Khayyam J. Math. 2 (2016), no. 1, 25–38

Khayyam Journal of Mathematics

emis.de/journals/KJM
kjm-math.org

ON SOME GENERALIZED SPACES OF INTERVAL NUMBERS WITH AN INFINITE MATRIX AND MUSIELAK-ORLICZ FUNCTION

KULDIP RAJ[1]* AND SURUCHI PANDOH[2]

Communicated by H.R. Ebrahimi Vishki

Abstract. In the present paper we introduce and study some generalized I-convergent sequence spaces of interval numbers defined by an infinite matrix and a Musielak-Orlicz function. We also make an effort to study some topological and algebraic properties of these spaces.

1. Introduction and preliminaries

Ideal convergence is a generalization of statistical convergence and any concept involving ideal convergence plays a vital role not only in pure mathematics but also in other branches of science involving mathematics, especially information theory, computer science, biological science, dynamical systems, geographic information systems, and motion planning in robotics. Kostyrko et al. [9] was initially introduced the notion of I-convergence based on the structure of admissible ideal I of subset of natural numbers $\mathbb{N}$.

Let N be a nonempty set. Then a family of sets $I \subseteq 2^N$ (Power set of N) is said to be an *ideal* if I is additive i.e. $A, B \in I \Rightarrow A \cup B \in I$ and $A \in I, B \subseteq A \Rightarrow B \in I$. A nonempty family of sets $\pounds(I) \subseteq 2^N$ is said to be a *filter* in N if and only if $\varnothing \notin \pounds(I)$, for $A, B \in \pounds(I)$ we have $A \cap B \in \pounds(I)$, and for each $A \in \pounds(I)$, $A \subseteq B$ implies $B \in \pounds(I)$.
An ideal $I \subseteq 2^N$ is called *non-trivial* if $I \neq 2^N$. A non-trivial ideal $I \subseteq 2^N$ is called *admissible* if $\big\{\{x\} : x \in \mathbb{N}\big\} \subseteq I$. A non-trivial ideal is *maximal* if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset. For each ideal I, there exist a filter $\pounds(I)$ corresponding to I i.e. $\pounds(I) = \{K \subseteq \mathbb{N} : K^c \in I\}$,

Date: Received: 22 August 2015; Revised: 21 June 2016; Accepted: 24 June 2016.

* Corresponding author.

2010 *Mathematics Subject Classification.* Primary 40J05; Secondary 40C05, 46A45.

Key words and phrases. ideal-convergence, Λ-convergence, interval number, Orlicz function, difference sequence.

where $K^c = N \backslash K$. Further details on ideal convergence can be found in [18], [20], [24], etc. The notion of I-convergent double sequences was initially introduced by Tripathy and Tripathy (see [25]).

Let $\lambda = (\lambda_m)$ be a non-decreasing sequence of positive numbers tending to ∞ such that

$$\lambda_{m+1} \leq \lambda_m + 1,\ \lambda_1 = 1.$$

The generalized de la Vallée Poussin mean is defined by

$$t_m(x) = \frac{1}{\lambda_m} \sum_{k \in J_m} x_k,$$

where $J_m = [m - \lambda_m + 1, m]$.
A sequence $x = (x_k)$ is said to be (V, λ)-summable to a number l (see [10]) if $t_m(x) \to l$, as $m \to \infty$. If $\lambda_m = m$, then (V, λ)-summability reduces to $(C, 1)$-summability. We write

$$[C, 1] = \Big\{x = (x_k) : \exists\, l \in \mathbb{R},\ \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} |x_k - l| = 0\Big\}$$

and

$$[V, \lambda] = \Big\{x = (x_k) : \exists\, l \in \mathbb{R},\ \lim_{m\to\infty} \frac{1}{\lambda_m} \sum_{k\in J_m} |x_k - l| = 0\Big\}.$$

Let $\lambda = (\lambda_m)$ and $\mu = (\mu_n)$ be two non-decreasing sequences of positive real numbers, each tending to ∞ such that $\lambda_{m+1} \leq \lambda_m + 1,\ \lambda_1 = 1;\ \mu_{n+1} \leq \mu_n + 1,\ \mu_1 = 1$. Let $J_m = [m - \lambda_m + 1, m]$, $J_n = [n - \mu_n + 1, n]$, $J_{mn} = J_m \times J_n$ and $\Lambda = \lambda_{mn} = \lambda_m \mu_n$. The generalized double de la Vallée-Poussin mean is defined by

$$t_{mn}(x) = \frac{1}{\lambda_{mn}} \sum_{(k,l)\in J_{mn}} x_{kl}.$$

Chiao [1] introduced sequences of interval numbers and defined usual convergence of sequences of interval numbers. Şengönül and Eryılmaz [21] introduced and studied bounded and convergent sequence spaces of interval numbers and showed that these spaces are complete metric spaces. Esi ([2], [3]) introduced and studied strongly almost λ-convergence and statistically almost λ-convergence of interval numbers and lacunary sequence spaces of interval numbers, respectively. Recently, Esi in [4] has studied double sequences of interval numbers.

A set consisting of a closed interval of real numbers x such that $a \leq x \leq b$ is called an interval number. A real interval can also be considered as a set. Thus we can investigate some properties of interval numbers, for instance arithmetic properties or analysis properties. We denote the set of all real-valued closed intervals by $\mathbb{R}$. Any element of $\mathbb{R}$ is called a closed interval and denoted by $\bar{x}$. That is $\bar{x} = \{x \in \mathbb{R} : a \leq x \leq b\}$. An interval number $\bar{x}$ is a closed subset of real numbers. Let x_l and x_r be first and last points of an interval number $\bar{x}$. For $\bar{x}_1, \bar{x}_2 \in \mathbb{R}$, we have $\bar{x}_1 = \bar{x}_2 \Leftrightarrow x_{1_l} = x_{2_l}, x_{1_r} = x_{2_r}$, $\bar{x}_1 + \bar{x}_2 = \{x \in \mathbb{R} : x_{1_l} + x_{2_l} \leq$

$x \leq x_{1_r} + x_{2_r}\}$, if $\alpha \geq 0$, then $\alpha\bar{x} = \{x \in \mathbb{R} : \alpha x_{1_l} \leq x \leq \alpha x_{1_r}\}$, if $\alpha < 0$, then $\alpha\bar{x} = \{x \in \mathbb{R} : \alpha x_{1_r} \leq x \leq \alpha x_{1_l}\}$, and

$$\bar{x}_1.\bar{x}_2 = \left\{ \begin{array}{c} x \in \mathbb{R} : \min\{x_{1_l}.x_{2_l} \ , x_{1_l}.x_{2_r} \ , x_{1_r}.x_{2_l} \ , x_{1_r}.x_{2_r}\} \leq x \\ \leq \min\{x_{1_l}.x_{2_l} \ , x_{1_l}.x_{2_r} \ , x_{1_r}.x_{2_l} \ , x_{1_r}.x_{2_r}\} \end{array} \right\}.$$

In [14], Moore proved that the set of all interval numbers is a complete metric space defined by the metric $d(\bar{x}_1, \bar{x}_2) = \max\{|x_{1_l} - x_{2_l}|, |x_{1_r} - x_{2_r}|\}$. In the special case $\bar{x}_1 = [a, a]$ and $\bar{x}_2 = [b, b]$, we obtain the usual metric of $\mathbb{R}$.

