Khayyam J. Math. 8 (2022), no. 2, 243-260
DOI: 10.22034/KJM.2022.286589.2263

Khayyam Journal of Mathematics
emis.de/journals/KJM
kjm-math.org

ORTHOGONAL SPLINE COLLOCATION METHODS FOR 1D-PARABOLIC PROBLEMS WITH INTERFACES

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Communicated by P.I. Naumkin

Abstract. Orthogonal spline collocation (OSC) methods are used to solve one-dimensional heat conduction problems with interfaces. Cubic monomial basis functions are used to approximate the solution for spatial discretization and the Crank–Nicolson method for time stepping. Existence and uniqueness of results are established for a discrete problem. This method is easily extended to monomials of a higher degree. We present the results of experiments involving several examples, which show the efficiency of the OSC method. For both cubic and quartic basis functions, the results of numerical experiments demonstrate fourth-order accuracy in L^∞ and L^2 norms and third-order accuracy in the H^1 norm. Moreover, sixth-order superconvergence in a nodal error of derivative of the OSC approximation for quartics is observed. The OSC approach gives rise to almost block diagonal linear systems, which are solved using standard software.

1. Introduction

We consider one-dimensional heat conduction in double-layers as follows:

$$u_t - (\beta(x)u_x)_x = f, \quad x \in [0, L], \quad t > 0, \tag{1.1}$$

subject to the initial and boundary conditions

$$u(x,0) = \begin{cases} u_1(x,0), & x \in [0,\ell), \\ u_2(x,0), & x \in [\ell, L], \end{cases}$$

$$\mu_a u_1(0,t) - \nu_a u_{1x}(0,t) = g_0(t), \quad \mu_b u_2(L,t) + \nu_b u_{2x}(L,t) = g_1(t), \tag{1.2}$$

Date: Received: 16 May 2021; Revised: 3 March 2022; Accepted: 29 June 2022.

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2020 *Mathematics Subject Classification.* Primary 65N22; Secondary 65F10, 65F25.

Key words and phrases. Heat conduction equation, Orthogonal cubic spline collocation methods, Discontinuous data, Cubic monomial basis functions, Almost block diagonal (ABD) matrix.

respectively, where the coefficient $\beta(x)$ is piecewise constant or piecewise continuous with finite jump across the interface $x = x_\ell$, where $x_\ell \in (0, L)$, μ_a, ν_a, μ_b, and ν_b are given constants.
The interface conditions at $x = x_\ell$ are given by

$$u_1(x_\ell, t) = u_2(x_\ell, t) \quad \text{and} \quad k_1 u_{1x}(x_\ell, t) = k_2 u_{2x}(x_\ell, t), \quad t \in (0, T],\ 0 < T < \infty.$$

Here, $\beta_i = \frac{k_i}{\rho_i c_i}$, k_i is the conductivity, ρ_i is the density, c_i is the specific heat, u_i is the temperature, and f_i is the source term, where $i = 1, 2$. We assume that

$$\beta(x) = \begin{cases} \beta_1(x), & x \in [0, \ell), \\ \beta_2(x), & x \in [\ell, L], \end{cases} \qquad f = \begin{cases} f_1(x,t), & x \in [0, \ell), \\ f_2(x,t), & x \in [\ell, L]. \end{cases}$$

Heat conduction in multilayered thin films is often encountered in engineering applications, such as laser process in a gold thin layer padding on a thin chromium layer for micro machining and patterning [8, 9, 12]. Predicting the temperature distribution in a multilayered film is essential for the precision of the laser process. There are several higher-order accurate finite difference schemes that have been developed for solving heat conduction equations with Neumann boundary conditions in one layer [5, 13]. Recently, Sun and Dai [11] have developed a higher-order accurate FD scheme for solving heat conduction in a double-layered thin film with Neumann boundary conditions.
In this paper, we examine the application of a fourth-order orthogonal cubic spline collocation (OSC) method to (1.1) with Dirichlet, Neumann, and also Robin boundary conditions. In this approach, the interface conditions and boundary conditions require no special treatment. The OSC method has been used to solve a wide variety of problems: see, for example, [2, 4, 10]. It yields a C^1 piecewise polynomial approximation, which is of optimal accuracy in the L^∞, L^2, and H^1 norms and possesses superconvergence properties of the nodal points of the mesh. The calculation of the elements of the coefficient matrix in the algebraic system determining the approximate solution is very fast since no integrals need to be evaluated or approximated.
We define

$$[u(x,t)]_{|x=x_\ell} = \lim_{x \to x_\ell^+} u(x,t) - \lim_{x \to x_\ell^-} u(x,t) = u^+(x_\ell, t) - u^-(x_\ell, t),$$

and

$$\begin{aligned} [\beta u_x(x,t)]_{|x=x_\ell} &= \lim_{x \to x_\ell^+} (\beta(x) u_x(x,t)) - \lim_{x \to x_\ell^-} (\beta(x) u_x(x,t)) \\ &= \beta_2(x_\ell^+) u_x(x_\ell^+, t) - \beta_1\left(x_\ell^-\right) u_x(x_\ell^-, t)\,. \end{aligned}$$

We divide the given interval $I = [0, L]$ into two parts namely $I^- = [0, x_\ell)$ and $I^+ = [x_\ell, L]$. Below, we show the grid formulation graphically.

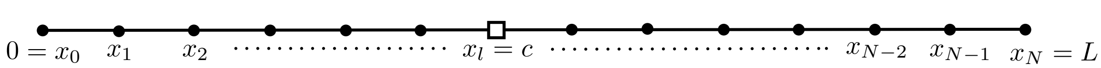

FIGURE 1. Dots represent regular grid points and □ represents interface grid point

An outline of this paper is as follows. In section 2, we prove some required lemmas, which will be frequently used in the corresponding sections. In section 3, we introduce basic notation and formulate the standard OSC approach for solving (1.1)–(1.2) using cubic monomial basis functions. This approach requires no modification at an interface. In section 4, we use this method to solve heat conduction problems with interfaces and present the results of experiments involving several examples, mainly from the literature [1, 11]. From these results, we observe that in each case expected orders of convergence are achieved, namely fourth-order accuracy in the L^∞ and L^2 norms, third-order in the H^1 norm, and fourth-order superconvergence in the first derivative at the nodal points. We conclude our findings in section 5.

2. Preliminary lemmas

In this section, we prove that some matrices are nonsingular, which will use to show the uniqueness of the coefficient matrix in the next section.

Lemma 2.1. *Suppose that*

$$0 < k_1 \le \beta(x) \le k_2, \qquad |\beta'(x)| \le k, \tag{2.1}$$

where k, k_1, k_2 are positive constants that are independent of the discretization parameter h. Then the matrix H_i, $i = 1, 2, \ldots, N$, defined in (3.14) is nonsingular provided h and Δt are sufficiently small.

