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REVIEWED TECHNIQUES IN AUTOMATIC CONTINUITY OF LINEAR FUNCTIONALS

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ABSTRACT. Some techniques, which were already used to derive automatic continuity results, are chosen and modified, and extended results, as well as generalized results, are obtained. A technique of using the open mapping theorem and a technique of using the Hahn Banach extension theorem are explained. Results in connection with measurable cardinals are also obtained. Results for multiplicative linear functionals, positive linear functionals, and uniqueness of topology are obtained. For example, sequential continuity of real multiplicative linear functionals on sequentially complete LMC algebras is obtained when Michael's open problem is concerned only with the boundedness of multiplicative linear functionals. The continuity of positive linear functionals on F-algebras with identity elements and involution is derived when these functionals are continuous on the set of all involution-symmetric elements. Possibilities of extending the concept of positive linear functionals are considered to derive results for the continuity of such functionals on topological groups and topological vector spaces with additional structures. The technique for the Carpenter's uniqueness theorem is modified to derive the boundedness of some homomorphisms. The entire article is oriented toward Michael's problem.

1. INTRODUCTION

If the continuity of a map T from a topological vector space X into a topological vector space Y is assured by imposing some algebraic conditions on X, Y, or T, then the continuity is called automatic continuity. All deviations are also accepted to bring results under the theory of automatic continuity. The classical closed graph theorem is a perfect first result in the theory of automatic continuity.

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Every basis in a complete metrizable topological vector space is a Schauder basis. This is another result of automatic continuity of functionals; see [31]. The first two results given in the article [35] of Ng and Warner are automatic continuity results for group homomorphisms and ring homomorphisms. Recent studies concentrate more on automatic continuity in the theory of topological groups (see [9, 10, 39]), although there are recent articles like [4, 14, 16, 19–23, 33, 36, 37, 44, 45, 47] for automatic continuity in the theory of topological algebras. Many results were derived for automatic continuity in topological algebras mainly because of Johnson's uniqueness theorem [27] to settle an open question about the uniqueness of complete norm topology on semisimple algebras, affirmatively, and because of the following two questions of Michael [32], which remain open.

1. Is every complex multiplicative linear functional on a complex commutative Fréchet algebra continuous?
2. Is every complex multiplicative linear functional on a complex commutative complete LMC algebra bounded?

A recent attempt at these equivalent problems may be found in [23]. Equivalence of these two problems was established by Dixon and Fremlin [11]. Many techniques were evolved to establish results for automatic continuity just for only one application: continuity need not be verified in lengthy ways unnecessarily when it is required for an application. Some techniques are chosen and applied to derive some generalized results. For readability, instead of writing all of them in this section, they are explained at the places where they are required. The same thing is done even for known results. There are recent articles [1, 2, 13, 34, 38, 43] that study the properties of topological algebras applicable for automatic continuity.

All vector spaces to be considered are over the complex field $\mathbb{C}$ and the real field $\mathbb{R}$. All topologies to be considered will be Hausdorff. A complete metrizable topological vector space is called an F-space. An F-space that is also a locally convex topological vector space is called a Fréchet space. Multiplication in a topological algebra is assumed to be jointly continuous. A complete metrizable topological algebra is called an F-algebra. A locally convex F-algebra is called a B_0-algebra. A locally convex topological algebra (A, τ) is called a locally multiplicatively convex algebra (LMC algebra), if the topology τ is induced by a family of seminorms $(p_\alpha)_{\alpha\in I}$ satisfying the relation $p_\alpha(xy) \leq p_\alpha(x)p_\alpha(y), \text{ for all } x, y \in A \text{ and } \text{ for all } \alpha \in I$. Such seminorms are called submultiplicative seminorms. An F-algebra that is also an LMC algebra is called a Fréchet algebra. A linear functional on an algebra is called a multiplicative linear functional, if it also preserves multiplication. A real linear functional f on an algebra A is called a positive linear functional, if $f(x^2) \geq 0, \text{ for all } x \in A$. A complex linear functional f on a complex algebra A with an involution $*$ is called a positive linear functional if $f(xx^*) \geq 0, \text{ for all } x \in A$. A multiplicative identity element in an algebra is denoted by e. Sometimes the notation e is also used for idempotent elements. A topological algebra is said to be functionally continuous, if each multiplicative linear functional on it is continuous, when the same field is fixed for both algebra and functional. The classical involution $*$ in a complex algebra A satisfies the condition $(xy)^* = y^*x^*, \text{ for all } x, y \in A$. Another one satisfying $(xy)^* = x^*y^*$ may also be considered, while deriving the following inequalities (see the proof of [42, Theorem 11.31]). If f is a real positive linear functional on a real algebra A, then $|f(xy)|^2 \leq f(x^2)f(y^2),\;$ for all $\;x, y \in A$, and $|f(x)|^2 \leq f(e)f(x^2),\;$ for all $\;x \in A$ when A contains an identity element e. For the classical involution, if f is a complex positive linear functional f on a complex algebra A with an involution $*$, then $|f(xy^*)|^2 \leq f(xx^*)f(yy^*),\;$ for all $\;x, y \in A$; and $f(x^*) = \overline{f(x)}$ and $|f(x)|^2 \leq f(e)f(xx^*),\;$ for all $\;x \in A$, when A contains multiplicative identity e. Let us use these inequalities without mentioning them. Let us consider only classical involutions in complex algebras.

2. Dixon–Fremlin technique: Illustrations

Let us begin with the Dixon–Fremlin technique given in [11], and with another known variation of this technique.

Theorem 2.1. *Let $(X, (p_\alpha)_{\alpha\in I})$ be a sequentially complete locally convex topological vector space with a family of seminorms $(p_\alpha)_{\alpha\in I}$, which induces the topology on X, and let (Y, d) be an F-space with a metric d, which induces the topology on Y. Let $T : (X, (p_\alpha)_{\alpha\in I}) \longrightarrow (Y, d)$ be a linear map having sequentially closed graph in $X \times Y$. Then T is bounded.*

Proof. (Outline of the Dixon–Fremlin technique)
Let E be a bounded subset of X. For each $n = 1, 2, \ldots$, let $I_n = \{\alpha \in I : p_\alpha(x) \leq n,\;$ for all $\;x \in E\}$, and let $p_n(x) = \sup\,\{p_\alpha(x) : \alpha \in I_n\},\;$ for all $\;x \in X$. If $Z = \{x \in X : p_n(x) < \infty,\;$ for all $\;n = 1, 2, 3, ..\}$, then $(Z, (p_n)_{n=1}^\infty)$ is a Fréchet space in which E is a bounded subset. Let $T|_Z$ denote the restriction of T to Z. Then $T|_Z : (Z, (p_n)_{n=1}^\infty) \longrightarrow (Y, d)$ is continuous, by the classical closed graph theorem. So, $T(E)$ is bounded.
Second proof:

Let E be a bounded subset of X. Let F be the closed absolute convex hull of E in X. Let μ_F be the Minkowski functional induced by F. Then $Z = \{x \in X : \mu_F(x) < \infty\}$. Thus (Z, μ_F) is a Banach space in which E is a bounded subset.

Verification: [42, Theorems 1.13, 1.35, 1.36, and 1.39] may be used along with the fact that F is bounded in the locally convex space X to verify that μ_F is a norm on Z. Since $E \subseteq F$ and since $\mu_F(x) \leq 1,\;$ for all $x \in F$, the set E is also a bounded subset of (Z, μ_F). If a Cauchy sequence $(x_n)_{n=1}^\infty$ in (Z, μ_F) is considered, then it is contained in a positive scalar multiple of F, say, γF, so that $\mu_F(x_n) \leq \gamma,\;$ for all n. This sequence should also be a Cauchy sequence in the sequentially complete space $(X, (p_\alpha)_{\alpha\in I})$, because F is bounded in $(X, (p_\alpha)_{\alpha\in I})$, and hence $(x_n)_{n=1}^\infty$ converges to some x in $(X, (p_\alpha)_{\alpha\in I})$. Then, this x should be in the sequentially closed set γF, a subset of Z. It now follows from the uniqueness of limit of the sequence $(x_n)_{n=1}^\infty$ in the completion of (Z, μ_F) that the sequence $(x_n)_{n=1}^\infty$ should converge to x in (Z, μ_F), where $\mu_F(x) \leq \gamma < \infty$ (Another approach: $\mu_F(x_n - x_m) \to 0$ as $n, m \to \infty$ and so $\mu_F(x_n - x) \to 0$ as $n \to \infty$, and $\mu_F(x) < \infty$ in $(X, (p_\alpha)_{\alpha\in I})$). Thus, (Z, μ_F) becomes a Banach space. Again, by the classical closed graph theorem, $T(E)$ is bounded. $\square$

Theorem 2.2. *Let A be a complex sequentially complete locally convex algebra. Let B be a complex commutative semisimple Banach algebra. Let E be a bounded subset of A such that $E^2 = \{xy : x \in E, y \in E\} \subseteq E$. Let $T : A \longrightarrow B$ be an algebra homomorphism. Then $T(E)$ is a bounded subset of B.*

Proof. Let F, μ_F, and Z be as in the second proof of the previous Theorem 2.1 with replacement of X and Y by A and B, respectively. Since $E^2 \subseteq E$, then $F^2 \subseteq F$ so that (Z, μ_F) is a Banach algebra. So, $T|_Z : (Z, \mu_F) \longrightarrow B$ is a continuous homomorphism (see [42, Theorem 11.10]). This proves that $T(E)$ is bounded. □

This technique of reducing locally convex algebras to normed algebras has been used directly and indirectly in many articles like [25]. Note that if $x \in Z$ in the proof of the previous Theorem 2.2, then the spectrum of x in Z contains the spectrum of x in A. Thus, for each $x \in E$, one should have finite spectral radius for spectrum of x in A. Finiteness of spectral radius has also been used in many articles like [15, 40].

