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# BIDERIVATIONS ON TRIVIAL EXTENSION ALGEBRAS

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ABSTRACT. We investigate biderivations, inner biderivations, and extremal biderivations on a trivial extensions algebra. Our results are examined for some special trivial extension algebras, such as triangular algebras and certain generalized matrix algebras, renovating some older results.

## 1. Introduction and preliminaries

Let $A$ be an algebra over a unital commutative ring, and let $X$ be an $A$-module. A linear map $D : A \longrightarrow X$ is called a derivation if

$$D(ab) = aD(b) + D(a)b \quad (a, b \in A).$$

A bilinear map $D$ from $A \times A$ into $X$ is called a biderivation, if it is a derivation with respect to both components; that is, the mappings ${}_aD, D_b : A \to X$ defined by ${}_aD(b) = D(a, b) = D_b(a)$ are derivations for every $a, b \in A$. Every biderivation of the form $D(a, b) = \lambda[a, b]$, where $\lambda \in Z(A, X) = \{\lambda \in X; a\lambda = \lambda a, \text{ for all } a \in A\}$ is called an inner biderivation. We say that $\Phi : A \times A \longrightarrow A$ is an extremal biderivation if there exists $x \in X$ such that $[x, [A, A]] = 0$ and $\Phi(a, b) = [a, [b, x]]$ for all $a, b \in A$

During the last decades various mappings on certain algebras have been studied by many authors; see, for example, [1,2,4,5,7,8,12,15–17] and references therein. In particular, Brešar, Martindale, and Miers [7] showed that every biderivation on a noncommutative prime ring is inner. After that, Benkovič [3] showed that under some conditions every biderivation on a triangular algebra is a sum of an extremal and an inner biderivation.

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Our main aim in this paper is to study biderivations on a trivial extension algebra. Let us introduce a trivial extension algebra. Let $A$ be an algebra with $X$ as an $A$-module. Then the direct product $A \times X$ equipped with the pointwise module operations and the multiplication

$$(a, x)(b, y) = (ab, ay + xb) \qquad (a, b \in A, x, y \in X) \tag{1.1}$$

forms an algebra, which is called a trivial extension algebra and will be denoted by $A \ltimes X$.

The most famous example of a trivial extension algebra is a triangular algebra $\mathfrak{T} = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$, as it can be identified with the trivial extension algebra $(A \oplus B) \ltimes X$ in a natural way; see [12]. Certain maps on a triangular algebra have been characterized by some authors; see, for example, [3, 8, 13–15, 17]. Some of their results have been extended to trivial extension algebras; see [1, 4, 10, 12, 16].

In this paper, we investigate biderivations, inner biderivations, and extremal biderivations on a trivial extensions algebra. In Section 2, we study the structures of biderivations and extremal biderivations on a trivial extension algebra, and we focus on those conditions under which a trivial extension algebra enjoys a nonzero extremal biderivation. Section 3 is devoted to the study of those conditions on a unital algebra with a nontrivial idempotent, under which biderivations can be presented as the sum of extremal biderivation with certain type of biderivations. Finally, inner biderivations on a trivial extension algebras are investigated.

## 2. Biderivations and extremal biderivations on $A \ltimes X$

We begin with the following straightforward result characterizing the structure of a derivation on $A \ltimes X$ (see [12, Theorem 3.1]).

**Lemma 2.1.** *Every derivation $D : A \ltimes X \to A \ltimes X$, enjoys the presentation*

$$D(a, x) = (D_A(a) + T_A(x), D_X(a) + T_X(x)) \quad (a \in A, x \in X), \tag{2.1}$$

*where $D_A : A \to A$ and $D_X : A \to X$ are derivations and $T_A : X \to A$ and $T_X : X \to X$ are linear maps satisfying the following conditions for all $a \in A$ and $x, y \in X$:*

- *$T_A$ is an $A$-module morphism.*
- *$T_X(ax) = aT_X(x) + D_A(a)x$ and $T_X(xa) = T_X(x)a + xD_A(a)$.*
- *$T_A(x)y + xT_A(y) = 0$.*

*Moreover, $D$ is an inner derivation, that is, $D = d_{(a_0,x_0)}$ for some $a_0 \in A$ and $x_0 \in X$, if and only if $D_A = d_{a_0}$ and $D_X = d_{x_0}$ are inner derivations, and also $T_A = 0$ and $T_X = \Delta_{a_0}$, where $\Delta_{a_0}(x) = xa_0 - a_0x$.*

In the following, we get the structure of a biderivation on module extension algebras.