Throughout the paper we shall denote the set of all real sequences by w.

Definition 1.1. A transformation f from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{R}$ by $i, j \mapsto f(i, j) = \bar{x}, \bar{x} = (\bar{x}_{ij})$ is called a sequence of double interval numbers, where $\bar{x}_{ij}$ denotes the $(i, j)^{th}$ term of the sequence $\bar{x} = (\bar{x}_{ij})$.

Definition 1.2. [1] A sequence $\bar{x} = (\bar{x}_k)$ of interval numbers is said to be *convergent* to the interval number $\bar{x}_0$ if for each $\epsilon > 0$ there exists a positive integer k_0 such that $d(\bar{x}_k, \bar{x}_0) < \epsilon$ for all $k \geq k_0$ and we denote it by $\lim_k \bar{x}_k = \bar{x}_0$. Thus, $\lim_k \bar{x}_k = \bar{x}_0 \Leftrightarrow \lim_k x_{k_l} = x_{0_l}$ and $\lim_k x_{k_r} = x_{0_r}$.

Definition 1.3. An interval valued double sequence $\bar{x} = (\bar{x}_{ij})$ is said to be convergent in the Pringsheim's sense or *P-convergent* to an interval number $\bar{x}_0$, if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d(\bar{x}_{ij}, \bar{x}_0) < \epsilon \text{ for all } i, j > N,$$

where $\mathbb{N}$ is the set of natural numbers, and we denote it by $P\text{-}\lim \bar{x}_{ij} = \bar{x}_0$. The interval number $\bar{x}_0$ is called the Pringsheim limit of $\bar{x} = (\bar{x}_{ij})$. More exactly, we say that a double sequence $\bar{x} = (\bar{x}_{ij})$ converges to a finite interval number $\bar{x}_0$ if $\bar{x}_{ij}$ tend to $\bar{x}_0$ as both i and j tends to ∞ independently of one another.

Definition 1.4. A sequence $(x_k) \in w$ is said to be *I-convergent* to a number L if for every $\epsilon > 0$, the set $\{k \in \mathbb{N} : |x_k - L| \geq \epsilon\} \in I$. In this case we write $I\text{-}\lim x_k = L$.

Definition 1.5. An interval valued double sequence $\bar{x} = (\bar{x}_{ij})$ is *bounded* if there exists a positive number M such that $d(\bar{x}_{ij}, \bar{x}_0) \leq M$ for all $i, j \in \mathbb{N}$. We shall denote all bounded sequences of double interval numbers by $\bar{l}^2_\infty$.

Definition 1.6. Two non-negative functions f, g are called *equivalent*, whenever $C_1 f \leq g \leq C_2 f$, for some $C_j > 0,\ j = 1, 2$ and in this case we write $f \approx g$.

Definition 1.7. A sequence space E is said to be *solid (or normal)* if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ and (α_k) is any sequence of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$.

Definition 1.8. Let $K = \{k_1 < k_2 < ...\} \subset \mathbb{N}$ and let E be a sequence space. A *K-step* space of E is a sequence space $\lambda^E_K = \{(x_{k_n}) \in w : (x_k) \in E\}$.

Definition 1.9. A canonical preimage of a sequence $(x_{k_n}) \in \lambda^E_K$ is a sequence $(y_k) \in w$ defined by

$$y_k = \begin{cases} x_k, & \text{if } k \in K \\ 0\ , & \text{otherwise.} \end{cases}$$

The canonical preimage of a step space λ_K^E is the set of canonical preimages of all the elements in λ_K^E, i.e., y is in the canonical preimage of λ_K^E if and only if y is a canonical preimage of some $x \in \lambda_K^E$.

Definition 1.10. A sequence space E is said to be *monotone* if it contains the canonical preimages of its step spaces.

Lemma 1.11. [7] *Every normal space is monotone.*

Let $A = (a_{mnij})$ denote a four dimensional summability method that maps the complex double sequences x into the double sequence Ax where the mn^{th} term of Ax is as follows:

$$(Ax)_{mn} = \sum_{i,j=1,1}^{\infty,\infty} a_{mnij} x_{ij}.$$

Such a transformation is said to be non-negative if a_{mnij} is non-negative for all m, n, i and j.

The notion of difference sequence spaces was introduced by Kızmaz [8], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Colak [6] by introducing the spaces $l_\infty(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$. Another type of generalization of the difference sequence spaces is due to Tripathy and Esi [23] who studied the spaces $l_\infty(\Delta_n^m)$, $c(\Delta_n^m)$ and $c_0(\Delta_n^m)$. Let m, n be non-negative integers, then for $Z = c, c_0$ and l_∞, we have sequence spaces

$$Z(\Delta_n^m) = \{x = (x_k) \in w : (\Delta_n^m x_k) \in Z\},$$

where $\Delta_n^m x = (\Delta_n^m x_k) = (\Delta_n^{m-1} x_k - \Delta_n^{m-1} x_{k+1})$ and $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_n^m x_k = \sum_{v=0}^{m} (-1)^v \begin{pmatrix} m \\ v \end{pmatrix} x_{k+nv}.$$

Taking $m = n = 1$, we get the spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ studied by Kızmaz [8]. Taking $n = 1$, we get the spaces $l_\infty(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$ studied by Et and Colak [6]. Similarly, we can define difference operators on double sequence spaces as:

$$\begin{aligned} \Delta x_{k,l} &= (x_{k,l} - x_{k,l+1}) - (x_{k+1,l} - x_{k+1,l+1}) \\ &= x_{k,l} - x_{k,l+1} - x_{k+1,l} + x_{k+1,l+1}, \end{aligned}$$

$$\Delta^m x_{k,l} = \Delta^{m-1} x_{k,l} - \Delta^{m-1} x_{k,l+1} - \Delta^{m-1} x_{k+1,l} + \Delta^{m-1} x_{k+1,l+1}$$

and

$$\Delta_n^m x_{k,l} = \Delta_n^{m-1} x_{k,l} - \Delta_n^{m-1} x_{k,l+1} - \Delta_n^{m-1} x_{k+1,l} + \Delta_n^{m-1} x_{k+1,l+1}.$$

For more details about sequence spaces see [13], [17], [19] and references therein.

An Orlicz function $M : [0, \infty) \to [0, \infty)$ is a continuous, non-decreasing and convex function such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$. If convexity of Orlicz function is replaced by $M(x+y) \leq M(x) + M(y)$, then this function is called modulus function. Lindenstrauss and Tzafriri [11] used the idea of Orlicz functions to define the following sequence space,

$$\ell_M = \Big\{x = (x_k) \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty, \text{ for some } \rho > 0\Big\}$$

which is known as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$||x|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}.$$

Also it was shown in [11] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to ℓ_p $(p \geq 1)$. An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t)dt,$$

where η is known as the kernel of M, is right differentiable for $t \geq 0$, $\eta(0) = 0$, $\eta(t) > 0$, η is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.