Proof. To show H_i, $i = 1, 2, \ldots, N$, are invertible, it is sufficient to show that $\det(H_i) \neq 0$. Now consider the determinant of H_i, $i = 1, 2, \ldots, N$, we obtain

$$\det(H_i) = H_{11} \;\; H_{22} - H_{12} \;\; H_{21},$$

where

$$\begin{aligned}
H_{11} &= \left(\frac{h}{2}\left(1-\frac{1}{\sqrt{3}}\right)\right)^2 - \Delta t\beta_j'(\xi_{2i-1})\left(\frac{h}{2}\left(1-\frac{1}{\sqrt{3}}\right)\right) - \Delta t\beta_j(\xi_{2i-1}),\\
H_{12} &= \left(\frac{h}{2}\left(1-\frac{1}{\sqrt{3}}\right)\right)^3\\
&\quad -\frac{3}{2}\Delta t\beta_j'(\xi_{2i-1})\left(\frac{h}{2}\left(1-\frac{1}{\sqrt{3}}\right)\right)^2 - 3\Delta t\beta_j(\xi_{2i-1})\left(\frac{h}{2}\left(1-\frac{1}{\sqrt{3}}\right)\right),\\
H_{21} &= \left(\left(\frac{h}{2}\left(1+\frac{1}{\sqrt{3}}\right)\right)^2 - \Delta t\beta_j'(\xi_{2i})\left(\frac{h}{2}\left(1+\frac{1}{\sqrt{3}}\right)\right) - \Delta t\beta_j(\xi_{2i})\right),\\
H_{22} &= \left(\frac{h}{2}\left(1+\frac{1}{\sqrt{3}}\right)\right)^3 - \frac{3}{2}\Delta t\beta_j'(\xi_{2i})\left(\frac{h}{2}\left(1+\frac{1}{\sqrt{3}}\right)\right)^2\\
&\quad -3\Delta t\beta_j(\xi_{2i})\left(\frac{h}{2}\left(1+\frac{1}{\sqrt{3}}\right)\right).
\end{aligned}$$

Using the hypothesis and simplifying, we have the following expression

$$\begin{aligned}
\det(H_i) &> \frac{h^5}{36\sqrt{3}} + \sqrt{3}\Delta t^2 k_1^2 h + (k_2 - k_1)\Delta t \frac{h^2}{2}\left(\frac{\sqrt{3}}{2}k\Delta t - h\right) \\
&\quad -(k_1 + k_2)\Delta t \frac{h^2}{2}\left(\Delta t k(k+1) + \frac{h}{3\sqrt{3}}\right) \\
&= \frac{h^5}{36\sqrt{3}} \\
&\quad +h\Delta t\left(\sqrt{3}k_1^2\Delta t + (k_2 - k_1)\frac{h}{2}\left(\frac{\sqrt{3}}{2}k\Delta t - h\right)\right) \\
&\quad -h\Delta t\left((k_1 + k_2)\frac{h}{2}\left(\Delta t k(k+1) + \frac{h}{3\sqrt{3}}\right)\right)
\end{aligned}$$

Now for sufficiently small h, we can have

$$\begin{aligned}
\det(H_i) > \frac{h^5}{36\sqrt{3}} &+ h\Delta t\left(\sqrt{3}k_1^2\Delta t + (k_2 - k_1)\frac{h}{2}\left(\frac{\sqrt{3}}{2}k\Delta t - h\right)\right) \\
&-h\Delta t\left((k_1 + k_2)\frac{h}{2}\left(\Delta t k(k+1) + \frac{h}{3\sqrt{3}}\right)\right) > 0.
\end{aligned}$$

Thus, we conclude that all the matrices H_i, $i = 1, 2, \ldots, N$, are invertible. $\square$

Lemma 2.2. *Let $\{h_{qr}\}$ be the entries of H_i, and assume that $|h_{qr}| \leq k$, where $q, r = 1, 2$. Furthermore, assume that $|\beta'(x)| \leq k$, where k is a positive constant that is independent of discretization parameter h. Then the matrices $\Gamma_i, i = 1, 2, \ldots, N$, in (3.17) are invertible.*

Proof. To show that Γ_i, $i = 1, 2, \ldots, N$, are invertible, it is sufficient to show that $\det(\Gamma_i) \neq 0$. Now consider the determinant of Γ_i, $i = 1, 2, \ldots, N$, we obtain

$$\det(\Gamma_i) = \Gamma_{11}\Gamma_{22} - \Gamma_{12}\Gamma_{21},$$

where

$$\begin{aligned}
\Gamma_{11} &= 1 - h_{11}h^2 - h_{21}h^3 - h_{12}h^2 - h_{22}h^3, \\
\Gamma_{12} &= h - (h^2 h_{11} + h^3 h_{21})\left(\left(\frac{h}{2} - \frac{h}{2\sqrt{3}}\right) - \frac{\Delta t}{2}\beta_j'(\xi_{2i-1})\right) \\
&\quad -(h^2 h_{12} + h^3 h_{22})\left(\left(\frac{h}{2} + \frac{h}{2\sqrt{3}}\right) - \frac{\Delta t}{2}\beta_j'(\xi_{2i})\right), \\
\Gamma_{21} &= -2h(h_{11} + h_{12}) - 3h^2(h_{21} + h_{22}), \\
\Gamma_{22} &= 1 - (2hh_{11} + 3h^2 h_{21})\left(\left(\frac{h}{2} - \frac{h}{2\sqrt{3}}\right) - \frac{\Delta t}{2}\beta_j'(\xi_{2i-1})\right) \\
&\quad -(2hh_{12} + 3h^2 h_{22})\left(\left(\frac{h}{2} + \frac{h}{2\sqrt{3}}\right) - \frac{\Delta t}{2}\beta_j'(\xi_{2i})\right).
\end{aligned}$$

Using the given hypothesis $|h_{qr}| \leq k$, we arrive at

$$\begin{aligned}\det(\Gamma_i) \geq & \left(1 - 2kh^2 - 2kh^3\right)\left(1 - 2kh(2h + 3h^2) - k^2\Delta t(2h + 3h^2)\right) \\ & + \left(h - kh(h^2 + h^3) - k^2\Delta t(h^2 + h^3)\right)\left(4kh + 6kh^2\right).\end{aligned}$$

After simplifying, we have the following expression:

$$\begin{aligned}\det(\Gamma_i) \geq & \ 1 - kh(2 + 3h)\left(2h + k\Delta t + 2kh^2(1 + h)(h + k\Delta t)\right) \\ & + kh(2 + 3h)\left(2kh^2(1 + h)(2h + k\Delta t) + 2h\right).\end{aligned}$$

Now for sufficiently small h, we have

$$\begin{aligned}\det(\Gamma_i) \geq & \ 1 - kh(2 + 3h)\left(2h + k\Delta t + 2kh^2(1 + h)(h + k\Delta t)\right) \\ & + kh(2 + 3h)\left(2kh^2(1 + h)(2h + k\Delta t) + 2h\right) \\ > & \ 0.\end{aligned}$$