**Theorem 2.2.** *Every biderivation $D : (A \ltimes X) \times (A \ltimes X) \to (A \ltimes X)$ has the form*

$$D\big((a, x), (b, y)\big) = \Big(D_A\left((a, b)\right) + T_A\left((x, y)\right) + \mu_1\left((a, y)\right) + \mu_2\left((x, b)\right), \tag{2.2}$$

$$D_X((a,b)) + T_X((x,y)) + \nu_1((a,y)) + \nu_2((x,b))\Big),$$

*whose component maps satisfy the following conditions for all $a, b \in A, x, y, z \in X$:*

- $D_A : A \times A \to A$ *and* $D_X : A \times A \to X$ *are biderivations.*
- $T_A : X \times X \to A$ *is an $A$-module bimorphism, and* $T_X : X \times X \to X$ *is bilinear.*
- $\mu_1, \mu_2^t : A \times X \to A$ *and* $\nu_1, \nu_2^t : A \times X \to X$ *are bilinear maps that are derivations on their first components. Furthermore, $\mu_1$ and $\mu_2^t$ are $A$-module morphisms on their second components.*
- $T_X(ax, y) = aT_X(x, y) + \mu_1(a, y)x$ *and* $T_X(xa, y) = T_X(x, y)a + x\mu_1(a, y)$.
- $T_X(x, by) = bT_X(x, y) + \mu_2(x, b)y$ *and* $T_X(x, yb) = T_X(x, y)b + y\mu_2(x, b)$.
- $xT_A(y, z) = yT_A(x, z) = -T_A(x, y)z$.
- $\nu_1(a, by) = b\nu_1(a, y) + D_A(a, b)y$ *and* $\nu_1(a, yb) = \nu_1(a, y)b + yD_A(a, b)$.
- $\nu_2(ax, b) = a\nu_2(x, b) + D_A(a, b)x$ *and* $\nu_2(xa, b) = \nu_2(x, b)a + xD_A(a, b)$.
- $\mu_1(a, y)z + y\mu_1(a, z) = 0$ *and* $\mu_2(x, b)z + x\mu_2(z, b) = 0$.

*Moreover, $D$ is an inner biderivation if and only if there are $f \in Z(A, A) \cap Z(X, A)$ and $g \in Z(A, X)$ such that $D_A(a, b) = f[a, b], \nu_1(a, x) = \nu_2^t(a, x) = f[a, x], D_X(a, b) = g[a, b]$ and $\mu_1, \mu_2, T_A,$ and $T_X$ are zero maps.*

*Proof.* The map $D$ is a biderivation if and only if ${}_{(a,x)}D$ and $D_{(b,y)}$ are derivations for every $(a, x), (b, y) \in A \bowtie X$. Then by Lemma 2.1, we can write

$${}_{(a,x)}D((b, y)) = \big({}^{(a,x)}D_A(b) +^{(a,x)} T_A(y),^{(a,x)} D_X(b) +^{(a,x)} T_X(y)\big),$$

and

$$D_{(b,y)}((a, x)) = \big(D_A^{(b,y)}(a) + T_A^{(b,y)}(x), D_X^{(b,y)}(a) + T_X^{(b,y)}(x)\big),$$

where ${}^{(a,x)}D_A, D_A^{(b,y)} : A \to A$ and ${}^{(a,x)}D_X, D_X^{(b,y)} : A \to X$ are derivations, and ${}^{(a,x)}T_A, T_A^{(b,y)} : X \to A$ and ${}^{(a,x)}T_X, T_X^{(b,y)} : X \to X$ are linear maps such that for all $a \in A, x, z \in X$,

(1) $T_A^{(b,y)}$ is an $A$-module map,
(2) $T_X^{(b,y)}(ax) = D_A^{(b,y)}(a)x + aT_X^{(b,y)}(x)$ and $T_X^{(b,y)}(xa) = xD_A^{(b,y)}(a) + T_X^{(b,y)}(x)a$,
(3) $T_A^{(b,y)}(x)z + xT_A^{(b,y)}(z) = 0$.