A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is said to be a Musielak-Orlicz function (see [12, 15]). A sequence $\mathcal{N} = (N_k)$ is defined by

$$N_k(v) = \sup\{|v|u - M_k(u) : u \geq 0\},\ k = 1, 2, \cdots$$

is called the complementary function of a Musielak-Orlicz function $\mathcal{M}$. For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows

$$t_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \ \text{ for some } \ c > 0\Big\},$$

$$h_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \ \text{ for all } \ c > 0\Big\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k),\ x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$||x|| = \inf\Big\{k > 0 : I_{\mathcal{M}}\Big(\frac{x}{k}\Big) \leq 1\Big\}$$

or equipped with the Orlicz norm

$$||x||^0 = \inf\Big\{\frac{1}{k}\Big(1 + I_{\mathcal{M}}(kx)\Big) : k > 0\Big\}.$$

A Musielak-Orlicz function $\mathcal{M} = (M_k)$ is said to satisfy Δ_2-*condition* if there exist constants $a,\ K > 0$ and a sequence $c = (c_k)_{k=1}^{\infty} \in l_+^1$ (the positive cone of l^1) such that the inequality

$$M_k(2u) \leq KM_k(u) + c_k$$

holds for all $k \in \mathbb{N}$ and $u \in \mathbb{R}^+$, whenever $M_k(u) \leq a$.
Let $\bar{w}^2$ be the set of all double sequences of interval numbers and I be an admissible ideal of $\mathbb{N} \times \mathbb{N}$. Suppose $\mathcal{M} = (M_{ij})$ is a Musielak-Orlicz function and $A = (a_{mnij})$ be a non-negative four-dimensional regular summability method. Also suppose that $p = (p_{ij})$ is a bounded double sequence of positive real numbers and $u = (u_{ij})$ be a double sequence of strictly positive real numbers. In the present paper we define the following new double sequence spaces for interval numbers as follows: ${}_2\bar{w}^I[\Lambda, \mathcal{M}, p, u, \Delta^r, A] =$

$$\Big\{\bar{x} = (\bar{x}_{ij}) \in \bar{w}^2 : \Big\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_{mn}} \sum_{(i,j)\in J_{mn}} a_{mnij}\Big[M_{ij}\Big(\frac{u_{ij}d(\Delta_s^r \bar{x}_{ij}, \bar{x}_0)}{\rho}\Big)\Big]^{p_{ij}}$$

$$\geq \epsilon\Big\} \in I, \forall\, \epsilon > 0,\ \bar{x}_0 \in \mathbb{R} \text{ and for some } \rho > 0\Big\},$$

${}_2\bar{w}_0^I[\Lambda, \mathcal{M}, p, u, \Delta_s^r, A] =$

$$\Big\{\bar{x} = (\bar{x}_{ij}) \in \bar{w}^2 : \Big\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_{mn}} \sum_{(i,j)\in J_{mn}} a_{mnij}\Big[M_{ij}\Big(\frac{u_{ij}d(\Delta_s^r \bar{x}_{ij}, \bar{0})}{\rho}\Big)\Big]^{p_{ij}}$$

$$\geq \epsilon\Big\} \in I,\ \forall\, \epsilon > 0 \text{ and for some } \rho > 0\Big\},$$

${}_2\bar{w}_\infty^I[\Lambda, \mathcal{M}, p, u, \Delta_s^r, A] =$

$$\Big\{\bar{x} = (\bar{x}_{ij}) \in \bar{w}^2 : \exists\, K > 0 \text{ s.t. } \Big\{(m,n) \in \mathbb{N}\times\mathbb{N} :$$

$$\frac{1}{\lambda_{mn}} \sum_{(i,j)\in J_{mn}} a_{mnij}\Big[M_{ij}\Big(\frac{u_{ij}d(\Delta_s^r \bar{x}_{ij}, \bar{0})}{\rho}\Big)\Big]^{p_{ij}} \geq K\Big\} \in I,\ \text{for some } \rho > 0\Big\},$$

and
${}_2\bar{w}_\infty[\Lambda, \mathcal{M}, p, u, \Delta_s^r, A] =$

$$\Big\{\bar{x} = (\bar{x}_{ij}) \in \bar{w}^2 : \sup_{m,n} \frac{1}{\lambda_{mn}} \sum_{(i,j)\in J_{mn}} a_{mnij}\Big[M_{ij}\Big(\frac{u_{ij}d(\Delta_s^r \bar{x}_{ij}, \bar{0})}{\rho}\Big)\Big]^{p_{ij}} < \infty,$$

$$\text{for some } \rho > 0\Big\}.$$

Remark 1.12. Let us consider a few special cases of the above sequence spaces:

(i) If $\mathcal{M} = M_{ij}(x) = x$ for all $i, j \in \mathbb{N}$, then we have
${}_2\bar{w}^I[\Lambda, \mathcal{M}, p, u, \Delta_s^r, A] = {}_2\bar{w}^I[\Lambda, p, u, \Delta_s^r, A]$, ${}_2\bar{w}_0^I[\Lambda, \mathcal{M}, p, u, \Delta_s^r, A] = {}_2\bar{w}_0^I[\Lambda, p, u, \Delta_s^r, A]$, ${}_2\bar{w}_\infty^I[\Lambda, \mathcal{M}, p, u, \Delta_s^r, A] = {}_2\bar{w}_\infty^I[\Lambda, p, u, \Delta_s^r, A]$ and ${}_2\bar{w}_\infty[\Lambda, \mathcal{M}, p, u, \Delta_s^r, A] = {}_2\bar{w}_\infty[\Lambda, p, u, \Delta_s^r, A]$.

(ii) If $p = (p_{ij}) = 1$, for all i, j then we have
${}_2\bar{w}^I[\Lambda, \mathcal{M}, p, u, \Delta_s^r, A] = {}_2\bar{w}^I[\Lambda, \mathcal{M}, u, \Delta_s^r, A]$, ${}_2\bar{w}_0^I[\Lambda, \mathcal{M}, p, u, \Delta_s^r, A] = {}_2\bar{w}_0^I[\Lambda, \mathcal{M}, u, \Delta_s^r, A]$, ${}_2\bar{w}_\infty^I[\Lambda, \mathcal{M}, p, u, \Delta_s^r, A] = {}_2\bar{w}_\infty^I[\Lambda, \mathcal{M}, u, \Delta_s^r, A]$ and ${}_2\bar{w}_\infty[\Lambda, \mathcal{M}, p, u, \Delta_s^r, A] = {}_2\bar{w}_\infty[\Lambda, \mathcal{M}, u, \Delta_s^r, A]$.

(iii) If $u = (u_{ij}) = 1$, for all i, j then we have
${}_2\bar{w}^I[\Lambda, \mathcal{M}, p, u, \Delta_s^r, A] = {}_2\bar{w}^I[\Lambda, \mathcal{M}, p, \Delta_s^r, A]$, ${}_2\bar{w}_0^I[\Lambda, \mathcal{M}, p, u, \Delta_s^r, A] = {}_2\bar{w}_0^I[\Lambda, \mathcal{M}, p, \Delta_s^r, A]$, ${}_2\bar{w}_\infty^I[\Lambda, \mathcal{M}, p, u, \Delta_s^r, A] = {}_2\bar{w}_\infty^I[\Lambda, \mathcal{M}, p, \Delta_s^r, A]$ and ${}_2\bar{w}_\infty[\Lambda, \mathcal{M}, p, u, \Delta_s^r, A] = {}_2\bar{w}_\infty[\Lambda, \mathcal{M}, p, \Delta_s^r, A]$.