Thus, we conclude that all matrices $\Gamma_i, \ \ i = 1, 2, \ldots, N$, are invertible. □

Lemma 2.3. *Let $\{h_{qr}\}$ be the entries of H_i, and assume that $|h_{qr}| \leq k$, where $q, r = 1, 2$ and k is a positive constant that is independent of discretization parameter h. Then the entries of matrices $\Gamma_i, i = 1, 2, \ldots, N$, in (3.17) are nonzero.*

Proof. To show the entries of $\Gamma_i, \ i = 1, 2, \ldots, N$, are nonzero. We consider the entries of Γ_i as

$$\Gamma_i = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix}.$$

We have the following expression for Γ_{11}:

$$\Gamma_{11} = 1 - h_{11}h^2 - h_{21}h^3 - h_{12}h^2 - h_{22}h^3.$$

Using the hypothesis of the lemma and simplifying, we obtain

$$\Gamma_{11} \geq 1 - 2kh^2(1 + h).$$

We have the following expression for Γ_{12}:

$$\begin{aligned}\Gamma_{12} = & \ h - (h^2h_{11} + h^3h_{21})\left(\frac{h}{2}\left(1 - \frac{1}{\sqrt{3}}\right)\right) + (h^2h_{11} + h^3h_{21})\frac{\Delta t}{2}\beta_j'(\xi_{2i-1}) \\ & - (h^2h_{12} + h^3h_{22})\left(\frac{h}{2}\left(1 + \frac{1}{\sqrt{3}}\right)\right) + (h^2h_{12} + h^3h_{22})\frac{\Delta t}{2}\beta_j'(\xi_{2i}), \\ \geq & \ h\left(1 - kh(1 + h)(h + k)\right).\end{aligned}$$

Simplifying, we obtain

$$\Gamma_{12} \geq h\left(1 - kh(1 + h)(h + k)\right).$$

We have the following expression for Γ_{21}:

$$\Gamma_{21} = -2h_{11}h - 3h_{21}h^2 - 2h_{12}h - 3h_{22}h^2.$$

Using the hypothesis and simplifying, we obtain

$$\Gamma_{21} \geq -2kh(2 + 3h).$$

We have the following expression for Γ_{22}:

$$\begin{aligned}\Gamma_{22} &= 1-(2hh_{11}+3h^2h_{21})\left(\frac{h}{2}\left(1-\frac{1}{\sqrt{3}}\right)\right)+(2hh_{11}+3h^2h_{21})\frac{\Delta t}{2}\beta_j'(\xi_{2i-1})\\ &\quad -(h^2h_{12}+h^3h_{22})\left(\frac{h}{2}\left(1+\frac{1}{\sqrt{3}}\right)\right)+(h^2h_{12}+h^3h_{22})\frac{\Delta t}{2}\beta_j'(\xi_{2i})\\ &\geq 1-kh(2+3h)(h+k\Delta t).\end{aligned}$$

Simplifying, we obtain

$$\Gamma_{22}\geq 1-kh(2+3h)(h+k\Delta t).$$

Now for sufficiently small h, we have

$$\Gamma_{11}>0,\qquad \Gamma_{12}>0,\qquad \Gamma_{21}<0,\quad \text{and}\quad \Gamma_{22}>0.$$

$\square$

Now we consider first three rows of the homogeneous system of (3.18):

$$L_b\boldsymbol{y}_0 = \boldsymbol{0}, \tag{2.2}$$
$$\Gamma_1\boldsymbol{y}_0+I_2\,\boldsymbol{y}_1 = \boldsymbol{0}, \tag{2.3}$$
$$\Gamma_2\boldsymbol{y}_1+I_2\,\boldsymbol{y}_2 = \boldsymbol{0}. \tag{2.4}$$

From (2.3) and (2.4), we have

$$\boldsymbol{y}_1=-\Gamma_1\boldsymbol{y}_0, \tag{2.5}$$
$$\boldsymbol{y}_2=-\Gamma_2\boldsymbol{y}_1. \tag{2.6}$$

Using (2.5) in (2.6), we obtain $\boldsymbol{y}_2=\Gamma_1\Gamma_2\boldsymbol{y}_0$. By continuing in this way, we get the following expression:

$$\boldsymbol{y}_N=(-1)^N(\Gamma_1\Gamma_2\ldots\tilde{\Gamma}_l\ldots\Gamma_{N-1}\Gamma_N)\boldsymbol{y}_0. \tag{2.7}$$

We rewrite (2.7) as

$$\boldsymbol{y}_N=\boldsymbol{V}\boldsymbol{y}_0, \tag{2.8}$$

where

$$y_N=\begin{bmatrix} y_{N1}\\ y_{N2}\end{bmatrix},\qquad \boldsymbol{V}=\begin{bmatrix} V_{11} & V_{12}\\ V_{21} & V_{22}\end{bmatrix},\qquad \text{and}\quad \boldsymbol{y}_0=\begin{bmatrix} y_{01}\\ y_{02}\end{bmatrix}.$$

Using Lemma 5.2, we note that the matrix $\boldsymbol{V}$ defined in (2.8) is nonsingular. We write (2.8) as

$$\begin{bmatrix} y_{N1}\\ y_{N2}\end{bmatrix}=\begin{bmatrix} V_{11} & V_{12}\\ V_{21} & V_{22}\end{bmatrix}\begin{bmatrix} y_{01}\\ y_{02}\end{bmatrix}.$$

Using Lemma 5.3, we can assume that the entries of the matrix $\boldsymbol{V}$ are nonzero, that is,

$$V_{11}>0,\qquad V_{12}>0,\qquad V_{21}<0,\quad \text{and}\quad V_{22}>0.$$

3. Orthogonal spline collocation method

We use the orthogonal spline collocation methods with cubic monomial basis functions to approximate the solution of (1.1).

Let

$$\rho : 0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = L$$

denote a partition of $\bar{I}$, and set

$$I_i = [x_{i-1}, x_i], \quad h_i = x_i - x_{i-1}, \quad i = 1, 2, \ldots, N,$$

and $h = \max_{1 \leq i \leq N} h_i$.

We define the required function space

$$\mathcal{M}^3_{-1} = \left\{ \Phi : \Phi \in L^2(\Omega),\ \Phi_{|I_i} \in P_3,\ i = 1, 2, \ldots, N \right\},$$

where $L^2(\Omega)$ denotes the space of square integrable functions and P_3 denotes the set of polynomials of degree less than or equal to 3.

The orthogonal spline collocation approximation for (1.1) is a map $u_h : [0,T] \to \mathcal{M}^3_{-1}$ such that

$$\begin{aligned} &\left\{u_{1ht} - (\beta_1(x) u_{1hx})_x\right\}(\xi_i, t) = f_1(\xi_i, t), \qquad x \in [0, x_l),\ t \in (0,T],\ i = 1, 2, \ldots, 2N, \\ &\left\{u_{2ht} - (\beta_2(x) u_{2hx})_x\right\}(\xi_i, t) = f_2(\xi_i, t), \qquad x \in [x_l, L],\ t \in (0,T],\ i = 1, 2, \ldots, 2N, \end{aligned}$$

where ξ_i are the collocation points.