Corollary 2.3. *Let A and E be as in Theorem 2.2. Then for every multiplicative linear functional f on A, the set $f(E)$ is a bounded set.*

Proof. Take $B = \mathbb{C}$ and $T = f$ in Theorem 2.2. □

Theorem 2.4. *Let A be a complex sequentially complete locally convex algebra with a continuous involution $*$. Let E be a bounded subset of A such that $E^2 \subseteq E = \{x^* : x \in E\}$. Let f be a complex positive linear functional on A satisfying $|f(x)|^2 \leq M f(xx^*)$ and $\overline{f(x)} = f(x^*)$, for all $x \in A$, for some $M \geq 0$. Then $f(E)$ is bounded.*

Proof. Take $B = \mathbb{C}$ and $T = f$ in the proof of Theorem 2.2. Since $E = \{x^* : x \in E\}$, then $F = \{x^* : x \in F\}$, and hence Z is closed under involution. That is, (Z, μ_F) is a Banach algebra with an involution, which is the restriction of the involution on A. It should be noted that the restriction of f to (Z, μ_F) is continuous because of a comment between Definition 10 and Theorem 11, and because of Theorem 11 in Section 37 of [7]. □

Remark 2.5. The Dixon–Fremlin method is more convenient, if a topological vector space X is endowed with a family of quasi-semi-norms $(p_\alpha)_{\alpha \in I}$ of the following type: (i) $p_\alpha(x) \geq 0$, for all $x \in X$; (ii) $p_\alpha(x + y) \leq p_\alpha(x) + p_\alpha(y)$, for all $x, y \in X$; and (iii) $p_\alpha(\lambda x) \leq |\lambda|^{\rho_\alpha} p_\alpha(x)$, for all $x \in X$, for all scalars λ, for some $\rho_\alpha \in (0, 1]$. If E is a bounded subset of X, then for positive integers m, n, we can consider

$$I_{m,n} = \{\alpha \in I : p_\alpha(x) \leq m, \text{ for all } x \in E, \frac{1}{n} < \rho_\alpha \leq 1\}.$$

This article does not consider such quasi-semi-norms. This article does not discuss the Borel graph theorem, for which the Dixon–Fremlin method can be applied. Some possible generalizations of these types are not recorded to maintain simplicity to some extent.

Let us next present statements with outlines for proofs for consequences of the uniform boundedness principle, which require corresponding classical theorems ([42, Theorems 2.6, 2.4, 2.8, and 2.17]) and the Dixon–Fremlin method.

Theorem 2.6. *Let $(F_\alpha)_{\alpha\in I}$ be a family of bounded linear transformations $F_\alpha : X \longrightarrow Y$, where X is a sequentially complete locally convex space and Y is a topological vector space. Let E be a bounded subset of X. Suppose that $\{F_\alpha(x) : \alpha \in I\}$ is bounded, for every $x \in X$. Then $\{F_\alpha(x) : \alpha \in I, x \in E\}$ is bounded.*

Proof. (An Outline): Construct the Fréchet space $(Z, (p_n)_{n=1}^\infty)$ as in the first proof of Theorem 2.1. By [42, Theorem 2.6], the restrictions of the members of the given family restricted to this Fréchet space forms an equicontinuous family. Now, the result follows from [42, Theorem 2.4]. □

Theorem 2.7. *Let $(F_i)_{i=1}^\infty$ be a sequence of bounded linear transformations $F_i : X \longrightarrow Y$, where X is a sequentially complete locally convex space and Y is a topological vector space. Suppose that $lim_{i\to\infty} F_i(x) = F(x)$ exists, for every $x \in X$, when $F : X \longrightarrow Y$ is a linear map. Then F is bounded.*

Proof. (An Outline): Let E be a bounded subset of X. Construct the Fréchet space $(Z, (p_n)_{n=1}^\infty)$ as in the first proof of Theorem 2.1. Then the restrictions of each F_i to $(Z, (p_n)_{n=1}^\infty)$ is continuous. By [42, Theorem 2.8], the restriction of F to $(Z, (p_n)_{n=1}^\infty)$ is continuous so that $F(E)$ is bounded. □

Theorem 2.8. *Let $F : \prod_{i=1}^n X_i \longrightarrow X$ be a multilinear map, where each X_i is a sequentially complete locally convex space and X is a topological vector space. If F is separately bounded, then F is jointly bounded.*

Proof. (An Outline): The definitions for multilinear mappings and separately boundedness can be stated naturally. Let us prove the statement for the case $n = 2$, from which the general case follows by induction. Let $Y = X_1 \times X_2$ so that the map F is considered on Y. Consider the given spaces X_1 and X_2 in the forms $(X_1, (p_\alpha)_{\alpha\in I_1})$ and $(X_2, (q_\beta)_{\beta\in I_2})$. Let $J = I_1 \times I_2$ and for each $\gamma = (\alpha, \beta) \in J$, and for each $y = (x_1, x_2) \in Y$, let $r_\gamma(y) = \max\{p_\alpha(x_1), q_\beta(x_2)\}$. Then Y may be considered in the form $(Y, (r_\gamma)_{\gamma\in J})$. Let E be a bounded subset of Y. Then $E \subseteq E_1 \times E_2$, for some bounded subsets E_1 and E_2 of X_1 and X_2, respectively. For each $(m, n) \in \mathbb{N} \times \mathbb{N}$, let $J_{m,n} = \{\gamma = (\alpha, \beta) \in J : p_\alpha(x_1) \le m, q_\beta(x_2) \le n, \text{ for all } x_1 \in E_1, \text{ for all } x_2 \in E_2\}$, and for each $y = (x_1, x_2) \in Y$, let $r_{m,n}(y) = \max\{r_\gamma(y) : \gamma \in J_{m,n}\}$. Let $Z = \{y \in Y : r_{m,n}(y) < \infty, \text{ for all } (m, n) \in \mathbb{N} \times \mathbb{N}\}$. Then $(Z, (r_{m,n})_{(m,n)\in\mathbb{N}\times\mathbb{N}})$ is a Fréchet space in which E is a bounded subset, because $\mathbb{N} \times \mathbb{N}$ is a countable set. It now follows from [42, Theorem 2.17] that the image of E under F is bounded. □

3. Application of the open mapping theorem

The classical open mapping theorem, the classical closed graph theorem, and the classical Banach-Steinhaus theorem are almost equivalent. They are derived from Baire's category theorem, and they have many generalizations. They cannot be excluded from the theory of automatic continuity. A method of applying the closed graph Theorem 2.1 is to be explained in the proof of Theorem 9.2. The next Theorem 3.1 is also an open mapping theorem, which is a particular case of [29, Corollary 3.3], and which is to be applied in this section.

Theorem 3.1. *Let (X, d) be an F-space with respect to an addition invariant metric d. Let X^+ be a closed subset of X such that*

(i) *If $x_1, x_2 \in X^+$, then $x_1 + x_2 \in X^+$.*

(ii) *If $x \in X^+$ and λ is a nonnegative scalar, then $\lambda x \in X^+$.*

(iii) *Each $x \in X$ has a representation $x = x_1 - x_2$ with $x_1 \in X^+$ and $x_2 \in X^+$.*

Define $T : X^+ \times X^+ \longrightarrow X$ by $T(x_1, x_2) = x_1 - x_2$, for all $(x_1, x_2) \in X^+ \times X^+$. Then T is an open mapping, when $X^+ \times X^+$ is endowed with the product topology induced from the topology on X.

Definition 3.2. For this section, let us call a subset X^+ satisfying (i),(ii), and (iii) of the previous Theorem 3.1 as a positive cone of X.

Example 3.3. Let f be a nonzero real continuous linear functional on a topological vector space X. Let $X^+ = \{x \in X : f(x) \geq 0\}$. Then X^+ is a closed positive cone of X according to our Definition 3.2. Observe that $X^+ \bigcap (-X^+) \neq \{0\}$ if $dim X \geq 2$. The space X is also a closed positive cone of X.

Remark 3.4. Let X^+ be a closed positive cone of a real topological vector space X, as given in Definition 3.2. Then $X^+ \cap (-X^+)$ is a real closed vector subspace of X.

Theorem 3.5. *Let X be a sequentially complete locally convex space. Let X^+ be a sequentially closed positive cone such that for a given bounded sequence $(x_n)_{n=1}^{\infty}$ in X, there are bounded sequences $(x_n^{'})_{n=1}^{\infty}$ and $(x_n^{''})_{n=1}^{\infty}$ in X^+ satisfying $x_n = x_n^{'} - x_n^{''}$, for all n. Let $f : X \longrightarrow \mathbb{R}$ be a linear functional on X such that $f(x) \geq 0$, for all $x \in X^+$. Then f is bounded.*

Proof. Suppose that f is not bounded. Then there exists a bounded sequence $(x_n)_{n=1}^{\infty}$ in X such that $f(x_n) \geq 2^n$, for all $n = 1, 2, \ldots$. Then there are bounded sequences $(x_n^{'})_{n=1}^{\infty}$ and $(x_n^{''})_{n=1}^{\infty}$ in X^+ such that $x_n = x_n^{'} - x_n^{''}$, for all n. Then $f(x_n^{'}) \geq f(x_n) \geq 2^n$, for all $n = 1, 2, \ldots$ Let $x = \sum_{n=1}^{\infty} \frac{1}{2^n} x_n^{'} \in X^+$. Then,

$$
\begin{aligned}
f(x) &= \sum_{n=1}^{m} \frac{f(x_n^{'})}{2^n} + f(\sum_{n=m+1}^{\infty} \frac{x_n^{'}}{2^n}) \\
&\geq \sum_{n=1}^{m} \frac{2^n}{2^n} + 0 \\
&= m,
\end{aligned}
$$

for all $m = 1, 2, \ldots$, because $\sum_{n=m+1}^{\infty} \frac{x_n^{'}}{2^n} \in X^+$. This is a contradiction. Hence f is bounded on X. □

Let us now apply the open mapping Theorem 3.1 along with the arguments used in the proof of the previous Theorem 3.5.

Theorem 3.6. *Let X be an F-space. Let X^+ be a closed positive cone. Let $f : X \longrightarrow \mathbb{R}$ be a linear functional on X such that $f(x) \geq 0$, for all $x \in X^+$. Then f is continuous.*

Proof. Suppose that f is not continuous. Then there exists a sequence $(x_n)_{n=1}^{\infty}$ in X such that $x_n \longrightarrow 0$ as $n \longrightarrow \infty$, and such that $f(x_n) \geq 2^n$, for all $n = 1, 2, \ldots$. Then, by Theorem 3.1, by passing to a subsequence of $(x_n)_{n=1}^{\infty}$, it can be assumed that there are sequences $(x_n^{'})_{n=1}^{\infty}$ and $(x_n^{''})_{n=1}^{\infty}$ in X^+ such that $d(0, x_n^{'}) < 2^{-n}$ and $x_n = x_n^{'} - x_n^{''}$, for all n. Then $f(x_n^{'}) \geq f(x_n) \geq 2^n$, for all $n = 1, 2, \ldots$. Let $x = \sum_{n=1}^{\infty} x_n^{'} \in X^+$. Then,

$$\begin{aligned} f(x) &= \sum_{n=1}^{m} f(x_n^{'}) + f(\sum_{n=m+1}^{\infty} x_n^{'}) \\ &\geq \sum_{n=1}^{m} 2^n + 0 \end{aligned}$$

for all $m = 1, 2, \ldots$, because $\sum_{n=m+1}^{\infty} x_n^{'} \in X^+$. This is a contradiction. Hence f is continuous. □

4. A technique of T.-sh. Hsia

One standard technique to get a contradiction is establishing that some quantities are bounded as well as unbounded. There are many ways to implement this technique. One way is the way of the Hsia [24], which uses the positiveness of function values at some elements as was done in the previous section. His method is illustrated in this section. The first result is a variation of [35, Theorem 1], and it is also a generalization. If (X, d) is an additive metrizable topological group, then d can be chosen such that it is an one-sided translation invariant metric. Hence if $f : (X, d) \to \mathbb{R}$ is an addition preserving group homomorphism, then it is not continuous if and only if there is a sequence $(x_n)_{n=1}^{\infty}$ converging to zero in (X, d) such that $f(x_n) \to +\infty$, as $n \to \infty$. Completeness of (X, d) is also defined only in terms of one-sided translation invariant metrics. See [5, Theorem 2.1.7] and the proof of [42, Theorem 1.28].