Also, there are similar properties such as (4)–(6) for the maps with the symbol $(a, x)$.

Now it is sufficient to define $D_A(a, b) = D_A^{(b,0)}(a)$, $T_A(x, y) = T_A^{(0,y)}(x)$, $\mu_1(a, x) =^{(a,0)} T_A(x)$, $\mu_2(x, a) = T_A^{(a,0)}(x)$, $\nu_1(a, x) =^{(a,0)} T_X(x)$, $\nu_2(x, a) = T_X^{(a,0)}(x)$, $D_X(a, b) = D_X^{(b,0)}(a)$, and $T_X(x, y) = T_X^{(0,y)}(x)$. Then the results follows from properties (1)–(6) and the equations ${}_{(a,x)}D = D_{(b,y)}$ and

$$\begin{aligned} D\left((a, x), (b, y)\right) =& D\left((a, 0), (b, 0)\right) + D\left((a, 0), (0, y)\right) \\ &+ D\left((0, x), (b, 0)\right) + D\left((0, x), (0, y)\right). \end{aligned}$$

Moreover, $D((a, x)(b, y)) = (f, g)[(a, x), (b, y)]$ for some

$$(f, g) \in Z(A \bowtie X, (A \bowtie X)) = (Z(A, A) \cap Z(X, A)) \bowtie Z(A, X),$$

if and only if $D_A((a, b)) = f[a, b], \nu_1(a, x) = \nu_2^t(a, x) = f[a, x]$, and $D_X(a, b) = g[a, b]$, and $\mu_1, \mu_2, T_A$, and $T_X$ are zero maps.

□

We recall that a bilinear map $D : A\times A \to X$ is called an extremal biderivation if $D(a, b) = [a, [b, x]]$, for some $x \in X$ with $[x, [A, A]] = 0$, where $[x, a] = xa - ax$, for each $a \in A, x \in X$. It is obvious that $D = 0$ is an extremal biderivation, which is implemented with $x = 0$.

**Proposition 2.3.** *If Span$[A, A] = A$, then the only extremal biderivation $D : A \times A \to X$ is zero.*

*Proof.* Let for some $x \in X$ we have $[x, [A, A]] = 0$. Then the assumption implies that $x \in Z(A, X)$. Therefore $[A, [A, x]] = [A, 0] = 0$. □

**Example 2.4.** Let $H$ be an infinite-dimensional Hilbert space and let $B(H)$ be the algebra of all bounded linear operators on $H$. It is known from [17, Lemma 5.7] that $\mathrm{Span}[B(H), B(H)] = B(H)$ and so by Proposition 2.3, there is no nonzero extremal biderivation from $B(H) \times B(H)$ to any $B(H)$-module $X$.

With the notations as in Theorem 2.2, we have the following characterization for an extremal biderivation.

**Theorem 2.5.** *A bilinear map $D : (A \rtimes X) \times (A \rtimes X) \to (A \rtimes X)$ is an extremal biderivation implemented with $(a_0, x_0) \in (A \rtimes X)$ if and only if*

- *$D_A$ and $D_X$ are extremal biderivations implemented with $a_0$ and $x_0$, respectively,*
- *$T_A, T_X, \mu_1, \mu_2$ are zero maps and $[a_0, [A, X]] = 0$, and*
- *$\nu_1(a, x) = [a, [x, a_0]], \nu_2(x, a) = [x, [a, a_0]]$, for every $a \in A, x \in X$.*