We choose the collocation points $\{\xi_i\}_{i=1}^{2N}$ on $[0, L]$ are two-point Gauss-Legendre quadrature points which are defined by

$$\xi_{2i-1} = x_{i-1} + \frac{1}{2}\left(1 - \frac{1}{\sqrt{3}}\right) h_i \text{ and } \xi_{2i} = x_{i-1} + \frac{1}{2}\left(1 + \frac{1}{\sqrt{3}}\right) h_i,\ i = 1, 2, \ldots, N.$$

If $x = \xi_{2i-1}$, then $x - x_{i-1} = \frac{1}{2}(1 + \rho_1)\, h$ where $\rho_1 = -\frac{1}{\sqrt{3}}$. Similarly, If $x = \xi_{2i}$, then $x - x_{i-1} = \frac{1}{2}(1 + \rho_2)\, h$, where $\rho_2 = -\rho_1$. To make the notation simple, we let

$$(x - x_{i-1}) = \begin{cases} a & \text{if } x = \xi_{2i-1}, \\ b & \text{if } x = \xi_{2i}. \end{cases}$$

The orthogonal spline collocation approximation for the problem (1.1) using Crank-Nicolson scheme for the time stepping is defined as follows: On the interval $[x_{i-1}, x_i)$, $\ i = 1, 2, \ldots, \ell$

$$\frac{u_h^{n+1} - u_h^n}{\Delta t} = \frac{\beta_1}{2}(u_{hxx}^{n+1} + u_{hxx}^n) + \frac{1}{2}(f_1^{n+1}(x,t) + f_1^n(x,t)), \tag{3.1}$$

and on the interval $[x_i, x_{i+1}]$, $\ i = \ell, \ell+1, \ldots, N-1$,

$$\frac{u_h^{n+1} - u_h^n}{\Delta t} = \frac{\beta_2}{2}(u_{hxx}^{n+1} + u_{hxx}^n) + \frac{1}{2}(f_2^{n+1}(x,t) + f_2^n(x,t)), \tag{3.2}$$

where $n = 0, 1, \ldots, K-1$ and $K\Delta t = T$.

Let the approximate solution $u_h(x,t) \in \mathcal{M}^3_{-1}$ have the following expression on each subinterval $[x_{i-1}, x_i]$, $\ i = 1, 2, \ldots, N$, at time $t = t_n$:

$$u_h^n(x,t) = y_{i1}^n(t) + (x - x_{i-1})\, y_{i2}^n(t) + (x - x_{i-1})^2\, z_{i1}^n(t) + (x - x_{i-1})^3\, z_{i2}^n(t). \tag{3.3}$$

Differentiating (3.3) on both sides with respect to x, we obtain

$$u_{hx}^n(x,t) = y_{i2}^n(t) + 2(x - x_{i-1})\, z_{i1}^n(t) + 3(x - x_{i-1})^2\, z_{i2}^n(t),$$

and

$$u_{hxx}^n(x,t) = 2\, z_{i1}^n(t) + 6(x - x_{i-1})\, z_{i2}^n(t). \tag{3.4}$$

Using (3.3) and (3.4), the collocation equation (3.1) on the interval $[x_{i-1}, x_i)$ at $x = \xi_{2i-1}$ and $x = \xi_{2i}$, for $i = 1, 2, \dots, \ell$, are, respectively,

$$\begin{aligned} y_{i1}^{n+1}(t) &+ a\, y_{i2}^{n+1}(t) + a^2\, z_{i1}^{n+1}(t) + a^3\, z_{i2}^{n+1}(t) - \frac{\beta_1}{2}\Delta t \left(2z_{i1}^{n+1}(t) + 6a\, z_{i2}^{n+1}(t)\right) \\ &= y_{i1}^n(t) + a\, y_{i2}^n(t) + a^2\, z_{i1}^n(t) + a^3\, z_{i2}^n(t) + \frac{\beta_1}{2}\Delta t \left(2z_{i1}^n(t) + 6a\, z_{i2}^n(t)\right) \\ &\quad + \frac{1}{2}\Delta t \left(f_1^{n+1}(\xi_{2i-1}, t) + f_1^n(\xi_{2i-1}, t)\right), \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} y_{i1}^{n+1}(t) &+ b\, y_{i2}^{n+1}(t) + b^2\, z_{i1}^{n+1}(t) + b^3\, z_{i2}^{n+1}(t) - \frac{\beta_1}{2}\Delta t \left(2z_{i1}^{n+1}(t) + 6b\, z_{i2}^{n+1}(t)\right) \\ &= y_{i1}^n(t) + b\, y_{i2}^n(t) + b^2\, z_{i1}^n(t) + b^3\, z_{i2}^n(t) + \frac{\beta_1}{2}\Delta t \left(2z_{i1}^n(t) + 6b\, z_{i2}^n(t)\right) \\ &\quad + \frac{1}{2}\Delta t \left(f_1^{n+1}(\xi_{2i}, t) + f_1^n(\xi_{2i}, t)\right). \end{aligned} \tag{3.6}$$

Here a and b are defined as the collocation points which are obtained from the solutions of Legendre polynomial of degree two. And since we use OSC for the interval of consideration, we divide the interval into sub-intervals having uniform mesh size, each of which has two collocation points namely, a and b. Similarly, for $i = \ell + 1, \dots, N$, the collocation equation (3.2) on the interval $[x_{i-1}, x_i]$ at $x = \xi_{2i-1}$ and $x = \xi_{2i}$, respectively, are

$$\begin{aligned} y_{i1}^{n+1}(t) &+ a\, y_{i2}^{n+1}(t) + a^2\, z_{i1}^{n+1}(t) + a^3\, z_{i2}^{n+1}(t) - \frac{\beta_2}{2}\Delta t \left(2z_{i1}^{n+1}(t) + 6a\, z_{i2}^{n+1}(t)\right) \\ &= y_{i1}^n(t) + a\, y_{i2}^n(t) + a^2\, z_{i1}^n(t) + a^3\, z_{i2}^n(t) + \frac{\beta_2}{2}\Delta t \left(2z_{i1}^n(t) + 6a\, z_{i2}^n(t)\right) \\ &\quad + \frac{1}{2}\Delta t \left(f_2^{n+1}(\xi_{2i-1}, t) + f_2^n(\xi_{2i-1}, t)\right), \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} y_{i1}^{n+1}(t) &+ b\, y_{i2}^{n+1}(t) + b^2\, z_{i1}^{n+1}(t) + b^3\, z_{i2}^{n+1}(t) - \frac{\beta_2}{2}\Delta t \left(2z_{i1}^{n+1}(t) + 6b\, z_{i2}^{n+1}(t)\right) \\ &= y_{i1}^n(t) + b\, y_{i2}^n(t) + b^2\, z_{i1}^n(t) + b^3\, z_{i2}^n(t) + \frac{\beta_2}{2}\Delta t \left(2z_{i1}^n(t) + 6b\, z_{i2}^n(t)\right) \\ &\quad + \frac{1}{2}\Delta t \left(f_2^{n+1}(\xi_{2i}, t) + f_2^n(\xi_{2i}, t)\right). \end{aligned} \tag{3.8}$$