(iv) If $A = (C, 1, 1) = 1$, i.e. the double Cesàro matrix, then the above classes of sequences reduce to the following sequence spaces
${}_2\bar{w}^I[\Lambda, \mathcal{M}, p, u, \Delta_s^r, A] = {}_2\bar{w}^I[\Lambda, \mathcal{M}, p, u, \Delta_s^r]$, ${}_2\bar{w}_0^I[\Lambda, \mathcal{M}, p, u, \Delta_s^r, A] = {}_2\bar{w}_0^I[\Lambda, \mathcal{M}, p, u, \Delta_s^r]$, ${}_2\bar{w}_\infty^I[\Lambda, \mathcal{M}, p, u, \Delta_s^r, A] = {}_2\bar{w}_\infty^I[\Lambda, \mathcal{M}, p, u, \Delta_s^r]$ and ${}_2\bar{w}_\infty[\Lambda, \mathcal{M}, p, u, \Delta_s^r, A] = {}_2\bar{w}_\infty[\Lambda, \mathcal{M}, p, u, \Delta_s^r]$.

(v) Let $A = (C, 1, 1) = 1$ and $u_{ij} = 1$ for all i, j. If, in addition, $\mathcal{M}(x) = M(x)$ and $r = 0$; $s = 1$, then the spaces ${}_2\bar{w}^I[\Lambda, \mathcal{M}, p, u, \Delta_s^r, A]$, ${}_2\bar{w}_0^I[\Lambda, \mathcal{M}, p, u, \Delta_s^r, A]$, ${}_2\bar{w}_\infty^I[\Lambda, \mathcal{M}, p, u, \Delta_s^r, A]$ and ${}_2\bar{w}_\infty[\Lambda, \mathcal{M}, p, u, \Delta_s^r, A]$ are reduced to ${}_2\bar{w}^I[\Lambda, M, p]$, ${}_2\bar{w}_0^I[\Lambda, M, p]$, ${}_2\bar{w}_\infty^I[\Lambda, M, p]$ and ${}_2\bar{w}_\infty[\Lambda, M, p]$ which were introduced and studied by Esi and Hazarika [5].

The following inequality will be used through out the paper. If $0 \leq p_{ij} \leq \sup p_{ij} = H$, $D = \max(1, 2^{H-1})$ then

$$|a_{ij} + b_{ij}|^{p_{ij}} \leq D(|a_{ij}|^{p_{ij}} + |b_{ij}|^{p_{ij}}) \tag{1.1}$$

for all i, j and $a_{ij}, b_{ij} \in \mathbb{C}$. Also $|a|^{p_{ij}} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

The main purpose of this paper is to introduce and study some generalized I-convergent difference sequence spaces of interval numbers by using an infinite matrix and a Musielak-Orlicz function $\mathcal{M} = (M_{ij})$. We also make an effort to study some topological and algebraic properties of new sequence spaces and obtain the inclusion relation related to these spaces.

2. Main Results

Theorem 2.1. *$X[\Lambda, \mathcal{M}, p, u, \Delta_s^{r-1}, A) \subset X[\Lambda, \mathcal{M}, p, u, \Delta_s^r, A]$ for $X = {}_2\bar{w}^I, {}_2\bar{w}_0^I, {}_2\bar{w}_\infty^I, {}_2\bar{w}_\infty$.*

Proof. Let $\bar{x} = (\bar{x}_{ij}) \in {}_2\bar{w}_\infty[\Lambda, \mathcal{M}, p, u, \Delta_s^{r-1}, A]$. Then for some $\rho > 0$, we have

$$\sup_{m,n} \frac{1}{\lambda_{mn}} \sum_{(i,j) \in J_{mn}} a_{mnij} \Big[M_{ij}\Big(\frac{u_{ij} d(\Delta_s^{r-1} \bar{x}_{ij}, \bar{0})}{\rho} \Big) \Big]^{p_{ij}} < \infty.$$

Now, by the continuity of $\mathcal{M} = (M_{ij})$, the result follows from the following relation

$$\sup_{m,n}\frac{1}{\lambda_{mn}}\sum_{(i,j)\in J_{mn}} a_{mnij}\Big[M_{ij}\Big(\frac{u_{ij}d(\Delta_s^r\bar{x}_{ij},\bar{0})}{\rho}\Big)\Big]^{p_{ij}}$$
$$\leq \sup_{m,n}\frac{1}{2\lambda_{mn}}\sum_{(i,j)\in J_{mn}} a_{mnij}\Big[M_{ij}\Big(\frac{u_{ij}d(\Delta_s^{r-1}\bar{x}_{ij},\bar{0})}{\rho}\Big)\Big]^{p_{ij}}$$
$$+ \sup_{m,n}\frac{1}{2\lambda_{mn}}\sum_{(i,j)\in J_{mn}} a_{mnij}\Big[M_{ij}\Big(\frac{u_{ij}d(\Delta_s^r\bar{x}_{ij},\bar{0})}{\rho}\Big)\Big]^{p_{ij}}.$$

This shows that $\bar{x} = (\bar{x}_{ij}) \in {}_2\bar{w}_\infty[\Lambda, \mathcal{M}, p, u, \Delta_s^r, A]$. □

The following example shows that the inclusion in Theorem 2.1 is strict, in general.

Example 2.2. Let $A = (a_{mnij}) = I$ and $r, s = 1$. Let $M_{ij}(x) = x$, $u = (u_{ij}) = 1$, $p = (p_{ij}) = 1$ for all $i, j \in \mathbb{N}\times\mathbb{N}$, $\rho = 1$ and $\lambda_{mn} = 1$. Consider the interval sequence $\bar{x} = (\bar{x}_{ij})$ defined by

$$\bar{x}_{ij} = [ij,\ ij+1] \text{ for all } i, j \in \mathbb{N}\times\mathbb{N}$$

and $\Delta\bar{x}_{ij} = -\bar{1}$. Thus $\bar{x} = (\bar{x}_{ij}) \in {}_2\bar{w}_\infty[\Lambda, \mathcal{M}, p, u, \Delta_s^r, A]$.