Theorem 4.1. *Let (X, d) be a complete metrizable topological group with an addition operation. Let $T : X \longrightarrow X$ be a continuous mapping such that $T(0) = 0$. Let $f : X \longrightarrow \mathbb{R}$ be an addition preserving group homomorphism such that $f(T(x)) \geq 0$, for all $x \in X$. Assume further that there is a constant $K \geq 1$ and there is a constant $M > 0$ such that $M \leq |f(x)| \leq Kf(T(x))$ whenever $|f(x)| \geq M$ and $x \in X$. Then f is a continuous mapping.*

Proof. Suppose that f is not a continuous mapping. Then there is a sequence $(x_n)_{n=1}^{\infty}$ in X such that $x_n \longrightarrow 0$, as $n \longrightarrow \infty$, and such that $f(x_n) \geq nK^n$, for all n. Then there is a subsequence $(y_n)_{n=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that for $z_{n,j} = y_j + T(y_{j+1} + T(y_{j+2} + \cdots + T(y_{n-1} + T(y_n)) \ldots))$, $1 \leq j \leq n-1$, the relation $d(z_{n,j}, z_{n-1,j}) < 2^{-n}$, for all $n = 2, 3, \ldots$, is true, because of the following reasons. First $T(0) = 0$, and then T is continuous at 0 so that $T(y_m) \to 0$ as $m \to \infty$, and also $T(y_{n-1} + T(y_n))$ can be adjusted such that it can be sufficiently close to $T(y_{n-1})$. Ultimately $z_{n,j}$ can be adjusted such that it can be sufficiently close to $z_{n-1,j}$, by choosing y_n very close to zero. Then for $n > m > j$,

$$\begin{aligned} d(z_{n,j}, z_{m,j}) &\leq d(z_{n,j}, z_{n-1,j}) + d(z_{n-1,j}, z_{n-2,j}) + \cdots + d(z_{m+1,j}, z_{m,j}) \\ &< 2^{-n} + 2^{-n+1} + 2^{-n+2} + \cdots + 2^{-m-1} \\ &< 2^{-m}. \end{aligned}$$

Therefore, for each $j = 1, 2, \ldots$, $(z_{n,j})_{n=j+1}^{\infty}$ is a Cauchy sequence, which converges to some z_j in X. Then $f(z_n) = f(y_n) + f(T(z_{n+1})) \geq nK^n \geq M$, for all $n \geq m$, for some m. Now

$$\begin{aligned} f(z_m) &= f(y_m) + f(T(z_{m+1})) \\ &\geq mK^m + \frac{1}{K} f(z_{m+1}) \\ &= mK^m + \frac{1}{K} f(y_{m+1}) + \frac{1}{K} f(T(z_{m+2})) \\ &\geq mK^m + (m+1)K^m + \frac{1}{K} f(T(z_{m+2})) \\ &\geq mK^m + (m+1)K^m + \frac{1}{K^2} f(z_{m+2}) \\ &\geq mK^m + (m+1)K^m + (m+2)K^m + \frac{1}{K^3} f(z_{m+2}) \\ &\geq K^m(m + (m+1) + (m+2) + \cdots + (m+j)), \end{aligned}$$

for any $j \geq 1$. This is a contradiction. This proves that f is continuous on X. □

Hence, $f(z_m)$ in the proof of the previous Theorem 4.1 is bounded, as well as unbounded in the sense that $f(z_m)$ is a fixed nonnegative number, and it is greater than or equal to $K^m(m+(m+1)+(m+2)+\cdots+(m+j))$, for all $j \geq 1$. The next Corollary 4.2 is [35, Theorem 1].

Corollary 4.2. *Let (X, d) and T be as in the previous Theorem 4.1. Let $f : X \longrightarrow \mathbb{R}$ be an additive group homomorphism such that $(f(x))^2 \leq Kf(T(x))$, for all $x \in X$, for some $K > 0$. Then f is continuous.*

Proof. Since $(f(x))^2 \geq 0$, then $f(T(x)) \geq 0$, for all $x \in X$. Moreover, if $|f(x)| \geq 1$, then $1 \leq |f(x)| \leq (f(x))^2 \leq Kf(T(x))$. □

The next Corollary 4.3 is also a theorem in [35], and which may be considered as a generalization of [12, Theorem 3.1], where the proof is based on a Mittag-Leffler technique.

Corollary 4.3. *If A is a complete metrizable topological ring, then every ring homomorphism $f : A \longrightarrow \mathbb{R}$ is continuous.*

Proof. Take $X = A$, $T(x) = x^2$, for all $x \in X$ and $K = 1$ in the previous Corollary 4.2. □

Michael's problem is about continuity of complex multiplicative linear functionals on commutative complex Fréchet algebras. Corollary 4.3 establishes a positive solution but for the case of real multiplicative linear functionals.

Corollary 4.4. *Let A be an F-algebra with a continuous involution $*$. Let $f : A \longrightarrow \mathbb{C}$ be a positive linear functional such that $f(xx^*) \geq 0, f(x^*) = \overline{f(x)}$ and $|f(x)|^2 \leq Kf(xx^*)$, for all $x \in A$, for some $K > 0$. Then f is continuous on A.*

Proof. Take $X = \{x \in A : x = x^*\}$, and take $T(x) = x^2$, for all $x \in X$, and consider f restricted to X. Then f is continuous on X, by Corollary 4.2. So, f is continuous on A, because $*$ is continuous. □

Remark 4.5. If A has a multiplicative identity, then any complex positive linear functional f satisfies the conditions of the previous Corollary 4.4. These types of functionals are called extendable positive linear functionals. This terminology is also used because such functionals can be extended as positive linear functionals on the unitization of the algebra. See the proof of [7, Section 37, Theorem 11].

Let us now apply the Dixon and Fremlin method and Theorem 4.1 in the proof of the next Theorem 4.6.

Theorem 4.6. *Let $(X, (p_\alpha)_{\alpha\in I})$ be a sequentially complete locally convex topological vector space. Let $S : X \times X \longrightarrow X$ be a mapping such that for any $D \subseteq X$ and for any $J \subseteq I$ satisfying*

$$\sup_{x\in D} \sup_{\alpha\in J} p_\alpha(x) < \infty,$$

and

$$\sup_{x,y\in D} \sup_{\alpha\in J} p_\alpha(S(x, y)) < \infty.$$

Let $T : X \longrightarrow X$ be a mapping such that $T(0) = 0$ and such that $p_\alpha(T(x)-T(y)) \leq K_\alpha p_\alpha(S(x, y))p_\alpha(x - y)$, for some $K_\alpha \geq 1$, for all $\alpha \in I$, for all $x, y \in X$. Let $f : X \longrightarrow \mathbb{R}$ be a linear mapping such that $f(T(x)) \geq 0$, for all $x \in X$. Assume further that there is a constant $K \geq 1$ and there is a constant $M > 0$ such that $M \leq |f(x)| \leq Kf(T(x))$ whenever $|f(x)| \geq M$. Then f is a bounded functional on $(X, (p_\alpha)_{\alpha\in I})$.

Proof. Let E be a bounded subset of X. For each $m, n \in \mathbb{N}$ (the set of natural numbers), let $I_{m,n} = \{\alpha \in I : K_\alpha \leq m, p_\alpha(x) \leq n,$ for all $x \in E\}$. If $I_{m,n} = \emptyset$, then define $q_{m,n}(x) = 0$, for all $x \in X$. If $I_{m,n} \neq \emptyset$, then define $q_{m,n}(x) = \sup_{\alpha\in I_{m,n}} p_\alpha(x)$, for all $x \in X$. Let $Y = \{x \in X : q_{m,n}(x) < \infty,$ for all $m, n\}$. Then $(Y, (q_{m,n})_{m,n})$ is a Fréchet space, in which E is a bounded subset. Let $(x_r)_{r=1}^\infty$ be a sequence in Y converging to x in $(Y, (q_{m,n})_{m,n})$. Let $D = \{x_r : r = 1, 2, \dots\} \bigcup \{x\}$. Since $\sup_{y\in D} q_{m,n}(y) < \infty$, by our assumption, $\sup_{z,y\in D} q_{m,n}(S(z, y)) < \infty$. Then the inequality

$$q_{m,n}(T(x_r) - T(x)) \leq m \left(\sup_{z,y\in D} q_{m,n}(S(z, y))) \times (q_{m,n}(x_r - x) \right)$$

implies that $T(x_r) - T(x) \longrightarrow 0$ in $(Y, (q_{m,n})_{m,n})$ as $r \longrightarrow \infty$. If $z \in Y$, then

$$\begin{aligned} q_{m,n}(T(z)) &= q_{m,n}(T(z) - T(0)) \\ &\leq mq_{m,n}(z-0) \inf\{q_{m,n}(S(z,0)), q_{m,n}(S(0,z))\} \\ &< +\infty. \end{aligned}$$

Thus $T(Y) \subseteq Y$ and $T|_Y : (Y, (q_{m,n})_{m,n}) \longrightarrow (Y, (q_{m,n})_{m,n})$ is continuous. So, by Theorem 4.1, $f(E)$ is a bounded set. That is, $f : (X, (p_\alpha)_{\alpha \in I}) \longrightarrow \mathbb{R}$ is bounded. □

The next two corollaries are known through [24] of Hsia and [11] of Dixon and Fremlin but for complete LMC algebras.