Combining equations (3.5)–(3.8), we obtain

$$\left[B - \frac{\Delta t}{2}\beta_j A\right] u_h^{n+1} = \left[B + \frac{\Delta t}{2}\beta_j A\right] u_h^n + \frac{1}{2}\Delta t \left(S(\xi_m, t_{n+1}) + S(\xi_m, t_n)\right), \tag{3.9}$$

where

$$S = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \qquad j = 1,2 \text{ and } m = 1,2,\ldots,2N.$$

Since, the continuity conditions are not in built into the approximate solution u_h^n and u_{hx}^n, we now impose the continuity conditions at $x = x_i,\ i = 1,2,\ldots,\ell-1,\ell+1,\ldots,N$, we obtain

$$y_i = C_i\, y_{i-1} + D_i\, z_{i-1}, \quad i = 1,2,\ldots,\ell-1,\ell+1,\ldots,N, \tag{3.10}$$

where C_i and D_i are 2×2 matrices of the form

$$C_i = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}, \qquad D_i = \begin{bmatrix} h^2 & h^3 \\ 2h & 3h^2 \end{bmatrix},$$

and

$$y_i = [y_{i1}(t),\ \ y_{i2}(t)]^T, \quad z_i = [z_{i1}(t),\ \ z_{i2}(t)]^T.$$

At the interface $x = x_\ell$, u_h and ku_{hx} are discontinuous giving

$$E_\ell \boldsymbol{y}_\ell = C_\ell \boldsymbol{y}_{\ell-1} + D_\ell \boldsymbol{z}_{\ell-1}, \tag{3.11}$$

where

$$E_\ell = \begin{bmatrix} 1 & 0 \\ 0 & k_2(x_\ell^+) \end{bmatrix},\ C_\ell = \begin{bmatrix} 1 & h \\ 0 & k_1(x_\ell^-) \end{bmatrix},\ D_\ell = \begin{bmatrix} h^2 & h^3 \\ 2hk_1(x_\ell^-) & 3h^2k_1(x_\ell^-) \end{bmatrix}.$$

On multiplying (3.11) by E_ℓ^{-1}, we obtain

$$\boldsymbol{y}_\ell = \tilde{C}_\ell \boldsymbol{y}_{\ell-1} + \tilde{D}_\ell \boldsymbol{z}_{\ell-1}, \tag{3.12}$$

where $\tilde{C}_\ell = E_\ell^{-1} C_\ell$ and $\tilde{D}_\ell = E_\ell^{-1} D_\ell$. Combining (3.9)–(3.10) and (3.12), we obtain an almost block diagonal linear system of order $4N+2$ of the form

$$\mathcal{A}(\Delta t)\mathbf{u}^{n+1} = \mathcal{A}(-\Delta t)\mathbf{u}^n + \Delta t\, \mathbf{S^{n+1/2}}. \tag{3.13}$$

Here

$$\mathcal{A}(\Delta t) = \begin{bmatrix} L_b \\ G_1 & H_1 \\ -C_1 & -D_1 & I_2 \\ & & \ddots \\ & & G_\ell & H_\ell \\ & & -\tilde{C}_\ell & -\tilde{D}_\ell & I_2 \\ & & & & \ddots \\ & & & & G_N & H_N \\ & & & & -C_N & -D_N & I_2 \\ & & & & & & R_b \end{bmatrix},$$

where

$$G_i = \begin{bmatrix} 1 & a \\ 1 & b \end{bmatrix}, \qquad H_i = \begin{bmatrix} a^2 & a^3 \\ b^2 & b^3 \end{bmatrix} - \Delta t\, \beta_j \begin{bmatrix} 1 & 3a \\ 1 & 3b \end{bmatrix}, \tag{3.14}$$

$i = 1,2,\ldots,N$, for $j = 1$, we choose the intervals $[x_0,x_1],[x_1,x_2],\ldots,[x_{\ell-1},x_\ell)$ and $j = 2$ we choose the intervals $[x_\ell,x_{\ell+1}],[x_{\ell+1},x_{\ell+2}],\ldots,\ [x_{N-1},x_N]$ and

$$\mathbf{S} = [g_0\ \ \mathbf{f_1}\ \ \mathbf{0}\ \ \mathbf{f_2}\ \ \ldots\ \ \mathbf{f_j}\ \ \mathbf{0}\ldots\mathbf{f_N}\ \ \mathbf{0}\ \ g_1]^T,$$

where $\mathbf{f_j} = \begin{bmatrix} f_1(\xi_{2j-1},t) \\ f_1(\xi_{2j},t) \end{bmatrix}$, $j = 1,\ldots,\ell$, and $\mathbf{f_j} = \begin{bmatrix} f_2(\xi_{2j-1},t) \\ f_2(\xi_{2j},t) \end{bmatrix}$, $j = \ell+1,\ldots,N$. Here L_b and R_b are the contributions from left and right boundary, respectively and I_2 is the identity matrix of size 2×2. Simplifying the right hand side of (3.13), we arrive at

$$\begin{bmatrix} L_b &&&&&&&& \\ G_1 & H_1 &&&&&&& \\ -C_1 & -D_1 & I_2 &&&&&& \\ && \ddots &&&&&& \\ && G_\ell & H_\ell &&&&& \\ && -\tilde{C}_\ell & -\tilde{D}_\ell & I_2 &&&& \\ &&&& \ddots &&&& \\ &&&& G_N & H_N && \\ &&&& -C_N & -D_N & I_2 \\ &&&&&& R_b \end{bmatrix} \begin{bmatrix} y_0 \\ z_0 \\ y_1 \\ \vdots \\ y_{\ell-1} \\ z_{\ell-1} \\ \vdots \\ y_{N-1} \\ z_{N-1} \\ y_N \end{bmatrix} = \begin{bmatrix} g_0 \\ F_1 \\ \mathbf{0} \\ \vdots \\ F_\ell \\ 0 \\ \vdots \\ F_N \\ \mathbf{0} \\ g_1 \end{bmatrix}. \quad (3.15)$$

We solve the system (3.15) by first condensing, that is, by eliminating the variables $\boldsymbol{z}_i,\ i = 1,2,\ldots,N$, in the following way.
From the system (3.15), the i^{th} equation is

$$G_i\,\boldsymbol{y}_{i-1} + H_i\boldsymbol{z}_{i-1} = \boldsymbol{F}_i, \quad (3.16)$$

From the Lemma 4.1, the matrix H_i is nonsingular, we have

$$\boldsymbol{z}_{i-1} = (H_i)^{-1}\left[\mathbf{F_i} - G_i\boldsymbol{y}_{i-1}\right].$$