*Proof.* It is easy to check that $D$ is extremal implemented with $(a_0, x_0) \in (A \rtimes X)$ if and only if for every $(a, x), (b, y) \in (A \rtimes X)$,

$$\begin{aligned} D((a, x), (b, y)) &= [(a, x), [(b, y), (a_0, x_0)]] \\ &= \big([a, [b, a_0]], [a, [b, x_0] + [y, a_0]] + [x, [b, a_0]]\big) \end{aligned}$$

with $[(a_0, x_0), [A \rtimes X, A \rtimes X]] = 0$. Then by the notations as in (2.2), we get $D_A(a, b) = [a, [b, a_0]]$ and $D_X(a, b) = [a, [b, x_0]]$ with $[a_0, [A, A]] = 0$ and $[x_0, [A, A]] = 0$. We further get that $\mu_1, \mu_2, T_A$ and $T_X$ are zero, and also, $[a_0, [A, X]] = 0$, $\nu_1(a, x) = [a, [x, a_0]]$, and $\nu_2(x, a) = [x, [a, a_0]]$. □

From this, we arrive at the following result concerning the existence of nonzero extremal biderivations.

**Corollary 2.6.** *There is a nonzero extremal biderivation form $(A \rtimes X) \times (A \rtimes X)$ to $(A \rtimes X)$ if and only if either there exists an element $a_0 \in A$ such that $a_0 \notin Z(A) \cap Z(X, A)$, $[a_0, [A, A]] = 0$, $[a_0, [A, X]] = 0$, or there exists an element $x_0 \notin Z(A, X)$ with $[x_0, [A, A]] = 0$.*

*Proof.* If $a_o \in A$ such that $a_0 \notin Z(A) \cap Z(X, A)$, $[a_0, [A, A]] = 0$ and $[a_0, [A, X]] = 0$, then by Theorem 2.5, the bilinear map $D : (A \rtimes X) \times (A \rtimes X) \to (A \rtimes X)$ defined by $D((a, x), (b, y)) = [[(a, x), (b, y)], (a_0, 0)] = ([a, [b, a_0]], [x, [b, a_0]] + [a, [y, a_0]])$ is a nonzero extremal biderivation. Similarly, if $x_0 \in X$ such that $x_0 \notin Z(A, X)$ and $[x_0, [A, A]] = 0$, then $D((a, x), (b, y)) = [[(a, x), (b, y)], (0, x_0)] = (0, [a, [b, x_0]])$ is the desired biderivation. The converse can also be derived from Theorem 2.5. □

We examine the latter result for two special cases. First for $A \rtimes A$, equipped with the product of $A$ as the module operations and then for $A \rtimes X_0$ with zero as the module operations of $X$.

**Corollary 2.7.** *Let $X_0$ denote $X$ as an $A$-module equipped with zero module operations. Then the following assertions are equivalent:*

(i) *The algebra $(A \rtimes A)$ enjoys a nonzero extremal biderivation.*
(ii) *$A$ enjoys a nonzero extremal biderivation.*
(iii) *The algebra $(A \rtimes X_0)$ enjoys a nonzero extremal biderivation, for some $X$ as $A$-module equipped with zero module operations.*

*Proof.* Follows immediately from Corollary 2.6. □

## 3. The unital case with a nontrivial idempotent

Let $A$ be a unital algebra and let $B$ be a subalgebra of $A$. We recall that a unital $A$-module $X$ is called left (resp., right) $B$-essential if $bX = 0$ (resp., $Xb = 0$), for some $b \in B$, implies that $b = 0$.

We begin with the following result.

**Proposition 3.1.** *Let $A$ be a unital algebra with a nontrivial idempotent $p$ and $q = 1 - p$. Let $X$ be a unital $A$-module with an $A$-submodule $M$ that is left $pAp$-essential, right $qAq$-essential, with $pm = mq$ for every $m \in M$. Then for every biderivation $D : A \times A \to X$ and each $a, b \in A$ with $[a, b] = 0$, we have*

$$D(a, b) = pD(a, b)q + qD(a, b)p.$$

*Proof.* For each $m \in M$, from [6, Corollary 2.4], we have

$$[p, pmq]D(a, b) = D(p, pmq)[a, b] = 0 = [a, b]D(p, pmq) = D(a, b)[p, pmq].$$

Now since $qp = 0$, we get

$$mqD(a, b) = pmqD(a, b) = 0 = D(a, b)pmq = D(a, b)pm,$$

and so $mqD(a, b)q = 0 = pD(a, b)pm$. This, together with the fact that $M$ is essential, implies that $qD(a, b)q = 0 = pD(a, b)p$. Therefore

$$\begin{aligned} D(a, b) &= pD(a, b)p + pD(a, b)q + qD(a, b)p + qD(a, b)q \\ &= pD(a, b)q + qD(a, b)p. \end{aligned}$$

□

The following result gives a presentation of a biderivation on certain unital algebra enjoying a nontrivial idempotent.