Theorem 2.3. *If $0 < p_{ij} \leq q_{ij} < \infty$ for each i and j, then we have ${}_2\bar{w}^I[\Lambda, \mathcal{M}, p, u, \Delta_s^r, A] \subset {}_2\bar{w}^I[\Lambda, \mathcal{M}, q, u, \Delta_s^r, A]$.*

Proof. Let $\bar{x} = (\bar{x}_{ij}) \in {}_2\bar{w}^I[\Lambda, \mathcal{M}, p, u, \Delta_s^r, A]$. Then there exists $\rho > 0$ such that

$$\frac{1}{\lambda_{mn}}\sum_{(i,j)\in J_{mn}} a_{mnij}\Big[M_{ij}\Big(\frac{u_{ij}d(\Delta_s^r\bar{x}_{ij},\bar{x}_0)}{\rho}\Big)\Big]^{p_{ij}} \geq \epsilon.$$

This implies that $a_{mnij}\left[M_{ij}\left(\frac{u_{ij}d(\Delta_s^r\bar{x}_{ij},\bar{x}_0)}{\rho}\right)\right]^{p_{ij}} < 1$, for sufficiently large values of i and j (see [22]). Since $\mathcal{M} = (M_{ij})$ is non-decreasing, we get

$$\frac{1}{\lambda_{mn}}\sum_{(i,j)\in J_{mn}} a_{mnij}\Big[M_{ij}\Big(\frac{u_{ij}d(\Delta_s^r\bar{x}_{ij},\bar{x}_0)}{\rho}\Big)\Big]^{q_{ij}}$$
$$\leq \frac{1}{\lambda_{mn}}\sum_{(i,j)\in J_{mn}} a_{mnij}\Big[M_{ij}\Big(\frac{u_{ij}d(\Delta_s^r\bar{x}_{ij},\bar{x}_0)}{\rho}\Big)\Big]^{p_{ij}}.$$

Therefore,

$$\Big\{(m,n)\in\mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_{mn}}\sum_{(i,j)\in J_{mn}} a_{mnij}\Big[M_{ij}\Big(\frac{u_{ij}d(\Delta_s^r\bar{x}_{ij},\bar{x}_0)}{\rho}\Big)\Big]^{q_{ij}} \geq \epsilon\Big\} \subseteq$$
$$\Big\{(m,n)\in\mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_{mn}}\sum_{(i,j)\in J_{mn}} a_{mnij}\Big[M_{ij}\Big(\frac{u_{ij}d(\Delta_s^r\bar{x}_{ij},\bar{x}_0)}{\rho}\Big)\Big]^{p_{ij}} \geq \epsilon\Big\} \in I.$$

Thus, $\bar{x} = (\bar{x}_{ij}) \in {}_2\bar{w}^I[\Lambda, \mathcal{M}, q, u, \Delta_s^r, A]$. This completes the proof. □

Theorem 2.4. *(i) Let* $0 < \inf p_{ij} \leq p_{ij} \leq 1$. *Then*

$$_2\bar{w}^I[\Lambda, \mathcal{M}, p, u, \Delta_s^r, A] \subseteq {_2\bar{w}^I}[\Lambda, \mathcal{M}, u, \Delta_s^r, A],$$

$$_2\bar{w}_0^I[\Lambda, \mathcal{M}, p, u, \Delta_s^r, A] \subseteq {_2\bar{w}_0^I}[\Lambda, \mathcal{M}, \Delta_s^r, A].$$

(ii) Let $1 \leq p_{ij} \leq \sup p_{ij} < \infty$. *Then*

$$_2\bar{w}^I[\Lambda, \mathcal{M}, u, \Delta_s^r, A] \subseteq {_2\bar{w}^I}[\Lambda, \mathcal{M}, p, u, \Delta_s^r, A],$$

$$_2\bar{w}_0^I[\Lambda, \mathcal{M}, u, \Delta_s^r, A] \subseteq {_2\bar{w}_0^I}[\Lambda, \mathcal{M}, p, u, \Delta_s^r, A].$$

Proof. *(i)* Let $\bar{x} = (\bar{x}_{ij}) \in {_2\bar{w}^I}[\Lambda, \mathcal{M}, p, u, \Delta_s^r, A]$. Since $0 < \inf p_{ij} \leq p_{ij} \leq 1$ we have

$$\tfrac{1}{\lambda_{mn}} \textstyle\sum_{(i,j)\in J_{mn}} a_{mnij} \Big[M_{ij}\Big(\tfrac{u_{ij} d(\Delta_s^r \bar{x}_{ij}, \bar{x}_0)}{\rho}\Big)\Big]$$

$$\leq \frac{1}{\lambda_{mn}} \sum_{(i,j)\in J_{mn}} a_{mnij} \Big[M_{ij}\Big(\frac{u_{ij} d(\Delta_s^r \bar{x}_{ij}, \bar{x}_0)}{\rho}\Big)\Big]^{p_{ij}}.$$

Therefore,

$$\Big\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_{mn}} \sum_{(i,j)\in J_{mn}} a_{mnij} \Big[M_{ij}\Big(\frac{u_{ij} d(\Delta_s^r \bar{x}_{ij}, \bar{x}_0)}{\rho}\Big)\Big] \geq \epsilon\Big\}$$

$$\subseteq \Big\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_{mn}} \sum_{(i,j)\in J_{mn}} a_{mnij} \Big[M_{ij}\Big(\frac{u_{ij} d(\Delta_s^r \bar{x}_{ij}, \bar{x}_0)}{\rho}\Big)\Big]^{p_{ij}} \geq \epsilon\Big\} \in I.$$

The other part can be proved in the similar way.

(ii) Let $\bar{x} = (\bar{x}_{ij}) \in {_2\bar{w}^I}[\Lambda, \mathcal{M}, u, \Delta_s^r, A]$. Since $1 \leq p_{ij} \leq \sup p_{ij} < \infty$, then for each $0 < \epsilon < 1$ there exists a positive integer n_0 such that

$$\tfrac{1}{\lambda_{mn}} \sum_{(i,j)\in J_{mn}} a_{mnij} \Big[M_{ij}\Big(\frac{u_{ij} d(\Delta_s^r \bar{x}_{ij}, \bar{x}_0)}{\rho}\Big)\Big] \leq \epsilon < 1 \quad \text{for all} \quad n \geq n_0, \text{ (see [16]).}$$

This implies that

$$\tfrac{1}{\lambda_{mn}} \sum_{(i,j)\in J_{mn}} a_{mnij} \Big[M_{ij}\Big(\frac{u_{ij} d(\Delta_s^r \bar{x}_{ij}, \bar{x}_0)}{\rho}\Big)\Big]^{p_{ij}}$$

$$\leq \frac{1}{\lambda_{mn}} \sum_{(i,j)\in J_{mn}} a_{mnij} \Big[M_{ij}\Big(\frac{u_{ij} d(\Delta_s^r \bar{x}_{ij}, \bar{x}_0)}{\rho}\Big)\Big].$$

Therefore, we have

$$\Big\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_{mn}} \sum_{(i,j)\in J_{mn}} a_{mnij} \Big[M_{ij}\Big(\frac{u_{ij} d(\Delta_s^r \bar{x}_{ij}, \bar{x}_0)}{\rho}\Big)\Big]^{p_{ij}} \geq \epsilon\Big\}$$

$$\subseteq \Big\{(m,n)\in\mathbb{N}\times\mathbb{N}: \frac{1}{\lambda_{mn}}\sum_{(i,j)\in J_{mn}} a_{mnij}\Big[M_{ij}\Big(\frac{u_{ij}d(\Delta_s^r\bar{x}_{ij},\bar{x}_0)}{\rho}\Big)\Big]\geq\epsilon\Big\}\in I.$$

The other part can be proved in the similar way. This completes the proof. □

Theorem 2.5. *Let $\mathcal{M}=(M_{ij})$ and $\mathcal{S}=(S_{ij})$ be two Musielak-Orlicz functions. Then*

$${}_2\bar{w}^I[\Lambda,\mathcal{M},p,u,\Delta_s^r,A]\cap{}_2\bar{w}^I[\Lambda,\mathcal{S},p,u,\Delta_s^r,A]\subset{}_2\bar{w}^I[\Lambda,\mathcal{M}+\mathcal{S},p,u,\Delta_s^r,A].$$