Corollary 4.7. *Let $(A, (p_\alpha)_{\alpha \in I})$ be a sequentially complete LMC algebra. Let f be a real multiplicative linear functional on A. Then f is bounded.*

Proof. Take $X = A$. Define $T : X \longrightarrow X$ by $T(x) = x^2$, for all $x \in X$, and define $S : X \times X \longrightarrow X$ by $S((x, y)) = x + y$, for all $(x, y) \in X \times X$. Note that

$$\begin{aligned} p_\alpha(T(x) - T(y)) &= p_\alpha(x^2 - y^2) \\ &= p_\alpha(\frac{1}{2}(x+y)(x-y) + \frac{1}{2}(x-y)(x+y)) \\ &\leq p_\alpha(x+y)p_\alpha(x-y) \\ &= p_\alpha(S((x,y)))p_\alpha(x-y), \end{aligned}$$

for all $\alpha \in I$, for all $x, y \in X$. Now, Corollary 4.7 follows from the previous Theorem 4.6 with $M = K = K_\alpha = 1$. □

Corollary 4.8. *Let $(A, (p_\alpha)_{\alpha \in I})$ be a sequentially complete LMC algebra with a sequentially continuous involution $*$. Let f be an extendable positive linear functional on A (Remark 4.5). Then f is bounded.*

Proof. Take $X = \{x \in A : x = x^*\}$. Then X is a sequentially complete locally convex space, because $*$ is sequentially continuous and A is sequentially complete. Define $T : X \longrightarrow X$ by $T(x) = x^2$, for all $x \in X$, and define $S : X \times X \longrightarrow X$ by $S((x, y)) = x + y$, for all $(x, y) \in X \times X$. Now Theorem 4.6 implies that $f|_X$ is bounded. Since the involution maps bounded sets onto bounded sets, f is a bounded map on A. □

These two Corollaries 4.7 and 4.8 are to be extended further for sequential continuity of f in Section 8.

5. A technique for noncontinuous involution

Corollaries 4.4 and 4.8 require the continuity and sequential continuity of involutions. There is a standard technique for the removal of the condition on the continuity of involutions in the case of Banach algebras, which is given in the proof of [42, Theorem 11.3], where the theorem proves that every complex positive linear functional on a Banach algebra with identity and with an involution is continuous. This technique is to be modified for F-algebras in the next Theorem 5.1, but for a different purpose.

Theorem 5.1. *Let A be a complex F-algebra with an involution $*$ and an identity e. Let f be a positive linear functional on A. Let $H = \{x \in A : x = x^*\}$. If $f|_H$ is continuous on H, then f is continuous on A.*

Proof. Suppose that $f|_H$ is continuous on H. Note that $f(H)$ is a subset of the real line. Let $\overline{H}$ be the closure of H in A. Then $f|_H$ has a unique continuous extension $g : \overline{H} \longrightarrow \mathbb{R}$. The natural mapping from $\overline{H} \times \overline{H}$ to A given by $(u, v) \mapsto u + iv$ is continuous, surjective, and open. Let $y \in \overline{H} \cap i\overline{H}$. Then there are sequences $(u_n)_{n=1}^{\infty}$ and $(v_n)_{n=1}^{\infty}$ in H such that $u_n \longrightarrow y$ and $iv_n \longrightarrow y$ as $n \longrightarrow \infty$. Then $u_n^2 + v_n^2 \longrightarrow 0$ as $n \longrightarrow \infty$ so that $f(u_n^2 + v_n^2) \longrightarrow 0$ as $n \longrightarrow \infty$. Note that $0 \leq |\ g(u_n)|^2 =|f(u_n)|^2 \leq f(e)f(u_n^2) \leq f(e)f(u_n^2 + v_n^2)$. So, $g(u_n) \longrightarrow 0$ when $g(u_n) \longrightarrow g(y)$ as $n \longrightarrow \infty$. Thus $g(y) = 0$ whenever $y \in \overline{H} \cap i\overline{H}$.

Let $u_n + iv_n \longrightarrow 0$ in A as $n \longrightarrow \infty$, when $u_n \in H$, $v_n \in H$, for all n. Then there are sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ in $\overline{H}$ such that $x_n + iy_n = u_n + iv_n$, for all n, and such that $x_n \longrightarrow 0$ and $y_n \longrightarrow 0$ as $n \longrightarrow \infty$ because the natural mapping from $\overline{H} \times \overline{H}$ onto A is open in view of the open mapping theorem. Then $g(x_n) \longrightarrow 0$ and $g(y_n) \longrightarrow 0$ as $n \longrightarrow \infty$ and $g(u_n - x_n) = 0 = g(v_n - y_n)$, because $u_n - x_n = i(y_n - v_n) \in \overline{H} \cap i\overline{H}$, for all n. Note that

$$\begin{aligned} 0 \leq |f(u_n + iv_n)| &= |f(u_n) + if(v_n)| = |g(u_n) + ig(v_n)| \\ &= |g(x_n) + ig(y_n)| \leq |g(x_n)| + |g(y_n)|, \end{aligned}$$

for all n. Therefore, $f(u_n + iv_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Thus, $f(u_n + iv_n) \longrightarrow 0$ whenever $u_n + iv_n \longrightarrow 0$ as $n \longrightarrow \infty$, when $u_n \in H$, $v_n \in H$, for all n. This proves that $f : A \longrightarrow \mathbb{C}$ is continuous. □

It should be noted that it is not known whether every complex positive linear functional on an F-algebra with identity and with a noncontinuous involution is continuous or not, even for commutative algebras.

6. Application of the Hahn Banach extension theorem

The Hahn Banach extension theorem can be partially considered as an automatic continuity result because continuous linear extensions are possible only for some classes of topological vector spaces. Let us use the classical Hahn Banach theorem to derive a variation of [42, Theorem 12.39] by using almost the same arguments. Otherwise, the next Theorem 6.2 cannot be result of automatic continuity. The next Lemma 6.1 is [42, Lemma 5.26].

Lemma 6.1. *Suppose that X is a normed space of bounded real (or complex) functions on a set, under the supremum norm. Suppose that T is a linear functional on X such that $\|T\|= 1 = T(1)$. Then $0 \leq Tf \leq 1$ whenever $0 \leq f \leq 1$ and $f \in X$. Hence $Tg \geq 0$, whenever $g \geq 0$ and $g \in X$.*

To a complex algebra A with identity, and for each $z \in A$, let $\sigma_A(z)$(or $\sigma(z)$, simply) denote the spectrum of z in A, and $r_A(z)$(or $r(z)$, simply) denote the spectral radius of z of the spectrum of z in A.

Theorem 6.2. *Let $(A, \| \cdot \|)$ be a complex Banach algebra with identity e such that $\|x^2\|=\|x\|^2$, for all $x \in A$. Let $z \in A$. Then there is a bounded linear functional*

$f : A \longrightarrow \mathbb{C}$ *such that* $f(z) = r(z) = \|z\|, f(x) \geq 0$ *whenever* $\sigma_A(x) \subseteq [0, \infty)$, *and* f *is a multiplicative linear functional on the smallest closed subalgebra containing* e *and* z.

Proof. Let A_z be a maximal commutative subalgebra containing z in A. Then the spectrum of z in A_z is the spectrum of z in A. Then there is a multiplicative linear functional g on A_z such that $g(z) = r(z)$, $g(e) = 1$, and $\|g\| \leq 1$, because $\|x^2\| = \|x\|^2$, for all $x \in A$. Extend this functional g to a bounded linear functional f on A such that $\|f\| \leq 1$. To complete the proof, fix $y \in A$ for which $\sigma_A(y) \subseteq [0, \infty)$. Let A_y be a maximal commutative subalgebra containing y in A. Let Δ_y be the maximal ideal space of A_y, with the Gelfand topology. For each $x \in A_y$, let $\widehat{x}$ denote the Gelfand transform of x. Let $\widehat{A_y} = \{\widehat{x} : x \in A_y\}$, which is a normed algebra of bounded functions on Δ_y, with the supremum norm $\| \ \|_\infty$. Define $h : \widehat{A_y} \longrightarrow \mathbb{C}$ by $h(\widehat{x}) = f(x)$, for all $x \in A_y$, which is well defined because $\|x^2\| = \|x\|^2$, for all $x \in A$. Then $h(1) = f(e) = 1$, and $|h(\widehat{x})| = |f(x)| \leq \|x\| = r(x) = \|\widehat{x}\|_\infty$, for all $\widehat{x} \in \widehat{A_y}$. By the previous Lemma 6.1, $h(\widehat{x}) \geq 0$ whenever $\widehat{x} \geq 0$. In particular $h(\widehat{y}) \geq 0$. That is, $f(y) \geq 0$, whenever $\sigma(y) \subseteq [0, \infty)$. □

7. Gleason–Kahane–Źelazko theorem

The Gleason–Kahane–Źelazko theorem [17, 28] is a famous theorem that is considered a theorem without applications. There are many versions and many generalizations of this theorem (for example, [26]). In this section, a version of this theorem for LMC algebras is presented. An indirect abstract application of this version to an automatic continuity result will be given in Section 11.
Let us recall the following two known Lemmas 7.1 and 7.2 (see [42, Lemma 10.8] and [5, Lemma 1.3.19]).

Lemma 7.1. *Let* f *be an entire function of one complex variable,* $f(0) = 1, f'(0) = 0$, *and* $0 < | \ f(\lambda) \ | \leq e^{|\lambda|}$ $(\lambda \in \mathbb{C})$. *Then* $f(\lambda) = 1$ *for all* $\lambda \in \mathbb{C}$.

Lemma 7.2. *Let* f *be a nonzero linear functional on an algebra* A *with identity* e. *Suppose that* $f(e) = 1$ *and that* $f(x^2) = 0$ *whenever* $f(x) = 0$. *Then* f *is a multiplicative linear functional on* A.

Let $SingA$ denote the collection of all singular elements in an algebra A with an identity. Let us now present a version of the Gleason–Kahane–Źelazko theorem for LMC algebras with the help of the Dixon–Fremlin technique.

Theorem 7.3. *Let* $(A, (p_\alpha)_{\alpha \in I})$ *be a complex sequentially complete LMC algebra with an identity* e. *It is assumed that* $p_\alpha(e) = 1$ *and* $p_\alpha(xy) \leq p_\alpha(x)p_\alpha(y)$, *for all* $x, y \in A$, *for all* $\alpha \in I$. *Let* f *be a complex bounded linear functional on* A *such that* $f(e) = 1$. *Suppose* $\ker f \subseteq SingA$. *Then* f *is a multiplicative linear functional.*

Proof. Let b be a nonzero element in $\ker f$. To every natural number n, let $I_n = \{\alpha \in I : p_\alpha(b) \leq n\}$. Let $q_n(x) = \sup_{\alpha \in I_n} p_\alpha(x)$, for all $x \in A$. Let $B = \{x \in A : q_n(x) < \infty$, for all $n = 1, 2, 3, \ldots\}$. Then $\left(B, (q_n)_{n=1}^\infty\right)$ is a

Fréchet algebra, and $f|_B$ is continuous on $\big(B,(q_n)_{n=1}^{\infty}\big)$, because f on $\big(A,(p_\alpha)_{\alpha\in I}\big)$ is bounded. Without loss of generality, assume that there is an integer m such that $|f(x)| \leq q_m(x)$, for all $x \in B$, and such that $q_m(b) \neq 0$. Let a be a scalar multiple of b such that $q_m(a) = 1$. Let

$$g(\lambda) = \sum_{n=0}^{\infty} \frac{f(a^n)}{n!}\lambda^n,$$

for all $\lambda \in \mathbb{C}$. Then $|g(\lambda)| \leq e^{|\lambda|}$, for all $\lambda \in \mathbb{C}$, because

$$|f(a^n)| \leq q_m(a^n) \leq 1,$$

for all n. Let

$$E(\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}a^n \in B \subseteq A.$$

Then $f(E(\lambda)) = g(\lambda)$, for all $\lambda \in \mathbb{C}$, because f is continuous on $\big(B,(q_n)_{n=1}^{\infty}\big)$. Since $E(\lambda)$ is invertible in A, $f(E(\lambda)) \neq 0$, for all $\lambda \in \mathbb{C}$. That is, $g(\lambda) \neq 0$, for all $\lambda \in \mathbb{C}$. Then by Lemma 7.1, $g(\lambda) = 1$, for all $\lambda \in \mathbb{C}$. Therefore $f(a^2) = 0$. Thus, if $f(b) = 0$, then$f(b^2) = 0$. Now, by Lemma 7.2, f should be a multiplicative linear functional on A. □

Observe that the Dixon–Fremlin technique was applied in the proof of Theorem 7.3.