Substituting $\boldsymbol{z}_{i-1}$ in (3.10), we obtain

$$-C_i\boldsymbol{y}_{i-1} - D_i(H_i)^{-1}\left[\mathbf{F}_i - G_i\boldsymbol{y}_{i-1}\right] + \boldsymbol{y}_i = \mathbf{0},$$

from which it follows that

$$\left[D_i\,H_i^{-1}\,G_i - C_i\right]\boldsymbol{y}_{i-1} + \boldsymbol{y}_i = D_i\,(H_i)^{-1}\,\mathbf{F}_i.$$

The condensed equations are then of the form

$$\Gamma_i\boldsymbol{y}_{i-1} + \boldsymbol{y}_i = D_i\,(H_i)^{-1}\,\mathbf{F}_i, \qquad i = 1,2,\ldots,\ell,$$

where

$$\Gamma_i = D_i\,H_i^{-1}G_i - C_i. \quad (3.17)$$

At the interface point $x = x_\ell$,

$$\boldsymbol{z}_\ell = (H_\ell)^{-1}\left[\mathbf{F}_\ell - G_\ell\boldsymbol{y}_\ell\right],$$

and on substituting in (3.16), we arrive at

$$-\tilde{C}_\ell\boldsymbol{y}_{\ell-1} - \tilde{D}_\ell(H_\ell)^{-1}\left[\mathbf{F}_\ell - G_\ell\boldsymbol{y}_{\ell-1}\right] + \boldsymbol{y}_\ell = (H_\ell)^{-1}.$$

That is,

$$\left[\tilde{D}_\ell H_\ell^{-1}G_\ell - \tilde{C}_\ell\right]\boldsymbol{y}_{\ell-1} + \boldsymbol{y}_\ell = \tilde{D}_\ell\,(H_\ell)^{-1}\mathbf{F}_\ell.$$

The condensed equations are then

$$\tilde{\Gamma}_\ell\boldsymbol{y}_{\ell-1} + \boldsymbol{y}_\ell = \tilde{D}_\ell(H_\ell)^{-1}\boldsymbol{F}_\ell + (H_\ell)^{-1},$$

where

$$\tilde{\Gamma}_\ell = D_\ell H_\ell^{-1} G_\ell - C_\ell.$$

Thus, the system (3.15) is reduced to the smaller ABD linear system of order $2N+2$ of the form:

$$\begin{bmatrix} L_a & & & & & & & \\ \Gamma_1 & I_2 & & & & & & \\ & \Gamma_2 & I_2 & & & & & \\ & & \ddots & & & & & \\ & & & \tilde{\Gamma}_l & I_2 & & & \\ & & & & \ddots & & & \\ & & & & & \Gamma_N & I_2 \\ & & & & & & R_b \end{bmatrix} \begin{bmatrix} \boldsymbol{y}_0 \\ \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \\ \vdots \\ \boldsymbol{y}_{\ell-1} \\ \vdots \\ \boldsymbol{y}_{N-1} \\ \boldsymbol{y}_N \end{bmatrix} = \begin{bmatrix} g_0 \\ D_1(H_1)^{-1}\mathbf{F}_1 \\ D_2(H_2)^{-1}\mathbf{F}_2 \\ \vdots \\ \tilde{D}_l(H_l)^{-1}\mathbf{F}_l \\ \vdots \\ D_N(H_N)^{-1}\mathbf{F}_N \\ g_1 \end{bmatrix}. \tag{3.18}$$

The system (3.18) is solved using a MATLAB version of the ABD solver in [6,7].

Below, we prove the almost block diagonal (ABD) linear system of order $2N+2$ defined in (3.18) has a unique solution.

Theorem 3.1. *Consider the boundary conditions* (1.2) *with*

$$\mu_a \nu_a \geq 0, \quad \mu_b \nu_b \geq 0, \quad |\mu_a| + |\nu_a| \neq 0, \quad |\mu_b| + |\nu_b| \neq 0, \quad |\mu_a| + |\mu_b| \neq 0.$$

For sufficiently small h, the almost block diagonal (ABD) linear system of order $2N+2$ defined in (3.18) *has a unique solution.*

Proof. For proof see [3]

□

4. Numerical results

We now present the results of several numerical experiments based on examples in [1,11]. These examples involve different types of boundary conditions and interface conditions. We compare the approximate solution with the exact solution and estimate the maximum-norm errors for all discretizations. We use grid refinement analysis to find the order of convergence at the grid points. For each problem, estimates of the error in the L^∞, L^2, and H^1 norms are computed. The L^∞ error is estimated by determining the maximum absolute error at 10 equally spaced points in each subinterval $I_j,\ j = 1, \ldots, N$. To estimate the L^2 and H^1 errors, composite three-point Gauss quadrature is used. The maximum absolute error at the nodes, the ℓ^∞ norm, of the approximation and its first derivative is also presented. In each case, the experimental convergence rate of the error is computed using

$$\text{Rate} = \frac{\log(E_N) - \log(E_{2N})}{\log 2},$$

where E_N denotes the norm of the error using N subintervals. In every example considered in this paper, the errors and convergence rates exhibit fourth-order accuracy in the L^∞ and L^2 norms, third-order in the H^1 norm, and fourth-order superconvergence in the ℓ^∞ norm of the first derivative.

Example 4.1 ([1]). *We consider*

$$u_t - (\beta(x)u_x)_x = f(x,t), \quad \beta(x) = \begin{cases} \beta_1(x) = 3e^{-10\left(x^2-\frac{x}{2}\right)^4}, & x \in [0, 0.5), \\ \beta_2(x) = 3, & x \in [0.5, 1], \end{cases}$$

with the initial condition

$$u(x,0) = \begin{cases} \sin(5\pi x), & x \in [0, 0.5), \\ 2\left(x - \frac{1}{2}\right)^7 + 1, & x \in [0.5, 1], \end{cases}$$

and the Dirichlet boundary conditions

$$u(0,t) = 0, \qquad u(1,t) = \frac{65}{64}e^{-t}, \quad t \in [0,1].$$

At the interface $x = 0.5$, both u and u_x are continuous.
The exact solution is

$$u(x,t) = \begin{cases} e^{-t}\sin(5\pi x), & x \in [0, 0.5), \\ \\ e^{-t}\left[2\left(x - \frac{1}{2}\right)^7 + 1\right], & x \in [0.5, 1], \quad t \in [0,1]. \end{cases}.$$

We present the errors and convergence rates using cubics in Tables 1 and 2.