**Theorem 3.2.** *Let $A$ be a unital algebra with a nontrivial idempotent $p$, let $q = 1 - p$, and let $X$ be a unital $A$-module. If $D : A \times A \to X$ is a biderivation such that $D(p, p) \neq 0$, $\big(pD(p, p)q + qD(p, p)p\big) \cap Z(A, X) \subseteq \{0\}$ and $pD(p, p)p + qD(q, q)q = 0$, then $D = \Phi + d$, where $\Phi(a, b) = [a, [b, D(p, p)]$ is an extremal biderivation and $d$ is a biderivation such that $\theta(p, p) = 0$.*

*Proof.* The identity

$$D(p,p)[a,b] = [p,p]D(a,b) = 0 = D(a,b)[p,p] = [a,b]D(p,p)$$

implies that $[D(p,p),[a,b]] = 0$. On the other hand, by a similar argument as Proposition 3.1, the identity $pD(p,p)p + qD(q,q)q = 0$ implies that $D(p,p) = pD(p,p)q + qD(p,p)p$. Now from $D(p,p) \neq 0$, we conclude $D(p,p) \notin Z(A,X)$. So if we define $\Phi$ by $\Phi(a,b) = [a,[b,D(p,p)]]$, for every $a,b \in A$, then it is an extremal biderivation. We also have

$$\begin{aligned}\Phi(p,p) &= [p,[p,D(p,p)]] = [p,[p,pD(p,p)q + qD(p,p)p]]\\ &= pD(p,p)q + qD(p,p)p\\ &= D(p,p),\end{aligned}$$

from which we get $d(p,p) = D(p,p) - \Phi(p,p) = 0$, as required. □

Applying Proposition 3.1 and Theorem 3.2 for a trivial extension algebra we arrive at the following result.

**Corollary 3.3.** *Let $A$ be a unital algebra with a nontrivial idempotent $p$, let $P = (p,0)$, and let $Q = (1,0) - P$. Let $X$ be a unital $A$-module such that $X$ has a left $P(A \ltimes X)P$-essential and right $Q(A \ltimes X)Q$-essential submodule $M$ with $Pm = mQ$ for each $m \in M$. If $D : (A \ltimes X) \times (A \ltimes X) \longrightarrow (A \ltimes X)$ is a biderivation with $D(P,P) \neq 0$ and $[PD(P,P)Q + QD(P,P)P] \cap Z(A \ltimes X) \subseteq \{0\}$, then $D = \Phi + d$, where $\Phi\big((a,x),(b,y)\big) = \big[(a,x),[(b,y),D(P,P)]\big]$, for all $(a,x),(b,y) \in A \ltimes X$, is an extremal biderivation and $d$ is a biderivation with $d(P,P) = 0$.*

We also use Proposition 3.1 and Theorem 3.2 for a generalized matrix algebra and a triangular algebra to obtain the following results of Du and Wang [9] and Benkovič [3].