8. Sequential continuity

Partially new techniques are used in the proof of the following two Propositions 8.1 and 8.3. Although more generalized results are to be presented, the next two results are presented just to understand the nature of the techniques in a simple way. Let us recall the arguments for "boundedness implied by sequential continuity" for linear functionals. Let f be a sequential continuous linear functional on a topological vector space X. If f is not bounded, then $f(B)$ is not bounded for some bounded subset B of X. Without loss of generality, let us assume that B is a countable infinite set, and let us write B in the form of a sequence $(x_n)_{n=1}^{\infty}$. Let us assume without loss of generality that $|f(x_n)| > n^2$, for all n, by passing to a subsequence. Let $\alpha_n = n^{-1}$, for all n. Then $(f(\alpha_n x_n))_{n=1}^{\infty}$ does not converge to zero. However, by [42, Theorem 1.30], $(\alpha_n x_n)_{n=1}^{\infty}$ converges to zero. By sequential continuity of f, the sequence $(f(\alpha_n x_n))_{n=1}^{\infty}$ should converge to zero. This is a contradiction. So, every sequential continuous linear functional on a topological vector space should be a bounded linear functional. This observation is necessary to understand the strength of the following results.

Proposition 8.1. *Let $\big(A,(p_\alpha)_{\alpha\in I}\big)$ be an LMC algebra. Let f be a bounded multiplicative linear functional on A. Then f is sequentially continuous.*

Proof. Suppose that f is not sequential continuous on A. Then there is a sequence $(x_n)_{n=1}^{\infty}$ converging to zero in A such that $f(x_n) \geq 2$, for all n. Since $p_\alpha(x_n) \longrightarrow 0$ as $n \longrightarrow \infty$, it is true that $p_\alpha(x_n^n) \longrightarrow 0$ as $n \longrightarrow \infty$, for all $\alpha \in I$, because $p_\alpha(x_n^n) \leq (p_\alpha(x_n))^n$, for all n, for all α. Indeed $f(x_n^n) \geq 2^n$, for all $n =$

$1, 2, \ldots$. So, $\{x_n^n : n = 1, 2, \ldots\}$ is bounded, when $\{f(x_n^n) : n = 1, 2, \ldots\}$ is unbounded. Thus f is not bounded on A. $\square$

Corollary 8.2. *Every real multiplicative linear functional on a sequentially complete LMC algebra is sequentially continuous.*

Proof. Use Corollary 4.7. $\square$

Proposition 8.3. *Let $\big(A, (p_\alpha)_{\alpha \in I}\big)$ be a complex LMC algebra with a sequential continuous involution $*$. Let f be a bounded extendable (Remark 4.5) positive linear functional on A such that $|f(x)\ |^2 \leq Kf(xx^*)$, for all $x \in A$, for some $K \geq 1$. Then f is sequentially continuous.*

Proof. Suppose that f is not sequential continuous on A. Then there is a sequence $(x_n)_{n=1}^\infty$ converging to zero in A such that $f(x_n) \geq 2K$, for all n, and such that $x_n = x_n^*$, for all n, because $*$ is sequentially continuous. Then $p_\alpha(x_n^{2^n}) \longrightarrow 0$ as $n \longrightarrow \infty$, for all $\alpha \in I$. However,

$$\begin{aligned} f(x_n^{2^n}) &\geq \frac{1}{K}|f(x_n^{2^{n-1}})|^2 \\ &\geq \frac{1}{K^n}|f(x_n)|^{2^n} \\ &\geq \frac{1}{K^n}2^{2^n}K^{2^n}, \end{aligned}$$

for all n. Thus $\{f(x_n^{2^n}) : n = 1, 2, \ldots\}$ is unbounded, when $\{x_n^{2^n} : n = 1, 2, \ldots\}$ is bounded. $\square$

Corollary 8.4. *Every extendable positive linear functional on a complex sequentially complete LMC algebra with a sequentially continuous involution is sequentially continuous.*

Proof. Use Corollary 4.8. $\square$

With this experience let us go for abstractions of these techniques.

Definition 8.5. Let (X, τ) be a topological vector space. Let $T : X \longrightarrow X$ be a mapping. The space X is said to be an s-topological vector space with respect to T, if the nth iteration $T^n(x_n) \longrightarrow 0$ as $n \longrightarrow \infty$ whenever $x_n \longrightarrow 0$ as $n \longrightarrow \infty$.

Lemma 8.6. *Let (X, τ) be a topological vector space, which is an s-topological vector space with respect to a mapping $T : X \longrightarrow X$. Let f be a linear functional on X such that $M \leq |f(x)| \leq K|f(T(x))|$ whenever $|f(x)| \geq M$, for some $M > 0$, for some $K \in (0, 1)$. Then f is sequentially continuous, whenever f is bounded.*

Proof. Suppose that f is not sequentially continuous. Then there is a sequence $(x_n)_{n=1}^\infty$ converging to zero such that $|f(x_n)| \geq M$, for all n. Then

$$|f(T(x_n))| \geq \frac{1}{K}|f(x_n)| \geq \frac{M}{K} \geq M,$$

$$|f(T^2(x_n))| \geq \frac{1}{K}|f(T(x_n))| \geq \frac{M}{K^2} \geq M,$$

and

$$|f(T^n(x_n))| \geq \frac{M}{K^n} \geq M,$$

for all n. Here $T^n(x_n) \longrightarrow 0$ as $n \longrightarrow \infty$ and $\{f(T^n(x_n)) : n = 1, 2, \ldots\}$ is unbounded, because $0 < K < 1$. Then f is not bounded. □

Example 8.7. Let $\big(A, (p_\alpha)_{\alpha \in I}\big)$ be an LMC algebra. Take $X = A$, $M = 1$, $T : X \longrightarrow X$ as the mapping defined by $T(x) = 2x^2$, and $K = \frac{1}{2}$ in the previous Lemma 8.6. Let f be a multiplicative linear functional on A. If $x_n \longrightarrow 0$, then $T^n(x_n) = 2^{2^n - 1} x_n^{2^n} \longrightarrow 0$ as $n \longrightarrow \infty$, because A is an LMC algebra. Also, if $|f(x)| \geq 1$, then

$$K|f(T(x))| = \frac{1}{2}|f(2x^2)| = |f(x^2)| \geq |f(x)| \geq 1.$$

Thus, Proposition 8.1 is a consequence of Lemma 8.6.

Example 8.8. Let $\big(A, (p_\alpha)_{\alpha \in I}\big)$ be a complex LMC algebra with a sequentially continuous involution $*$. Let $X = A$. Let f be an extendable (Remark 4.5) positive linear functional on X such that $|f(x)|^2 \leq Lf(xx^*)$ for all $x \in X$, for some $L \geq 1$. Let $T : X \longrightarrow X$ be defined by $T(x) = 2Lxx^*$. Take $M = 1$ and $K = \frac{1}{2}$. If $|f(x)| \geq 1$, then

$$K|f(T(x))| = \frac{1}{2}|f(2Lxx^*)| = Lf(xx^*) \geq |f(x)|^2 \geq |f(x)| \geq 1.$$

Thus by Lemma 8.6, f is sequential continuous, whenever f is bounded, because $*$ is sequential continuous. Thus, Proposition 8.3 is a consequence of Lemma 8.6.

Lemma 8.9. *Let (X, τ) be an s-topological vector space with respect to a mapping $T : X \longrightarrow X$. Let $\big(Y, (q_\alpha)_{\alpha \in J}\big)$ be a locally convex space. Let $S : Y \longrightarrow Y$ be a mapping such that*

$$q_\alpha(S^n(y)) \geq (q_\alpha(S(y)))^n,$$

for all n, for all $y \in Y$, for all $\alpha \in J$ and such that $q_\alpha(S(y)) \geq q_\alpha(y) \geq M$ whenever $q_\alpha(y) \geq M$, for some $M > 0$, for all $\alpha \in J$. Let $f : X \longrightarrow Y$ be a linear mapping such that

$$q_\alpha(f(T^n(x))) \geq q_\alpha(S^n(f(x))),$$

for all n, for all α, for all $x \in X$. If f is bounded, then f is sequentially continuous.

Proof. Choose $L \geq 1$ such that $LM \geq 2$. Suppose that f is not sequentially continuous. Then there is a sequence $(x_n)_{n=1}^\infty$ converging to zero in X such that $q_\alpha(f(x_n)) \geq LM \geq M$, for all n, for some $\alpha \in J$. Then

$$q_\alpha(f(T^n(x_n))) \geq q_\alpha(S^n(f(x_n))) \geq (q_\alpha(S(f(x_n))))^n \geq (q_\alpha(f(x_n)))^n \geq (LM)^n,$$

for all $n = 1, 2, \ldots$. Then f is not bounded. □

Example 8.10. Let $\big(A,(p_\alpha)_{\alpha\in I}\big)$ be an LMC algebra with a sequentially continuous involution. Let $\big(B,(q_\alpha)_{\alpha\in J}\big)$ be a locally convex algebra with an involution such that $q_\alpha(xx^*) = (q_\alpha(x))(q_\alpha(x^*))$, and $q_\alpha(x) = q_\alpha(x*)$, for all $\alpha \in I$, for all $x \in B$. Take $X = A$ and $Y = B$. Let $T : X \longrightarrow X$ and $S : Y \longrightarrow Y$ be defined by $T(x) = xx^*$ and $S(y) = yy^*$, for all $x \in X$, for all $y \in Y$. Let $f : X \longrightarrow Y$ be a linear mapping such that $f(x^*) = (f(x))^*$ and $f(xx^*) = f(x)f(x^*)$, for all $x \in X$. Then all conditions of Lemma 8.9 are satisfied with $M = 1$, because A is an s-topological vector space with respect to T. In fact, the equalities $q_\alpha(S^n(y)) = (q_\alpha(S(y)))^n$ and $q_\alpha(f(T^n(x))) = q_\alpha(S^n(f(x)))$ hold good. Thus, if f is bounded, then f is sequentially continuous.

