0^{-1}\tilde{g}(X)X^{-n},0,\dots,0,a_n^{-1}\tilde{g}(X)).$$

It follows from Cramer's rule that $\det A$ (which is a monic polynomial at $(X^{-1}+X)$) is equal to the determinant of the matrix obtained from A by replacing its nth column by ${}^{\mathrm{t}}(a_0^{-1}\tilde{g}(X)X^{-n},0,\dots,0,a_n^{-1}\tilde{g}(X))$. Thus, denoting by $\tilde{h}$ the determinant of the matrix obtained from A by replacing its nth column by ${}^{\mathrm{t}}(a_0^{-1}X^{-n},0,\dots,0,a_n^{-1})$, we obtain $\det A = \tilde{g}\,\tilde{h}$, where $\det A$ is a monic polynomial at $(X^{-1}+X)$ with coefficients in $\mathbb{Z}[a_0^{\pm},a_1,\dots,a_{n-1},a_n^{\pm}]$ and of degree $2n-1$. As $X^m(X^{-m-1}+X^{-m+1}) = (X^{-1}+X)$, we conclude that

$$(X^{-1}+X)\cdot\det A = g\cdot(X^{-m-1}+X^{-m+1})\cdot\tilde{h},$$

is a monic polynomial at $(X^{-1}+X)$ with coefficients in $\mathbb{Z}[a_0^{\pm},a_1,\dots,a_{n-1},a_n^{\pm}]$ and of degree $2n$.

Now, let us go back to our example $g = X^{-2}+2X^{-1}+3-X = X^{-2}(1+2X+3X^2-X^3) = X^{-2}\,\tilde{g}$ with $\tilde{g} = 1+2X+3X^2-X^3$. Keeping the notation as above, we obtain

$$\begin{aligned}
\det A &= \begin{vmatrix} 2+(X^{-1}+X) & 2 & -1 & 0 & 0 \\ -1 & (X^{-1}+X) & -1 & 0 & 0 \\ 0 & -1 & (X^{-1}+X) & -1 & 0 \\ 0 & 0 & -1 & (X^{-1}+X) & -1 \\ 0 & 0 & -1 & -3 & -3+(X^{-1}+X) \end{vmatrix} \\
&= 1 - X - 4X^2 - 16X^3 - 9X^4 - 17X^5 - 9X^6 - 16X^7 - 4X^8 - X^9 + X^{10} \\
&= \tilde{g}(X)\begin{vmatrix} 2+(X^{-1}+X) & 2 & X^{-3} & 0 & 0 \\ -1 & (X^{-1}+X) & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & (X^{-1}+X) & -1 \\ 0 & 0 & -1 & -3 & -3+(X^{-1}+X) \end{vmatrix} \\
&= \tilde{g}(X)\cdot(1-3X-X^2-4X^3-X^4-4X^5-2X^6-X^7),
\end{aligned}$$

and finally,

$$\begin{aligned}
(1-3X-X^2-4X^3-X^4-4X^5-2X^6-X^7)(X+X^3)\cdot g &= (X^{-1}+X)\cdot\det A \\
&= p(X^{-1}+X)
\end{aligned}$$

with $p(t) = t^6 - t^5 - 9t^4 - 12t^3 + 8t^2 + 13t$.

3. A direct algorithm

We propose in this section a new simple proof (an algorithm) for Proposition 2.2 based on the symmetrization of the considered doubly unitary Laurent polynomial.

Definition 3.1. Let $\mathbf{R}$ be a ring. A Laurent polynomial $f(X) \in \mathbf{R}[X,X^{-1}]$ is said to be *symmetric at X and X^{-1}* (or, simply, symmetric) if $f(X^{-1}) = f(X)$.