Proof. Let $\bar{x}=(\bar{x}_{ij})\in{}_2\bar{w}^I[\Lambda,\mathcal{M},p,u,\Delta_s^r,A]\cap{}_2\bar{w}^I[\Lambda,\mathcal{S},p,u,\Delta_s^r,A]$. Then for every $\epsilon>0$, we have

$$\Big\{(m,n)\in\mathbb{N}\times\mathbb{N}: \frac{1}{\lambda_{mn}}\sum_{(i,j)\in J_{mn}} a_{mnij}\Big[M_{ij}\Big(\frac{u_{ij}d(\Delta_s^r\bar{x}_{ij},\bar{x}_0)}{\rho_1}\Big)\Big]^{p_{ij}}\geq\epsilon\Big\}\in I,$$

for some $\rho_1>0$ and

$$\Big\{(m,n)\in\mathbb{N}\times\mathbb{N}: \frac{1}{\lambda_{mn}}\sum_{(i,j)\in J_{mn}} a_{mnij}\Big[S_{ij}\Big(\frac{u_{ij}d(\Delta_s^r\bar{x}_{ij},\bar{x}_0)}{\rho_2}\Big)\Big]^{p_{ij}}\geq\epsilon\Big\}\in I,$$

for some $\rho_2>0$. Let $\rho=\max\{\rho_1,\rho_2\}$. The result follows from the inequality

$$\begin{aligned}
&\tfrac{1}{\lambda_{mn}}\sum_{(i,j)\in J_{mn}} a_{mnij}\Big[(M_{ij}+S_{ij})\Big(\frac{u_{ij}d(\Delta_s^r\bar{x}_{ij},\bar{x}_0)}{\rho}\Big)\Big]^{p_{ij}}\\
&\quad= \frac{1}{\lambda_{mn}}\sum_{(i,j)\in J_{mn}} a_{mnij}\Big[M_{ij}\Big(\frac{u_{ij}d(\Delta_s^r\bar{x}_{ij},\bar{x}_0)}{\rho}\Big)\Big]^{p_{ij}}\\
&\quad+ \frac{1}{\lambda_{mn}}\sum_{(i,j)\in J_{mn}} a_{mnij}\Big[S_{ij}\Big(\frac{u_{ij}d(\Delta_s^r\bar{x}_{ij},\bar{x}_0)}{\rho}\Big)\Big]^{p_{ij}}\\
&\quad\leq D\frac{1}{\lambda_{mn}}\sum_{(i,j)\in J_{mn}} a_{mnij}\Big[M_{ij}\Big(\frac{u_{ij}d(\Delta_s^r\bar{x}_{ij},\bar{x}_0)}{\rho}\Big)\Big]^{p_{ij}}\\
&\quad+ D\frac{1}{\lambda_{mn}}\sum_{(i,j)\in J_{mn}} a_{mnij}\Big[S_{ij}\Big(\frac{u_{ij}d(\Delta_s^r\bar{x}_{ij},\bar{x}_0)}{\rho}\Big)\Big]^{p_{ij}},\ \text{see (1.1)}.
\end{aligned}$$

Thus, $\bar{x}=(\bar{x}_{ij})\in{}_2\bar{w}^I[\Lambda,\mathcal{M}+\mathcal{S},p,u,\Delta_s^r,A]$. This completes the proof. □

Theorem 2.6. *Let $\mathcal{M}=(M_{ij})$ and $\mathcal{S}=(S_{ij})$ be two Musielak-Orlicz functions. Then*

$${}_2\bar{w}^I[\Lambda,\mathcal{M},p,u,\Delta_s^r,A]\subset{}_2\bar{w}^I[\Lambda,\mathcal{S}\circ\mathcal{M},p,u,\Delta_s^r,A].$$

Proof. Let $\inf p_{ij}=H_0$. For given $\epsilon>0$, we first choose $\epsilon_0>0$ such that $\max\{\epsilon_0^H,\epsilon_0^{H_0}\}<\epsilon$. Now using the continuity of (S_{ij}) choose $0<\delta<1$ such that $0<t<\delta$ implies $S_{ij}(t)<\epsilon_0$.

Let $\bar{x} = (\bar{x}_{ij}) \in {}_2\bar{w}^I[\Lambda, \mathcal{M}, p, u, \Delta_s^r, A]$. Now from definition of ${}_2\bar{w}^I[\Lambda, \mathcal{M}, p, u, \Delta_s^r, A]$, for some $\rho > 0$

$$\bar{A}(\delta) = \Big\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_{mn}} \sum_{(i,j)\in J_{mn}} a_{mnij}\Big[M_{ij}\Big(\frac{u_{ij}d(\Delta_s^r\bar{x}_{ij}, \bar{x}_0)}{\rho}\Big)\Big]^{p_{ij}} \geq \delta^H\Big\} \in I.$$

Thus if $(m,n) \notin \bar{A}(\delta)$, then we have

$$\begin{aligned}
&\tfrac{1}{\lambda_{mn}} \sum_{(i,j)\in J_{mn}} a_{mnij}\Big[M_{ij}\Big(\frac{u_{ij}d(\Delta_s^r\bar{x}_{ij}, \bar{x}_0)}{\rho}\Big)\Big]^{p_{ij}} < \delta^H \\
&\qquad \Rightarrow \quad \sum_{(i,j)\in J_{mn}} a_{mnij}\Big[M_{ij}\Big(\frac{u_{ij}d(\Delta_s^r\bar{x}_{ij}, \bar{x}_0)}{\rho}\Big)\Big]^{p_{ij}} < \lambda_{mn}\delta^H \\
&\qquad \Rightarrow \quad a_{mnij}\Big[M_{ij}\Big(\frac{u_{ij}d(\Delta_s^r\bar{x}_{ij}, \bar{x}_0)}{\rho}\Big)\Big]^{p_{ij}} < \delta^H, \text{ for all } i,j = 1,2,3,\ldots \\
&\qquad \Rightarrow \quad a_{mnij}\Big[M_{ij}\Big(\frac{u_{ij}d(\Delta_s^r\bar{x}_{ij}, \bar{x}_0)}{\rho}\Big)\Big]^{p_{ij}} < \delta, \text{ for all } i,j = 1,2,3,\ldots\ .
\end{aligned}$$

Hence from the above inequality and using continuity of (S_{ij}), we must have

$$a_{mnij}\Big[S_{ij}\Big(M_{ij}\Big(\frac{u_{ij}d(\Delta_s^r\bar{x}_{ij}, \bar{x}_0)}{\rho}\Big)\Big)\Big] < \epsilon_0 \text{ for all } i,j = 1,2,3,\ldots$$

which consequently implies that

$$\begin{aligned}
&\sum_{(i,j)\in J_{mn}} a_{mnij}\Big[S_{ij}\Big(M_{ij}\Big(\frac{u_{ij}d(\Delta_s^r\bar{x}_{ij}, \bar{x}_0)}{\rho}\Big)\Big)\Big]^{p_{ij}} < \lambda_{mn}\max\{\epsilon_0^H, \epsilon_0^{H_0}\} < \lambda_{mn}\epsilon \\
&\qquad \Rightarrow \quad \frac{1}{\lambda_{mn}} \sum_{(i,j)\in J_{mn}} a_{mnij}\Big[S_{ij}\Big(M_{ij}\Big(\frac{u_{ij}d(\Delta_s^r\bar{x}_{ij}, \bar{x}_0)}{\rho}\Big)\Big)\Big]^{p_{ij}} < \epsilon.
\end{aligned}$$