The following Theorem 8.11 of Arens [3] is to be extended. This result is known as the best partial solution to the problems of Michael [32] mentioned in the first section.

Theorem 8.11. *Let f be a complex multiplicative linear functional on a complex commutative Fréchet algebra A. Let $x_1, x_2, \dots, x_n \in A$. Then there is a complex continuous multiplicative linear functional g on A such that $f(x_i) = g(x_i)$, for all $i = 1, 2, \dots, n$. Hence, f is continuous on the smallest closed subalgebra containing any finite subset of A.*

This Theorem 8.11 of Arens [3] is to be used along with the Dixon–Fremlin method and Proposition 8.1 in the proof of the next theorem.

Theorem 8.12. *Let $\big(A,(p_\alpha)_{\alpha\in I}\big)$ be a complex commutative sequentially complete LMC algebra. Let A_1 be the smallest subalgebra generated by a finite subset $\{u_1, u_2, \dots, u_m\}$ of A. Let f be a complex multiplicative linear functional on A. If $(w_n)_{n=1}^\infty$ is a sequence in A_1, which converges to some w in A, then $f(w_n) \longrightarrow f(w)$, as $n \longrightarrow \infty$.*

Proof. Let E be a bounded subset of A_1. For each n, let $I_n = \{\alpha \in I : p_\alpha(x) \le n,$ for all $x \in E \cup \{u_1, u_2, \dots, u_m\}\}$, and let $q_n(x) = \sup\,\{p_\alpha(x) : \alpha \in I_n\}$, for all $x \in A$. Let $A_2 = \{x \in A : q_n(x) < \infty,$ for all $n\}$. Then $(A_2, (q_n)_{n=1}^\infty)$ is a Fréchet algebra, $A_1 \subseteq A_2$ and E is a bounded subset of $(A_2, (q_n)_{n=1}^\infty)$. Let A_3 be the closure of A_1 in $(A_2, (q_n)_{n=1}^\infty)$. Then, by Theorem 8.11, $f|_{A_3}$ is continuous with respect to $(q_n)_{n=1}^\infty$, and hence $f(E)$ is a bounded set. That is, f is bounded on $(A_1, (p_\alpha)_{\alpha\in I})$. Let $w \in A$ and let $(w_n)_{n=1}^\infty$ be a sequence in A_1 converging to w in $(A, (p_\alpha)_{\alpha\in I})$. Let A_4 be the smallest subalgebra generated by $\{u_1, u_2, \dots, u_m, w\}$. Then f is bounded on A_4, and hence f is sequentially continuous on $(A_4, (p_\alpha)_{\alpha\in I})$, by Proposition 8.1. So, $f(w_n) \longrightarrow f(w)$, as $n \longrightarrow \infty$. □

9. Uniqueness of topology

Carpenter [8] proved that every complex commutative semisimple Fréchet algebra has a unique Fréchet algebra topology based on a bounded-unbounded-contradiction technique. This result is to be modified by extending the technique. Carpenter [8] used the Shilov idempotent theorem for commutative Fréchet algebras, which was observed by Rosenfeld [41]. The same arguments are applicable to establish the following Lemma 9.1, which was used in [44].

Let $\big(A, (p_\alpha)_{\alpha\in I}\big)$ be a complex commutative LMC algebra with an identity. For each $\alpha \in I$, let A_α be the completion of the normed algebra $A/p_\alpha^{-1}(0)$, where the norm is induced by p_α. The norm on A_α will also be denoted by p_α. Let $M(A)$, $M(A_\alpha)$ denote the collection of all nonzero continuous multiplicative linear functionals on A and A_α, respectively. Consider each $M(A_\alpha)$ naturally as a topological subspace of $M(A)$ under the Gelfand topology. In fact, if $\varphi \in M(A_\alpha)$, and $\pi_\alpha : A \to (A/{p_\alpha}^{-1}(0)) \subseteq A_a$ is the quotient mapping, then $\varphi \circ \pi_\alpha \in M(A)$, and $M(A_\alpha)$ is identified as a subset of $M(A)$ through the mapping $\varphi \mapsto \varphi \circ \pi_\alpha$.

The arguments used in [8, 41] to prove the Shilov idempotent theorem for complex commutative Fréchet algebras with identity element are applicable even to establish the next Lemma 9.1.

Lemma 9.1 (Shilov idempotent theorem)**.** *Let $\big(A, (p_\alpha)_{\alpha\in I}\big)$ be a complex commutative complete LMC algebra with an identity element. Let E and F be nonempty subsets of $M(A)$ such that $E\cup F = M(A)$. Suppose that $E\cap M(A_\alpha)$ and $F\cap M(A_\alpha)$ are closed in $M(A_\alpha)$, for all $\alpha \in I$. Then there is an idempotent e in A such that $f(e) = 1$ and $g(e) = 0$, for all $f \in E$ and $g \in F$.*

Proof. (An Outline): For each $\alpha \in I$, and for each $f \in M(A_\alpha)$ let us use the notation f' for the natural corresponding multiplicative linear functional on $A_\alpha/(\text{Radical of } A_\alpha)$. By the classical Shilov idempotent theorem for Banach algebras ([7, Section 21, Theorem 5]), for each $\alpha \in I$, there is a unique idempotent ${e_\alpha}'$ in $A_\alpha/(\text{Radical of } A_\alpha)$ such that $f'({e_\alpha}') = 1$, for all $f \in E \cap M(A_\alpha)$ and such that $g'({e_\alpha}') = 0$, for all $g \in F \cap M(A_\alpha)$. By using [7, Section 8, Theorem 14], for each $\alpha \in I$, from ${e_\alpha}'$, let us construct an idempotent e_α in A_α such that $f(e_\alpha) = 1$, for all $f \in E \cap M(A_\alpha)$ and such that $g(e_\alpha) = 0$, for all $g \in F \cap M(A_\alpha)$. Let $e = (e_\alpha)_{\alpha\in I}$. Then $e \in A$, a projective limit of $(A_\alpha)_{\alpha\in I}$, and this is a required idempotent. ☐

Theorem 9.2. *Let $\big(A, (p_\alpha)_{\alpha\in I}\big)$ be a complex commutative semisimple complete LMC algebra with identity such that $M(A)$ has no isolated points. Let (B, d) be a complex F-algebra with identity. Let $T : B \longrightarrow A$ be a surjective homomorphism. Then T has a closed graph in $B \times A$. Moreover,* $\ker T$ *is closed. Let $T_1 : B/\ker T \longrightarrow (A, (p_\alpha)_{\alpha\in I})$ be the natural map defined by $T_1(y + \ker T) = T(y)$, for all $y \in B$, when $B/\ker T$ is endowed with the quotient topology induced by d. Then T_1 has a closed graph, and hence $T_1^{-1}(E)$ is bounded whenever E is a bounded subset of $(A, (p_\alpha)_{\alpha\in I})$.*

Proof. Let $S = \{f \in M(A) : f \circ T \text{ is continuous on } B\}$. Let us first assume that the closure of S is not equal to $M(A)$ with respect to the Gelfand topology. Let f be in $M(A)$, which is not in the closure of S in $M(A)$. Since f is not isolated in $M(A)$, then f is not isolated in $M(A_\alpha)$, for some $\alpha \in I$, by Lemma 9.1, the Shilov idempotent theorem. For, if f were isolated in every $M(A_\beta)$ in which f is a member, then there would be an ideompotent e such that $f(e) = 1$ and $g(e) = 0$, for all $g \in M(A) \setminus \{f\}$ so that f would be isolated in $M(A)$ with respect to the Gelfand topology. Then $M(A_\alpha) \setminus (\text{Closure of } S)$ is a nonempty open set in $M(A_\alpha)$ containing f, which is not isolated with respect to the Hausdorff Gelfand topology. So, there is a sequence of distinct functionals $f_1, f_2, \ldots$ in $M(A_\alpha)$ such that they are not in the closure of S. By [8, Lemma 2], there is a sequence $z_1, z_2, \dots$ in B such that $f_i \circ T(z_k) = 0$ for $i < k$ and $f_i \circ T(z_k) \neq 0$ for $i \geq k$. Use induction to construct a sequence $x_1, x_2, \dots$ in B such that

(i) $\max_{1\leq j\leq i} d(0, z_j z_{j+1} \dots z_i x_i) < 2^{-i}$,

(ii) $|f_i \circ T(x_i)| > (|f_i \circ T(\sum_{j=1}^{i-1} z_1 z_2 \dots z_j x_j)| + i)|f_i \circ T(z_1 z_2 \dots z_i)|^{-1}$.

It is possible to construct these x_i, because each $f_i \circ T$ is not continuous at zero. Let $x = \sum_{i=1}^{\infty} z_1 z_2 \dots z_i x_i$. For each positive integer $k > 1$, it is true that

$$\begin{aligned} f_k \circ T(x) =& f_k \circ T(\sum_{i=1}^{k-1} z_1 z_2 \dots z_i x_i) + f_k \circ T(z_1 z_2 \dots z_k x_k) \\ &+ f_k \circ T(z_1 z_2 \dots z_{k+1} x_{k+1}) + f_k \circ T((z_1 z_2 \dots z_{k+1}) \sum_{i=k+2}^{\infty} z_{k+2} \dots z_i x_i). \end{aligned}$$

Therefore, when (ii), with $i = k$ in the form $|f_k \circ T(z_1 z_2 \dots z_k)||f_k \circ T(x_k)| - |f_k \circ T(\sum_{j=1}^{k-1} z_1 z_2 \dots z_j)| > k$ is applied,

$$|f_k \circ T(x)| \geq |f_k \circ T(z_1 z_2 \dots z_k x_k)| - |f_k \circ T(\sum_{i=1}^{k-1} z_1 z_2 \dots z_i x_i)| > k,$$

for all $k > 1$. This is impossible, because $f_k \in M(A_\alpha)$, and $|f_k(T(x))| \leq p_\alpha(T(x))$, for all k. So, S is dense in $M(A)$, and hence T has closed graph, because A is semisimple. (*Verification*: Consider a net $(x_\alpha)_{\alpha\in D}$ converging to zero in (B, d) for which $(T(x_\alpha))_{\alpha\in D}$ converges to some y in $(A, (p_\alpha)_{\alpha\in I})$. It should be proved that $y = 0$. If $g \in S$, then $(g \circ T(x_\alpha))_{\alpha\in D}$ converges to zero, as well as to $g(y)$, so that $g(y) = 0$. Let g be in $M(A)$, the closure of S. Fix $\epsilon > 0$. Find $h \in S$ such that $|h(y) - g(y)| < \epsilon$, where $h(y) = 0$. So, $|g(y)| < \epsilon$, for all $\epsilon > 0$. Thus, $g(y) = 0$, for all $g \in M(A)$. Since A is semisimple, $y = 0$.)