TABLE 1. ℓ^∞ and L^∞ errors and convergence rates using cubics for Example 4.1

N	$\beta_1 = 3e^{-10\left(x^2-\frac{x}{2}\right)^4}$		$\beta_2 = 3$			
	$\|u-U\|_{\ell^\infty}$	Rate	$\|u_x-U_x\|_{\ell^\infty}$	Rate	$\|u-U\|_{L^\infty}$	Rate
20	1.1877(-4)		8.6167(-4)		3.6389(-5)	
40	1.1082(-5)	4.0385	5.2656(-5)	4.0325	2.0353(-6)	4.1602
60	2.1917(-6)	3.9970	1.0357(-5)	4.0106	3.9223(-7)	4.0609
80	6.9385(-7)	3.9981	3.2726(-6)	4.0046	1.2293(-7)	4.0331
100	2.8400(-7)	4.0031	1.3403(-6)	4.0005	5.0101(-8)	4.0224
120	1.3685(-7)	4.0043	6.4707(-7)	3.9941	2.4092(-8)	4.0157

TABLE 2. L^2 and H^1 errors and convergence rates using cubics for Example 4.1

N	$\beta_1 = 3e^{-10\left(x^2-\frac{x}{2}\right)^4}$		$\beta_2 = 3$	
	$\|u-U\|_{L^2}$	Rate	$\|u-U\|_{H^1}$	Rate
20	1.7794(-4)		1.2843(-2)	
40	1.0927(-5)	4.0254	1.5993(-3)	3.0055
60	2.1511(-6)	4.0083	4.7347(-4)	3.0021
80	6.7995(-7)	4.0035	1.9972(-4)	3.0004
100	2.7835(-7)	4.0026	1.0224(-4)	3.0007
120	1.3420(-7)	4.0015	5.9165(-5)	3.0003

We present the errors and convergence rates using quartics in Tables 3 and 4.

TABLE 3. ℓ^∞, L^∞ errors and convergence rates using quartics for Example 4.1

N	$\beta_1 = 3e^{-10\left(x^2-\frac{x}{2}\right)^4}$				$\beta_2 = 3$	
	$\Vert u-U\Vert_{\ell^\infty}$	Rate	$\Vert u_x-U_x\Vert_{\ell^\infty}$	Rate	$\Vert u-U\Vert_{L^\infty}$	Rate
10	2.0380(-5)		1.0495(-4)		2.2330(-4)	
20	2.8529(-7)	6.1586	1.6098(-6)	6.0267	7.9079(-6)	4.8196
30	2.4447(-8)	6.0597	1.4701(-7)	5.9028	1.0641(-6)	4.9467
40	4.3148(-9)	6.0291	2.6014(-8)	6.0200	2.5434(-7)	4.9751
50	1.1286(-9)	6.0101	6.7403(-9)	6.0522	8.3602(-8)	4.9860

TABLE 4. L^2 and H^1 errors and convergence rates using quartics for Example 4.1

N	$\beta_1 = 3e^{-10\left(x^2-\frac{x}{2}\right)^4}$		$\beta_2 = 3$	
	$\Vert u-U\Vert_{L^2}$	Rate	$\Vert u-U\Vert_{H^1}$	Rate
10	1.8087(-4)		1.0122(-2)	
20	5.5388(-6)	5.0292	6.3976(-4)	3.9838
30	7.2709(-7)	5.0078	1.2665(-4)	3.9947
40	1.7236(-7)	5.0078	4.0102(-5)	3.9973
50	5.6451(-8)	5.0022	1.6432(-5)	3.9984

Example 4.2. *We consider the following problem:*

$$u_t - (\beta(x)u_x)_x = f, \quad \beta(x) = \begin{cases} \beta_1, & x \in [0, 0.5\pi), \\ \beta_2, & x \in [0.5\pi, \pi], \end{cases}$$

with the initial condition

$$u(x,0) = \begin{cases} 1-\cos(x), & x \in [0, 0.5\pi,) \\ (1+\cos(3x))^2, & x \in [0.5\pi, \pi], \end{cases}$$

and the Robin boundary conditions

$$u(0,t) + u_x(0,t) = 0, \qquad u(\pi,t) + u_x(\pi,t) = 0, \quad t \in [0,1],$$

and the interface condition at $x = 0.5\pi$,

$$u(0.5\pi^-, t) = u(0.5\pi^+, t), \quad u_x(0.5\pi^-, t) = \frac{1}{6}u_x(0.5\pi^+, t), \quad t \in [0,1].$$

The exact solution is

$$u(x,t) = \begin{cases} e^{-t}(1-\cos(x)), & x \in [0, 0.5\pi), \\ e^{-t}(1+\cos(3x))^2, & x \in [0.5\pi, \pi], \quad t \in [0,1]. \end{cases}$$

We present the errors and convergence rates using cubics in Tables 5–6.

TABLE 5. ℓ^∞ and L^∞ errors and convergence rates using cubics for Example 4.2

N	$\beta_1 = 1$		$\beta_2 = 5$			
	$\lVert u-U\rVert_{\ell^\infty}$	Rate	$\lVert u_x-U_x\rVert_{\ell^\infty}$	Rate	$\lVert u-U\rVert_{L^\infty}$	Rate
20	1.5529(-4)		4.2657(-4)		1.1911(-4)	
40	9.2919(-6)	4.0629	2.0642(-5)	4.3691	6.9517(-7)	4.0988
60	1.8463(-6)	3.9854	4.0563(-6)	4.0128	1.3557(-7)	4.0315
80	5.8453(-7)	3.9979	1.2836(-6)	3.9996	4.2704(-8)	4.0156
100	2.3898(-7)	4.0083	5.2584(-7)	3.9993	1.7477(-8)	4.0037
120	1.1505(-7)	4.0097	2.5313(-7)	4.0098	8.4230(-9)	4.0036

TABLE 6. L^2 and H^1 errors and convergence rates using cubics for Example 4.2

N	$\beta_1 = 1$		$\beta_2 = 5$	
	$\lVert u-U\rVert_{L^2}$	Rate	$\lVert u-U\rVert_{H^1}$	Rate
20	2.8909(-4)		8.0334(-3)	
40	1.7597(-5)	4.0381	9.9605(-4)	3.0117
60	3.4623(-6)	4.0048	1.2444(-4)	3.0005
80	1.0940(-6)	4.0048	1.2444(-4)	3.0005
100	4.4759(-7)	4.0051	6.3677(-5)	3.0025
120	2.1574(-7)	4.0029	3.6842(-6)	3.0012

We present the errors and convergence rates using quartics in Tables 7–8.