This shows that

$$\Big\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_{mn}} \sum_{(i,j)\in J_{mn}} a_{mnij}\Big[S_{ij}\Big(M_{ij}\Big(\frac{u_{ij}d(\Delta_s^r\bar{x}_{ij}, \bar{x}_0)}{\rho}\Big)\Big)\Big]^{p_{ij}} \geq \epsilon\Big\} \subset \bar{A}(\delta)$$

and so belongs to I. This completes the proof. $\square$

Theorem 2.7. *Let $\mathcal{M} = (M_{ij})$ be a Musielak-Orlicz function and let $A = (a_{mnij})$ be a non-negative four-dimensional regular summability method. Suppose that $\beta = \lim_{t\to\infty} \frac{M_{ij}(t)}{t} < \infty$. Then ${}_2\bar{w}^I[\Lambda, p, u, \Delta_s^r, A] = {}_2\bar{w}^I[\Lambda, \mathcal{M}, p, u, \Delta_s^r, A]$.*

Proof. In order to prove that ${}_2\bar{w}^I[\Lambda, p, u, \Delta_s^r, A] = {}_2\bar{w}^I[\Lambda, \mathcal{M}, p, u, \Delta_s^r, A]$, it is sufficient to show that ${}_2\bar{w}^I[\Lambda, \mathcal{M}, p, u, \Delta_s^r, A] \subset {}_2\bar{w}^I[\Lambda, p, u, \Delta_s^r, A]$. Now, let $\beta > 0$. By definition of β, we have $M_{ij}(t) \geq \beta t$ for all $t \geq 0$. Since $\beta > 0$, we have $t \leq \frac{1}{\beta}M_{ij}(t)$ for all $t \geq 0$. Let $\bar{x} = (\bar{x}_{ij}) \in {}_2\bar{w}^I[\Lambda, \mathcal{M}, p, u, \Delta_s^r, A]$. Thus, we have

$$\frac{1}{\lambda_{mn}} \sum_{(i,j)\in J_{mn}} a_{mnij}\Big[\Big(\frac{u_{ij}d(\Delta_s^r\bar{x}_{ij}, \bar{x}_0)}{\rho}\Big)\Big]^{p_{ij}}$$

$$\leq \frac{1}{\beta}\frac{1}{\lambda_{mn}}\sum_{(i,j)\in J_{mn}} a_{mnij}\Big[M_{ij}\Big(\frac{u_{ij}d(\Delta_s^r\bar{x}_{ij},\bar{x}_0)}{\rho}\Big)\Big]^{p_{ij}}.$$

Therefore, we have

$$\Big\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_{mn}}\sum_{(i,j)\in J_{mn}} a_{mnij}\Big[\Big(\frac{u_{ij}d(\Delta_s^r\bar{x}_{ij},\bar{x}_0)}{\rho}\Big)\Big]^{p_{ij}}\geq\epsilon\Big\}$$

$$\subseteq\Big\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\beta\lambda_{mn}}\sum_{(i,j)\in J_{mn}} a_{mnij}\Big[M_{ij}\Big(\frac{u_{ij}d(\Delta_s^r\bar{x}_{ij},\bar{x}_0)}{\rho}\Big)\Big]^{p_{ij}}\geq\epsilon\Big\}\in I,$$

which implies that $\bar{x}=(\bar{x}_{ij})\in {}_2\bar{w}^I[\Lambda,p,u,\Delta_s^r,A]$. This completes the proof. $\square$

Theorem 2.8. *Let $\mathcal{M}=(M_{ij})$ and $\mathcal{S}=(S_{ij})$ be two Musielak-Orlicz functions such that $M_{ij}\approx S_{ij}$ for all i,j. Then $Z[\Lambda,\mathcal{M},p,u,\Delta_s^r,A]=Z[\Lambda,\mathcal{S},p,u,\Delta_s^r,A]$, for $Z={}_2\bar{w}^I,\ {}_2\bar{w}_0^I,\ {}_2\bar{w}_\infty^I$ and ${}_2\bar{w}_\infty$.*

Theorem 2.9. *The sequence spaces ${}_2\bar{w}^I[\Lambda,\mathcal{M},p,u,\Delta_s^r,A]$, ${}_2\bar{w}_0^I[\Lambda,\mathcal{M},p,u,\Delta_s^r,A]$, ${}_2\bar{w}_\infty^I[\Lambda,\mathcal{M},p,u,\Delta_s^r,A]$ and ${}_2\bar{w}_\infty[\Lambda,\mathcal{M},p,u,\Delta_s^r,A]$ are solid as well as monotone.*

Proof. We give the proof of the theorem for ${}_2\bar{w}_0^I[\Lambda,\mathcal{M},p,u,\Delta_s^r,A]$ only. Let $\bar{x}=(\bar{x}_{ij})\in{}_2\bar{w}_0^I[\Lambda,\mathcal{M},p,u,\Delta_s^r,A]$ and (α_{ij}) be a scalar sequence such that $|\alpha_{ij}|\leq 1$ for all $i,j\in\mathbb{N}$. Then for every $\epsilon>0$ we have

$$\Big\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_{mn}}\sum_{(i,j)\in J_{mn}} a_{mnij}\Big[M_{ij}\Big(\frac{u_{ij}d(\Delta_s^r\alpha_{ij}\bar{x}_{ij},\bar{0})}{\rho}\Big)\Big]^{p_{ij}}\geq\epsilon\Big\}$$

$$\subseteq\Big\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{G}{\lambda_{mn}}\sum_{(i,j)\in J_{mn}} a_{mnij}\Big[M_{ij}\Big(\frac{u_{ij}d(\Delta_s^r\bar{x}_{ij},\bar{0})}{\rho}\Big)\Big]^{p_{ij}}\geq\epsilon\Big\}\in I,$$

where $G=\max\{1,|\alpha_{ij}|^H\}$. Hence $(\alpha\bar{x})\in{}_2\bar{w}_0^I[\Lambda,\mathcal{M},p,u,\Delta_s^r,A]$. By Lemma 1.11, the space ${}_2\bar{w}_0^I[\Lambda,\mathcal{M},p,u,\Delta_s^r,A]$ is monotone. This completes the proof. $\square$

In what follows, we show that the classes of interval numbers ${}_2\bar{w}^I[\Lambda,\mathcal{M},p,u,\Delta_s^r,A]$, ${}_2\bar{w}_0^I[\Lambda,\mathcal{M},p,u,\Delta_s^r,A]$, ${}_2\bar{w}_\infty^I[\Lambda,\mathcal{M},p,u,\Delta_s^r,A]$ and ${}_2\bar{w}_\infty[\Lambda,\mathcal{M},p,u,\Delta_s^r,A]$ are not symmetric, in general.