Let $(x_n)_{n=1}^{\infty}$ be a sequence in $\ker T$ such that $x_n \longrightarrow x$ in (B, d), as $n \longrightarrow \infty$. Since $f \circ T(x_n) \longrightarrow f \circ T(x)$, as $n \longrightarrow \infty$, for all $f \in S$, then $f \circ T(x) = 0$, for all $f \in S$. Hence it is true for every f in the closure of S. For, if $g \in M(A)$, then for a given $\epsilon > 0$, there is $f \in S$ such that $|f \circ T(x) - g \circ T(x)| < \epsilon$, where $f \circ T(x) = 0$. That is, $f \circ T(x) = 0$, for all $f \in M(A)$. So, if x is in the closure of $\ker T$, then, by [32, Corollary 5.5], $T(x) \in \bigcap_{f\in M(A)} f^{-1}(0) = \{0\}$. Hence $x \in \ker T$. This proves that $\ker T$ is closed. The earlier part implies that T_1 has a closed graph. Then the bijective mapping T_1^{-1} also has a closed graph so that Theorem 2.1 is applicable for T_1^{-1}. Now, Theorem 2.1 implies the remaining part of the statement. □

Corollary 9.3. *For every countable bounded subset E of A, there is a countable bounded subset F of B such that $T(F) = E$ in the previous theorem. If, in addition, B is a Banach algebra, then for every bounded subset E of A, there is a bounded subset F of B such that $T(F) = E$.*

10. A Technique of Źelazko and Goldmann

It was mentioned that Theorem 8.11 is the best partial affirmative answer to the questions raised by Michael [32], given in Section 1. One more interesting result is due to Źelazko [46]. He proved that every complex commutative Fréchet algebra A with a countable maximal ideal space is functionally continuous. Goldmann [18] observed that the technique of Źelazko has generalizations by stating and proving the following Lemma 10.1 for Fréchet algebras when the arguments are extendable to F-algebras.

Lemma 10.1. *Let $(f_n)_{n=1}^{\infty}$ be a sequence of distinct nonzero continuous multiplicative linear functionals on an F-Algebra A. Then there is a point $x \in A$ such that $f_i(x) \neq f_j(x)$ whenever $i \neq j$.*

Let us now observe an abstraction of the Źelazko-Goldmann technique.

Theorem 10.2. *Let (A, τ_1) be a topological algebra. Let (A, τ_2) be an F-algebra. Suppose that the followings are true.*

(i) *Every τ_1-continuous multiplicative linear functional on A is τ_2-continuous.*
(ii) *There is a finite subset $\{x_1, x_2, \ldots, x_n\}$ in A such that to every multiplicative linear functional f on A, the set $F_f = \{g : g$ is a continuous multiplicative linear functional on (A, τ_1), $g(x_i) = f(x_i)$, for all $i = 1, 2, \ldots, n\}$ is at most countable.*
(iii) *For each subset $\{x_{n+1}, x_{n+2}\}$ of A, and for each multiplicative linear functional f on A, the set $\{g : g$ is a continuous multiplicative linear functional on (A, τ_1), $g(x_i) = f(x_i)$, for all $i = 1, 2, \ldots, n+2\}$ is nonempty.*

Then (A, τ_1) is functionally continuous.

Proof. Let f be any given multiplicative linear functional on A. Then, by Lemma 10.1, there is an element x_{n+1} in A such that $g(x_{n+1}) \neq h(x_{n+1})$ whenever $g, h \in F_f$ and $g \neq h$, because each g in F_f is τ_2-continuous. Fix x_{n+2} in A arbitrarily. Then there is a continuous multiplicative linear functional g on (A, τ_1), such that $g(x_i) = f(x_i)$, for all $i = 1, 2, \ldots, n+2$. Since $g \in F_f$, it is unique in F_f with the property $g(x_i) = f(x_i)$, for all $i = 1, 2, \ldots, n+1$. For this unique g, it is true that $f(z) = g(z)$, for all $z \in A$, because x_{n+2} is arbitrary. That is, $f = g$ on A, and hence f is continuous on (A, τ_1). □

Let us observe that the following Corollary 10.3 is derivable from the previous Theorem 10.2. This corollary is the main result of [46].

Corollary 10.3. *Every complex commutative Fréchet algebra with a countable maximal ideal space is functionally continuous.*

Proof. Let (A, τ_1) be a given complex commutative Fréchet algebra with identity and with a countable maximal ideal space $M(A)$. Let f be a multiplicative linear functional on A. Fix $x_1 \in A$, arbitrarily. Then $\{g \in M(A) : f(x_1) = g(x_1)\}$ is countable, because $M(A)$ is countable. By Theorem 8.11, for any x_2, x_3 in A, $\{g \in M(A) : f(x_i) = g(x_i), i = 1, 2, 3\}$ is nonempty. Take $\tau_2 = \tau_1$ in the previous Theorem 10.2 to conclude that f is continuous on (A, τ_1). □

Remark 10.4. The conditions (ii) and (iii) in Theorem 10.2 can be replaced by the following single condition.

(ii)′ To any given multiplicative linear functional f on A, there is a finite set $\{x_1, x_2, \ldots, x_n\}$ in A such that the set $F_f = \{g : g$ is a continuous multiplicative linear functional on (A, τ_1), $f(x_i) = g(x_i)$, for all $i = 1, 2, \ldots, n\}$ is at most countable and such that for each subset $\{x_{n+1}, x_{n+2}\}$ of A, the set $\{g : g$ is a continuous multiplicative linear functional on (A, τ_1), $f(x_i) = g(x_i)$, for all $i = 1, 2, \ldots, n+2\}$ is nonempty.

11. Techniques for measurable cardinal numbers

A nonempty set X is said to have a measurable cardinal number, if there is a countably additive measure $\mu : \mathbb{P}(X) \longrightarrow \{0,1\}$ on the power set $\mathbb{P}(X)$ of X such that $\mu(X) = 1$ and $\mu(\{x\}) = 0$, for all $x \in X$. According to this definition, if a set X has a measurable cardinal number (or, has measurable cardinality), then it should be an uncountable set. The following Theorem 11.1 was proved by Larotonda and Zalduendo [30].

Theorem 11.1. *Let I be a nonempty set. Then the followings are equivalent.*

(a) *The cardinality of I is nonmeasurable.*
(b) *For every family $(A_i)_{i\in I}$ of algebras, every multiplicative linear functional f of the product $A = \prod_{i\in I} A_i$ factors through some A_k. That is, $f = f_k \circ \pi_k$, where $\pi_k : A \longrightarrow A_k$ is the projection and f_k is a multiplicative linear functional on A_k.*
(c) *Every product $A = \prod_{i\in I} A_i$ of functionally continuous algebras is functionally continuous.*

As usual, the topologies on products are product topologies and algebraic operations on products are coordinate-wise operations. Let us recall the followings, which are already known in connection with measurable cardinals.

Consider a countably additive measure $\mu : \mathbb{P}(X) \to \{0,1\}$ such that $\mu(X) = 1$. Let $M_0 = \{E \subseteq X : \mu(E) = 0\}$ and let $M_1 = \{E \subseteq X : \mu(E) = 1\}$. If $\mu(\{x\}) = 0$, for all $x \in X$, then X has a measurable cardinality, and M_0 and M_1 are subcollections of $\mathbb{P}(X)$ having the following properties.

(1) If $A \subseteq X$, then either $A \in M_0$ or $A \in M_1$, exclusively.
(2) If $A \subseteq X$, then either $A \in M_0$ or $X \setminus A \in M_0$, exclusively.
(3) If $A \subseteq B \subseteq X$, and $B \in M_0$, then $A \in M_0$.
(4) If $x \in X$, then $\{x\} \in M_0$.
(5) If $A_1, A_2, \ldots$ are (pairwise disjoint) members M_0, then $\cup_{n=1}^{\infty} A_n \in M_0$.

On the other hand, if M_0 and M_1 are nonempty subcollections of $\mathbb{P}(X)$ satisfying the conditions (1)-(5),then let us define $\mu : \mathbb{P}(X) \to \{0,1\}$ by $\mu(E) = 0$, for all $E \in M_0$ and $\mu(E) = 1$, for all $E \in M_1$ to obtain a countably additive measure μ such that $\mu(\{x\}) = 0$, for all $x \in X$. Now, let us consider another condition/property.

$(5')$ If $(A_\alpha)_{\alpha\in I}$ is a collection of (pairwise disjoint) members of M_0, and I has no measurable cardinality, then $\cup_{\alpha\in I} A_\alpha \in M_0$.

First, let us assume that M_0 and M_1 satisfy (1)–(5). Consider a collection $(A_\alpha)_{\alpha\in I}$ of pairwise disjoint members of M_0. Let us also assume that $\cup_{\alpha\in I} A_\alpha \in M_1$. Define $M_0{}' = \{J \subseteq I : \cup_{\alpha\in J} A_\alpha \in M_0\}$ and $M_1{}' = \{J \subseteq I : \cup_{\alpha\in J} A_\alpha \in M_1\}$. Then $M_0{}'$ and $M_1{}'$ also satisfy the (corresponding) conditions (1)–(5). This means that I has measurable cardinality. Thus, if (1)–(5) are true, then $(5')$ should also be true. Note that $\mathbb{N}$ has no measurable cardinality so that (5′) implies (5). It is further established that if I has no measurable cardinality, then $\mathbb{P}(I)$ has no measurable cardinality. (*Verification*: Let us begin with a nonzero countably additive measure μ on the collection of all subsets of $\{0,1\}^I$ such that it takes values in $\{0,1\}$ and such that it assumes value 0 for each singleton subset of $\{0,1\}^I$. Consider the corresponding M_0 and M_1. For each $\alpha \in I$, let $B_\alpha = \{(x_\beta)_{\beta\in I} \in \{0,1\}^I : x_\alpha = 0\}$, let $C_\alpha = \{(x_\beta)_{\beta\in I} \in \{0,1\}^I : x_\alpha = 1\}$, and let $D_\alpha = B_\alpha$ or C_α when $\mu(B_\alpha) = 1$ or when $\mu(C_\alpha) = 1$, respectively. Then by (5′), $\cap_{\alpha\in I} D_\alpha \in M_1$. Indeed, $\cap_{\alpha\in I} D_\alpha$ is a singleton subset of $\{0,1\}^I$ so that it is a member of M_0.)

In particular, $\mathbb{R}$ has no measurable cardinality. A generalization may be seen in [6, p. 43]. Moreover, any finite Cartesian product of sets having nonmeasurable cardinality should also have nonmeasurable cardinality. (*Verification*: Suppose that I and J have nonmeasurable cardinality. Let K be one among these two sets for which the cardinality is greater than or equal to the cardinality of the other one. Then the cardinality of $I \times J$ is less than or equal to the cardinality of $\{0,1\}^K$. This proves that $I \times J$ has no measurable cardinality.)