TABLE 7. ℓ^∞ and L^∞ errors and convergence rates using quartics for Example 4.2

N	$\beta_1 = 1$		$\beta_2 = 5$			
	$\lVert u-U\rVert_{\ell^\infty}$	Rate	$\lVert u_x-U_x\rVert_{\ell^\infty}$	Rate	$\lVert u-U\rVert_{L^\infty}$	Rate
4	1.4947(-5)		2.6649(-5)		4.7895(-5)	
8	1.6913(-7)	6.4656	4.2086(-7)	5.9846	1.2243(-6)	5.2899
12	1.4080(-8)	6.1309	3.7605(-8)	5.9565	1.5646(-7)	5.0750
16	2.4639(-9)	6.0589	6.7163(-9)	5.9879	3.6752(-8)	5.0354
20	6.4659(-10)	5.9952	1.7562(-9)	6.0113	1.1987(-8)	5.0209
24	2.1667(-10)	5.9968	5.8767(-10)	6.0045	4.8059(-9)	5.0138

TABLE 8. L^2 and H^1 errors and convergence rates using quartics for Example 4.2

	$\beta_1 = 1$		$\beta_2 = 5$	
N	$\|u-U\|_{L^2}$	Rate	$\|u-U\|_{H^1}$	Rate
4	3.1309(-5)		1.4947(-5)	
8	1.0327(-6)	4.9221	1.6913(-7)	4.0283
12	1.3645(-7)	4.9917	1.4080(-8)	3.9886
16	3.2349(-8)	5.0034	2.4639(-9)	3.9929
20	1.0585(-8)	5.0063	6.4659(-10)	3.9955
24	4.2485(-9)	5.0070	2.1667(-10)	3.9969

Example 4.3. *Lastly, we consider the following problem, which has two interfaces:*

$$u_t - (\beta(x)u_x)_x = f(x,t), \quad \beta(x) = \begin{cases} \beta_1, & x \in [0, 0.2), \\ \beta_2, & x \in [0.2, 0.6), \\ \beta_3, & x \in [0.6, 1], \end{cases}$$

with the initial condition

$$u(x,0) = \begin{cases} \cos(\pi x), & x \in [0, 0.2), \\ \cos(11\pi x), & x \in [0.2, 0.6), \\ \cos(\pi x), & x \in [0.6, 1], \end{cases}$$

and the Neumann boundary conditions

$$u_x(0,t) = u_x(1,t) = 0, \quad t \in [0,1].$$

The interface conditions at $x = 0.2$

$$u(0.2^-,t) = u(0.2^+,t), \quad u_x(0.2^-,t) = \frac{1}{11}u_x(0.2^+,t), \quad t \in [0,1],$$

and at $x = 0.6$,

$$u(0.6^-,t) = u(0.6^+,t), \quad \frac{1}{11}u_x(0.6^-,t) = u_x(0.6^+,t), \quad t \in [0,1].$$

The exact solution is

$$u(x,t) = \begin{cases} e^{-\pi^2 t}\cos(\pi x), & x \in [0, 0.2), \\ e^{-\pi^2 t}\cos(11\pi x), & x \in [0.2, 0.6), \\ e^{-\pi^2 t}\cos(\pi x), & x \in [0.6, 1]. \end{cases}$$

We present the errors and convergence rates using cubics in Tables 9–10.

TABLE 9. ℓ^∞ and L^∞ errors and convergence rates using cubics for Example 4.3

N	$\beta_1 = 1$		$\beta_2 = 5$		$\beta_3 = 2$	
	$\|u-U\|_{\ell^\infty}$	Rate	$\|u_x-U_x\|_{\ell^\infty}$	Rate	$\|u-U\|_{L^\infty}$	Rate
20	4.0236(-3)		1.1579(-2)		7.5203(-3)	
40	2.2382(-4)	4.1681	6.0640(-4)	4.2550	4.7152(-4)	3.9426
60	4.3382(-5)	4.0498	1.1882(-4)	4.0199	9.4490(-5)	3.9645
80	1.3613(-5)	4.0244	3.7289(-5)	4.0285	2.9977(-5)	3.9907
100	5.5883(-6)	3.9902	1.5288(-5)	3.9958	1.2280(-5)	3.9996
120	2.7019(-6)	3.9860	7.3700(-6)	4.0020	5.9191(-6)	4.0025

TABLE 10. L^2 and H^1 errors and convergence rates using cubics for Example 4.3

N	$\beta_1 = 1$	$\beta_2 = 5$	$\beta_3 = 2$	
	$\|u-U\|_{L^2}$	Rate	$\|u-U\|_{H^1}$	Rate
20	6.5662(-4)		1.4704(-1)	
40	3.7737(-4)	4.1210	1.8353(-2)	3.0009
60	7.3459(-5)	4.0361	5.4395(-3)	2.9992
80	2.3125(-5)	4.0177	2.2950(-3)	2.9997
100	9.4467(-6)	4.0119	1.1745(-3)	3.0020
120	4.5494(-7)	4.0076	6.7957(-4)	3.0009

We present the errors and convergence rates using quartics in Tables 11–12.

TABLE 11. ℓ^∞ and L^∞ errors and convergence rates using quartics for Example 4.3

N	$\beta_1 = 1$		$\beta_2 = 5$		$\beta_3 = 2$	
	$\|u-U\|_{\ell^\infty}$	Rate	$\|u_x-U_x\|_{\ell^\infty}$	Rate	$\|u-U\|_{L^\infty}$	Rate
10	4.1045(-5)		2.0076(-4)		2.8321(-4)	
20	4.8475(-7)	6.0469	2.9074(-6)	6.0642	7.8108(-6)	4.9309
30	3.9460(-8)	6.1355	2.3544(-7)	6.0537	1.0640(-6)	4.9386
40	6.8889(-9)	6.0670	4.1579(-8)	6.0270	2.5444(-7)	4.9733
50	1.7901(-9)	6.0394	1.0865(-8)	6.0144	8.3660(-8)	4.9847

TABLE 12. L^2 and H^1 errors and convergence rates using quartics for Example 4.3

N	$\beta_1 = 1$ $\|u - U\|_{L^2}$	$\beta_2 = 5$ Rate	$\beta_3 = 2$ $\|u - U\|_{H^1}$	Rate
10	3.3892(-4)		5.9221(-3)	
20	9.8997(-6)	5.0974	3.6313(-4)	4.0276
30	1.2936(-6)	5.0191	7.1662(-5)	4.0023
40	3.0609(-7)	5.0101	2.2659(-5)	4.0023
50	1.0019(-7)	5.0050	9.2802(-6)	4.0005

5. Concluding remarks

OSC methods have been used successfully to solve, in a straightforward manner, parabolic problems in one space variable with all kinds of interfaces and more general boundary conditions. In comparison with [1], in Example 4.1, OSC converges faster and gives superconvergent results for the solution derivatives at grid points, even for the variable interfaces. In comparison with [1], OSC handles all kinds of interfaces and all types of boundary conditions, effectively demonstrated by Examples 4.2 and 4.3. The errors obtained in OSC are relatively lower and simultaneously decreasing at a faster rate. Additionally, OSC handles Robin boundary conditions easily and gives fourth-order accuracy when u and flux $(\beta(x)u_x)$ are discontinuous. The obtained results can be easily extended to multiple interface points and higher dimensional parabolic partial differential equations. Especially noteworthy are its super convergence properties in a space, which, for example, in the case of quartics, yield sixth-order approximations to both the solution and its first derivative at the nodal points.

Acknowledgement.

We acknowledge the computational expertise and timely assistance of Sahasransu Mohanty, Dipartimento di Ingegneria e Scienze dell' Informazione e Matematica, Università degli Studidell'Aquila, L'Aquila, Italy that lead to a precise formulation of this research manuscript.

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