Lemma 3.2. *Let* $\mathbf{R}$ *be a ring. Then,*

$$\mathbf{R}[X^{-1}+X] = \{f \in \mathbf{R}[X,X^{-1}] \mid f \text{ is symmetric at } X \text{ and } X^{-1}\}.$$

In particular, any doubly monic symmetric Laurent polynomial is a monic polynomial at $X^{-1}+X$ *(i.e., it can be expressed as* $g(X^{-1}+X)$ *with a monic polynomial* $g \in \mathbf{R}[X]$*).*

Proof. We clearly have

$$\mathbf{R}[X^{-1}+X] \subseteq \{f \in \mathbf{R}[X,X^{-1}] \mid f \text{ is symmetric at } X \text{ and } X^{-1}\}.$$

Conversely, let $f \in \mathbf{R}[X,X^{-1}] \setminus \{0\}$ be a symmetric Laurent polynomial at X and X^{-1} of degree $2n$ (the degree of a symmetric Laurent polynomial is necessarily even). We proceed by induction on n. If $n = 0$, then $f = aX^m$ for some $a \in \mathbf{R} \setminus \{0\}$. As it is symmetric, necessarily $m = 0$, and thus, $f \in \mathbf{R} \subseteq \mathbf{R}[X^{-1}+X]$. Now, suppose that $n \geq 1$. The polynomial $g = f - a(X^{-1}+X)^n$, where a is the head coefficient of f, is also symmetric with $\deg(g) < \deg(f)$. The induction hypothesis applies and gives the desired result.

□

From the above proof, the following algorithm follows immediately.

Algorithm 3.3. (Computing the source of a symmetric Laurent polynomial)
Input: A symmetric Laurent polynomial $f \in \mathbf{R}[X,X^{-1}]$ of degree $2n$.
Output: A polynomial $\tilde{f} \in \mathbf{R}[X]$ of degree n such that $f = \tilde{f}(X^{-1}+X)$ ($\tilde{f}$ will be called the *source* of f).

```
sourcesymm(Laurent polynomial f) {
  if (deg(f) ≤ 0) {
  return f;
  }
  return 𝔥(f)X^(deg(f)/2)+sourcesymm(f − 𝔥(f)(X^{-1}+X)^(deg(f)/2));
}
```

A direct constructive proof of Proposition 2.2. By virtue of Lemma 3.2, just take $h(X) = \mathfrak{t}(g)^{-1}\mathfrak{h}(g)^{-1}g(X^{-1})$.

From the above proof, the following algorithm follows immediately.

Algorithm 3.4. (Computing a multiple of a doubly unitary Laurent polynomial which is a monic polynomial at $X^{-1}+X$)
Input: A doubly unitary Laurent polynomial $g \in \mathbf{R}[X,X^{-1}]$ of degree n.
Output: $[h,f]$ where $h \in \mathbf{R}[X,X^{-1}]$ and $f \in \mathbf{R}[X]$ monic of degree n such that $gh = f(X^{-1}+X)$.

```
symmdoub(doubly unitary Laurent polynomial g) {
  return [𝔱(g)^{-1}𝔥(g)^{-1}g(X^{-1}), sourcesymm(𝔱(g)^{-1}𝔥(g)^{-1}g(X)g(X^{-1}))];
}
```

Going back to the example $g = X^{-2} + 2X^{-1} + 3 - X$ computed with the algorithm given in Section 2, we find the following result from Algorithm 3.4:

$$(X^{-1} - 3 - 2X - X^2) \cdot g = q(X^{-1} + X) \text{ with } q(t) = t^3 - t^2 - 8t - 13$$

of degree 3 instead of degree 6 found by the algorithm given in Section 2.

References

1. M. Amidou and I. Yengui, *An algorithm for unimodular completion over Laurent polynomial rings*, Linear Algebra Appl. **429** (2008), no. 7, 1687–1698.
2. S. Barhoumi and I. Yengui, *On a localization of the Laurent polynomial ring*, JP J. Algebra Number Theory Appl. **5** (2005), no. 3, 591–602.
3. H. Lombardi and C. Quitté, *Commutative Algebra: Constructive Methods. Finite Projective Modules,* Updated and revised edition. Translated from the French by T.K. Roblot, Algebra and Applications 20. Springer, Dordrecht, 2015.
4. I. Yengui, *Constructive Commutative Algebra. Projective Modules Over Polynomial Rings and Dynamical Gröbner Bases*, Lecture Notes in Math. 2138, Springer, Cham, 2015.

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