**Corollary 3.4** (see [9, Proposition 4.2]). *Let $\mathfrak{G} = \begin{bmatrix} A & M \\ N & B \end{bmatrix}$ be a generalized matrix algebra, whose corner algebras $A, B$ are unital. Let $M$ be a left $A$-essential and a right $B$-essentiall module and $N$ a $(B,A)$-bimodule. Let $D : \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$ be a biderivation. If $D(p,p) \neq 0$, where $p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then $D = \Phi + d$, where $\Phi(x,y) = [x,[y,D(p,p)]]$ is an extremal biderivation and $d$ is a biderivation that satisfies $d(p,p) = 0$.*

**Corollary 3.5** (see [3, Proposition 4.10]). *Let $\mathfrak{T} = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$ be a triangular algebra, whose corner algebras $A, B$ are unital and let $M$ be a left $A$-essential and a right $B$-essentiall module. Let $D : \mathfrak{T} \times \mathfrak{T} \to \mathfrak{T}$ be a biderivation. If $D(p,p) \neq 0$, where $p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then $D = \Phi + d$, where $\Phi(x,y) = [x,[y,D(p,p)]]$ is an extremal biderivation and $d$ is a biderivation that satisfies $d(p,p) = 0$.*

We conclude this section with a result concerning the innerness of biderivations on a trivial extension algebra $A \ltimes X$. For this purpose, we need some prerequisites. We first define the subset $K_0(A \ltimes X)$ of $A$ by

$K_0(A \rtimes X) = \{a \in A |\ a' \in A$ exists such that $ax = xa', xa = a'x$ for all $x \in X\}$.

It is clear that, in the case where $X$ is an essential $A$-module, $K_0(A \rtimes X) \subseteq Z(A)$.

**Theorem 3.6.** *Let $A$ be a unital algebra with a nontrivial idempotent $p$, let $q = 1 - p$, and let $X$ be a unital $A$-module satisfying the identity $px = xq$ for all $x \in X$.Let the following conditions hold:*

(i) $K_0(A \rtimes X) = Z(A) \neq A$;

(ii) *If $\alpha(a, x) = 0$, for some $\alpha \in Z(A \rtimes X)$ and $(a, x) \neq 0$, then $\alpha = 0$;*

(iii) *Every derivation from $A \rtimes X$ to $A \rtimes X$ is inner.*

*Then every biderivation $D : (A \rtimes X) \times (A \rtimes X) \longrightarrow (A \rtimes X)$ that satisfies $D\big((p, 0),(p, 0)\big) = 0$, is inner.*

*Proof.* By Theorem 2.2, for every $a, b \in A,\ x, y \in X$, the biderivation $D$ has the form

$$D\big((a, x),(b, y)\big) = \Big(D_A((a, b)) + T_A((x, y)) + \mu_1((a, y)) + \mu_2((x, b)),$$
$$D_X((a, b)) + T_X((x, y)) + \nu_1((a, y)) + \nu_2((x, b))\Big).$$

Also, condition (i) implies that $p \in Z(A)$. Therefore, for all $a \in A$, we have

$$D_A((p, a)) = pD_A((p, a)) + D_A((p, a))p = 2pD_A((p, a)),$$

and so

$$pD_A((p, a)) = p(2pD_A((p, a)) = 2pD_A((p, a)).$$

Thus $pD_A((p, a)) = 0$, and hence $D_A((p, a)) = 0$.

Since $D_{(p,0)} : A \rtimes X \longrightarrow A \rtimes X$ is a derivation and according to $(iii)$, there exists $(a_1, x_1)$ such that $D_{(p,0)}(a, x) = [(a, x),(a_1, x_1)]$. Therefore $a_1 \in Z(A)$, $\mu_2(x, p) = 0$, $D_X(a, p) = [a, x_1]$, and $\nu_2(x, p) = [x, a_1] = xa_1 - a_1x = \acute{a_1}x - a_1x = (\acute{a_1} - a_1)x = \alpha x$, where $\alpha \in Z(A)$. Similarly, we can show that $\nu_1(p, x) = \beta x$ for some $\beta \in Z(A)$.