Example 2.10. Let $A=(a_{mnij})=I$ and $r,s=1$. Let $M_{ij}(x)=x$, $u=(u_{ij})=1$, $p=(pij)=1$ for all $i,j\in\mathbb{N}\times\mathbb{N}$, $\rho=1$ and $\lambda_{mn}=1$. Consider the interval sequence $\bar{x}=(\bar{x}_{ij})$ defined by

$$\bar{x}_{ij}=\Big[ij,\ ij+\frac{1}{2}\Big]\ \text{ for all } i,j\in\mathbb{N}\times\mathbb{N}$$

and $\Delta\bar{x}_{ij}=-\bar{1}$. Thus $\bar{x}=(\bar{x}_{ij})\in{}_2\bar{w}_\infty[\Lambda,\mathcal{M},p,u,\Delta_s^r,A]$.

Let the sequence of interval numbers $\bar{y}=(\bar{y}_{ij})$ be a rearrangement of the sequence of interval numbers $\bar{x}=(\bar{x}_{ij})$ defined as follows:

$$\bar{y}=(\bar{y}_{ij})=\left\{\begin{array}{c}\bar{x}_1,\bar{x}_2,\bar{x}_4,\bar{x}_3,\bar{x}_9,\bar{x}_5,\bar{x}_{16},\\ \bar{x}_6,\bar{x}_{25},\bar{x}_7,\bar{x}_{36},\bar{x}_8,\bar{x}_{49},\ldots\end{array}\right\}$$

i.e.

$$\Delta\bar{y}_{ij} = \begin{cases} \bar{x}_{(\frac{ij+1}{2})^2}, & \text{for all } i,j \text{ odd;} \\ \\ \bar{x}_{(mn+\frac{ij}{2})}, & \text{for all } i,j \text{ even and} \\ & m,n \text{ satisfies } mn(mn-1) < ij \leq mn(mn+1). \end{cases}$$

Thus for all i,j odd and $m,n \in \mathbb{N}\times\mathbb{N}$, satisfying $mn(mn-1) < \frac{ij+1}{2} \leq mn(mn+1)$, we have

$$\Delta\bar{y}_{ij} = \Big[\Big(mn+\frac{ij}{2}\Big) - \Big(\frac{ij+2}{2}\Big)^2 - \frac{1}{2}, \Big(mn+\frac{ij}{2}\Big) - \Big(\frac{ij+2}{2}\Big)^2 + \frac{1}{2}\Big].$$

From the last two equations, it is clear that $(\Delta\bar{y}_{ij})$ is unbounded, thus $\bar{y} = (\bar{y}_{ij}) \notin {}_2\bar{w}_\infty[\Lambda, \mathcal{M}, p, u, \Delta_s^r, A]$. Therefore, the class ${}_2\bar{w}_\infty[\Lambda, \mathcal{M}, p, u, \Delta_s^r, A]$ is not symmetric.

Acknowledgement. The authors thank the referee for their valuable suggestions which improve the presentation of the paper.

References

1. K.P. Chiao, *Fundamental properties of interval vector max-norm*, Tamsui Oxf. J. Math. Sci. **18** (2002), 219–233.
2. A. Esi, *Strongly almost λ-convergence and statistically almost λ-convergence of interval numbers*, Sci. Magna, **7** (2011), 117–122.
3. A. Esi, *Lacunary sequence spaces of interval numbers*, Thai J. Math. **10** (2012), 445–451.
4. A. Esi, *Some double sequence spaces of interval numbers defined by Orlicz functions*, Acta Comment. Univ. Tartu. Math. **17** (2013), no. 1, 57–64.
5. A. Esi and B. Hazarika, *Some ideal convergence of double Λ-interval number sequences defined by Orlicz function*, Global Journal of Mathematical Analysis, **1** (2013), no. 3, 110–116.
6. M. Et and R. Çolak, *On generalized difference sequence spaces*, Soochow J. Math. **21** (1995), 377–386.
7. P.K. Kamthan and M. Gupta, *Sequence Spaces and Series*, Marcel Dekker Inc., New York, (1980).
8. H. Kızmaz, *On certain sequence spaces*, Canad. Math. Bull. **24** (1981), 169–176.
9. P. Kostyrko, T. Šalát , and W. Wilczyński, *I-convergence*, Real Anal. Exchange, **26** (2000-2001), no. 2, 669–686.
10. L. Leindler, *Über die de la Vallée Pousinsche Summierbarkeit allgenmeiner Othogonalreihen*, Acta Math. Acad. Sci. Hungar. **16** (1965), 375–387.
11. J. Lindenstrauss, L. Tzafriri, *On Orlicz sequence spaces*, Israel J. Math. **10** (1971), 379–390.
12. L. Maligranda, *Orlicz spaces and interpolation*, Seminars in Mathematics, **5**, Polish Academy of Science, 1989.
13. S.A. Mohiuddine, A. Alotaibi and M. Mursaleen, *Statistical convergence of double sequences in locally solid Riesz spaces*, Abstr. Appl. Anal. **(2012)**, Article ID 719729, 9 pages.
14. R.E. Moore, *Automatic Error Analysis in Digital Computation*, LSMD-48421, Lockheed Missiles and Space Company, 1959.
15. J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes in Math. **1034**, Springer Verlag, 1983.
16. S.D. Parashar and B. Choudhary, *Sequence spaces defined by Orlicz functions*, Indian J. Pure Appl. Math. **25** (1994), 419–428.
17. K. Raj and S. Pandoh, *Some difference sequence spaces defined by a modulus function and an infinite matrix*, J. Inequal. Spec. Funct. **5** (2014), no. 1, 21–32.

18. K. Raj and S.K. Sharma, *Ideal Convergence sequence spaces defined by a Musielak-Orlicz function*, Thai J. Math. **11** (2013), 577–587.
19. K. Raj and S.K. Sharma, *Some multiplier double sequence spaces*, Acta Math. Vietnam., **37** (2012), 391–406.
20. T. Šalát, B.C. Tripathy and M. Ziman, *On some properties of I-convergence*, Tatra Mt. Math. Publ. **28** (2004) 279–286.
21. M. Şengönül and A. Eryılmaz, *On the sequence spaces of interval numbers*, Thai J. Math. **8** (2010), no. 3, 503–510.
22. B.C. Tripathy, Y. Altin and M. Et, *Generalized difference sequence spaces on seminormed space defined by Orlicz functions*, Math. Slovaca, **58** (2008), 315–324.
23. B.C. Tripathy and A. Esi, *A new type of difference sequence spaces*, Int. J. Sci. Tech. **1** (2006), 11–14.
24. B.C. Tripathy and B. Hazarika, *I-monotonic and I-convergent sequences*, Kyungpook Math. J. **51** (2011), 233–239.
25. B.K. Tripathy and B.C. Tripathy, *On I-convergent double sequences*, Soochow J. Math. **31** (2005), no. 4, 549–560.

[1,2] Department of Mathematics, Shri Mata Vaishno Devi University, Katra-182320, J & K (India).

E-mail address: kuldipraj68@gmail.com

E-mail address: suruchi.pandoh87@gmail.com