Fix $x_0 \in X$. Then the map $D_{(0,x_0)} : A \rtimes X \longrightarrow A \rtimes X$ is a derivation, and it must be inner. It follows that, $T_A = 0$, $\mu_1 = 0$, and there exists $a_3 \in Z(A)$ such that $T_X(x, x_0) = [x, a_3]$. Now condition (i) implies that $T_X(x, x_0) = \alpha(x_0)x$. Since $Z(A) \neq A$, we can choose $b_1, b_2 \in A$ such that $[b_1, b_2] \neq 0$. Hence

$$\begin{aligned} 0 = D((b_1, 0),(b_2, 0))[(0, x),(0, x_0)] &= [(b_1, 0),(b_2, 0)]D((0, x),(0, x_0)) \\ &= (0, [b_1, b_2]T_X(x, x_0)) = (0, [b_1, b_2]\alpha(x_0)x). \end{aligned}$$

Therefore,

$$\begin{aligned} (0,(\alpha + \beta)[b_1, b_2]x) &= (0, [b_1, b_2](\nu_1((p, x)) + \nu_2((x, p)))) \\ &= [(b_1, 0),(b_2, 0)]D((p, x),(p, x)) \\ &= D((b_1, 0),(b_2, 0))[(p, x),(p, x)] = 0. \end{aligned}$$

Now, the essentiality of $X$ implies that $[b_1, b_2]\alpha(x_0) = 0 = (\alpha + \beta)[b_1, b_2]$. From the assumption (ii) it follows that $\alpha(x_0) = 0$ and so $T_X = 0$. Moreover, $\alpha + \beta = 0$ or equivalently $\nu_1(p, x) = \beta x = -\nu_2(x, p)$.

Since $p \in Z(A)$ and $qx = xp$, we get $\nu_1((a,x)) = \alpha_0[a,x]$. Indeed,

$$\begin{aligned}
\nu_1((a,x)) &= \nu_1((pap,x)) + \nu_1((qaq,x)) \\
&= pa\nu_1((p,x)) + p\nu_1((a,x))\,p + \nu_1((p,x))\,ap \\
&\quad + qa\nu_1((q,x)) + q\nu_1((a,x))\,q + \nu_1((q,x))\,aq \\
&= pa\beta x + \beta xap - qa\beta x - \beta xaq \\
&= \beta(p-q)[a,x] = \alpha_0[a,x],
\end{aligned}$$

where $\alpha_0 = \beta(p-q)$. Now, similar to $\mu_1$, we can show that $\mu_2 = 0$ and so, if we show that $D_A((a,b)) = \alpha_0[a,b]$ and $D_X = 0$, then the proof is complete. For this end, since $\nu_1((a,bx)) = b\nu_1((a,x)) + D_A((a,b))\,x$, we have $\alpha_0[a,bx] = b\alpha_0[a,x] + D_A((a,b))\,x$. Then $D_A((a,b)) = \alpha_0[a,b]$. Also, from $D_X((p,p)) = 0$ we have

$$\begin{aligned}
D_X((p,b)) = D_X((p,bp)) + D_X((p,bq)) &= D_X((p,bp)) + D_X((p,qb)) \\
&= D_X((p,b))\,p + qD_X((p,b)) \\
&= 2D_X((p,b))p.
\end{aligned}$$

Multiplying $p$ of the right side, we conclude $D_X((p,b))\,p = 0$, and so $D_X((p,b)) = 0$. Hence

$$\begin{aligned}
D_X((a,b)) &= D_X((pa+aq, bp+qb)) \\
&= pD_X((a,bp+qb)) + D_X((a,bp+qb))\,q \\
&= pD_X((a,b))\,p + qD_X((a,b))\,q = 0,
\end{aligned}$$

and this completes the proof. □

Following [8], a linear map $L$ on an algebra $A$ is called a commuting map if $[L(a),a] = 0$ for all $a \in A$. It is easy to check that for every $\lambda$ in the center $Z(A)$ of $A$, and every linear map $\mu$ from $A$ to $Z(A)$, the map

$$L(a) = \lambda a + \mu(a) \quad (a \in A), \tag{3.1}$$

is a commuting map. The commuting maps of this type are called proper.

There is a close relation between commuting maps and biderivations. Indeed, for every commuting map $L : A \longrightarrow A$, the map $D_L : A \times A \longrightarrow A$, which is defined by $D_L(x,y) = [x, L(y)]$ for all $x, y \in A$, is a biderivation, and the innerness of $D_L$ is equivalent to the properness of $L$. This provides a way for characterizing proper commuting maps on a trivial extension algebra by employing Theorem 3.6, which studies innerness of biderivation under certain conditions.

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