Let us again begin with a countably additive measure $\mu : \mathbb{P}(X) \to \{0,1\}$ such that $\mu(X) = 1$ and such that $\mu(\{x\}) = 0,$ for all $x \in X$, and let us consider the corresponding M_0 and M_1 satisfying (1)–(5). Then (5′) is also true. This means that if $\mu(A_\alpha) = 0,$ for all $\alpha \in I$, and if I has no measurable cardinality, then $\mu(\cup_{\alpha\in I} A_\alpha) = 0$. When the definition of unordered sum is assumed, it is possible to write the relation $\mu(\cup_{\alpha\in I} A_\alpha) = \sum_{\alpha\in I} \mu(A_\alpha)$, whenever I has no measurable cardinality, and $A_\alpha \subseteq X,$ for all $\alpha \in I$.

Let I be a nonempty set. Let $A = \mathbb{R}^I$. For each subset J of I, let $e_J = (x_\alpha)_{\alpha\in I}$, when $x_\alpha = 0$ for $\alpha \notin J$ and $x_\alpha = 1$ for $\alpha \in J$. Let f be a nonzero real multiplicative linear functional on A. Then, by Corollary 8.2, f is sequentially continuous on $\mathbb{R}^I$. Define $\mu_f : \mathbb{P}(I) \longrightarrow \{0,1\}$ by $\mu_f(J) = 0$ if $f(e_J) = 0$, $\mu_f(J) = 1$ if $f(e_J) = 1$, and $\mu_f(\emptyset) = 0$ (in accordance with the relation $f(0) = 0$). Since f is sequentially continuous, μ is a nonzero countably additive measure. If I has no measurable cardinality, then f should factor through some coordinate $\mathbb{R}$, by Theorem 11.1. In the general case, for each $(x_\alpha)_{\alpha\in I}$, there is $J \subseteq I$ such that $\mu_f(J) = 1$ and $x_\alpha = x_\beta$ whenever $\alpha, \beta \in J$. On the other hand, if $\mu : \mathbb{P}(I) \longrightarrow \{0,1\}$ is a countably additive measure such that $\mu(I) = 1$ and $\mu(\{\alpha\}) = 0,$ for all $\alpha \in I$, then for given $(x_\alpha)_{\alpha\in I} \in \mathbb{R}^I$, on defining $f_\mu((x_\alpha)_{\alpha\in I}) = x_\beta,$ for all $\beta \in J$, when $J \subseteq I$ such that $x_\beta = x_\gamma,$ for all $\beta, \gamma \in J$ and such that $\mu(J) = 1$, the functional f_μ becomes a multiplicative linear functional on $\mathbb{R}^I$. (*Verification*: Since $\mathbb{R}$ has no measurable cardinality, for any fixed element $x = (x_\alpha)_{\alpha\in I} \in \mathbb{R}^I$, there is unique $i_x \in \mathbb{R}$ such that $E_x = \{\alpha \in I : x_\alpha = i_x\}$ has μ measure value 1. In this case, $f_\mu(x) = i_x$. Now, it can be verified with the help of the properties (1)–(5) that f_μ is a multiplicative linear functional.) All these things are known (see [30]). Indeed, sequential continuity obtained through Corollary 8.2 simplifies the arguments.

Let us first concentrate on positive linear functionals on $\mathbb{R}^I$. Let us use the notation e_J defined in the previous paragraph.

Lemma 11.2. *Let f be a nonzero real linear functional on $\mathbb{R}^I$ such that $f((x_\alpha)_{\alpha\in I}) \geq 0$, whenever $x_\alpha \geq 0$, for all $\alpha \in I$, and such that $f(e_J) = 0$ or 1, for all $J \subseteq I$. Then f should be a multiplicative linear functional on $\mathbb{R}^I$.*

Proof. By considering the natural extension of f to $\mathbb{C}^I$, by Corollary 8.4, it is concluded that f is sequentially continuous on $\mathbb{R}^I$. Hence $\mu : \mathbb{P}(I) \longrightarrow \{0, 1\}$ defined by $\mu(J) = f(e_J)$ is a countably additive measure. The functional f_μ defined above coincides with f. Then f is a multiplicative linear functional on $\mathbb{R}^I$.
Another proof:
This lengthy proof is provided as an application of the Gleason–Kahane–Źelazko theorem. Let us consider again f as a restriction of a positive linear functional g on $\mathbb{C}^I$. Let $(x_\alpha)_{\alpha\in I} \in \ker g$. It is claimed that $x_\alpha = 0$ for some $\alpha \in I$. Since $(Re\ x_\alpha)_{\alpha\in I} \in \ker g$ and $(Im\ x_\alpha)_{\alpha\in I} \in \ker g$, to establish this claim, let us assume without loss of generality that x_α are real, for all $\alpha \in I$. That is, let us assume that $(x_\alpha)_{\alpha\in I} \in \ker f$. Suppose that the claim is not true for this $(x_\alpha)_{\alpha\in I}$. Let $J = \{\alpha \in I : x_\alpha > 0\}$. Then $I \setminus J = \{\alpha \in I : x_\alpha < 0\}$. Define $(y_\alpha)_{\alpha\in I}$, $({y_\alpha}^{'})_{\alpha\in I}$, $(z_\alpha)_{\alpha\in I}$ and $({z_\alpha}^{'})_{\alpha\in I}$ by

$$y_\alpha = \begin{cases} \sqrt{x_\alpha} & \text{if } \alpha \in J, \\ 0 & \text{if } \alpha \notin J, \end{cases}$$

$${y_\alpha}^{'} = \begin{cases} \frac{1}{y_\alpha} & \text{if } \alpha \in J, \\ 0 & \text{if } \alpha \notin J, \end{cases}$$

$$z_\alpha = \begin{cases} -\sqrt{-x_\alpha} & \text{if } \alpha \notin J, \\ 0 & \text{if } \alpha \in J, \end{cases}$$

$${z_\alpha}^{'} = \begin{cases} \frac{1}{z_\alpha} & \text{if } \alpha \notin J, \\ 0 & \text{if } \alpha \in J. \end{cases}$$

By the Cauchy–Schwarz inequality,

$$0 \leq |f(e_J)|^2 = |f((y_\alpha^{'})_{\alpha\in I} e_J (y_\alpha)_{\alpha\in I})|^2 \leq f(({y_\alpha^{'}}^2)_{\alpha\in I}) f(e_J (x_\alpha)_{\alpha\in I})$$

and

$$0 \leq |f(e_{I\setminus J})|^2 = |f((z_\alpha^{'})_{\alpha\in I} e_{I\setminus J} (z_\alpha)_{\alpha\in I})|^2 \leq f(({z_\alpha^{'}}^2)_{\alpha\in I}) f(-e_{I\setminus J} (x_\alpha)_{\alpha\in I}).$$

So, if $e_J(x_\alpha)_{\alpha\in I} \in \ker f$ then $f(e_J) = 0$, and, if $e_{I\setminus J}(x_\alpha)_{\alpha\in I} \in \ker f$ then $f(e_{I\setminus J}) = 0$. On the other hand, by the Cauchy–Schwarz inequality, if $f(e_J) = 0$, then $e_J(x_\alpha)_{\alpha\in I} \in \ker f$, and, if $f(e_{I\setminus J}) = 0$, then $e_{I\setminus J}(x_\alpha)_{\alpha\in I} \in \ker f$. If $e_J(x_\alpha)_{\alpha\in I} \in \ker f$, then $e_{I\setminus J}(x_\alpha)_{\alpha\in I} \in \ker f$, because $(e_J + e_{I\setminus J})(x_\alpha)_{\alpha\in I} = (x_\alpha)_{\alpha\in I} \in \ker f$. Similarly, if $e_{I\setminus J}(x_\alpha)_{\alpha\in I} \in \ker f$, then $e_J(x_\alpha)_{\alpha\in I} \in \ker f$. Therefore, $f(e_J) = 0$ if and only if $f(e_{I\setminus J}) = 0$. This is a contradiction, because $f(e_I) = 1$. So, $x_\alpha = 0$ for some α. Thus $\ker f \subseteq \ker g \subseteq Sing\mathbb{C}^I$. By Theorem 7.3, g is a multiplicative linear functional on $\mathbb{C}^I$, because it is sequentially continuous in view of Corollary 8.4. So, f should be a multiplicative linear functional on $\mathbb{R}^I$. $\square$

Theorem 11.3. *Let f be a nonzero real positive linear functional on $\mathbb{R}^I$ (That is, $f((x_\alpha)_{\alpha\in I}) \geq 0$, whenever $x_\alpha \geq 0$, for all $\alpha \in I$). Then there are finitely many positive constants $k_1, k_2, \ldots, k_l$ and there are finitely many nonzero distinct multiplicative linear functionals $f_1, f_2, \ldots, f_l$ such that $f = k_1 f_1 + k_2 f_2 + \cdots + k_l f_l$.*

Proof. Suppose that there are infinitely many pairwise disjoint nonempty subsets $B_1, B_2, \ldots$ of I such that $f(e_{B_i}) \neq 0$, for every $i = 1, 2, \ldots$. Since f is a positive linear functional, this is impossible because

$$f(\sum_{i=1}^{\infty} \frac{1}{f(e_{B_i})} e_{B_i}) \geq f(\sum_{i=1}^{n} \frac{1}{f(e_{B_i})} e_{B_i}) = n,$$

for every n. So, there are only finitely many pairwise disjoint nonempty subsets $B_1, B_2, \ldots, B_l$ of I such that $f(e_{B_i}) \neq 0$, for all $i = 1, 2, \ldots, l$, and such that if $B \subseteq B_i$, for some i, then $f(e_B) = 0$ or $f(e_B) = f(e_{B_i})$ so that $f(e_{I\setminus(B_1\cup B_2\cup\cdots\cup B_l)}) = 0$. Define $k_i = f(e_{B_i})$ and

$$f_i((x_\alpha)_{\alpha\in I}) = \frac{1}{f(e_{B_i})} f(e_{B_i}(x_\alpha)_{\alpha\in I}),$$

for all $i = 1, 2, \ldots, l$, and for all $(x_\alpha)_{\alpha\in I} \in \mathbb{R}^I$. By the previous Lemma 11.2, each f_i is a multiplicative linear functional on $\mathbb{R}^I$. Since $f(e_{I\setminus(B_1\cup B_2\cup\cdots\cup B_l)}) = 0$, it follows from the Cauchy–Schwarz inequality

$$|f(e_{I\setminus(B_1\cup B_2\cup\cdots\cup B_l)}(x_\alpha)_{\alpha\in I})|^2 \leq f(e_{I\setminus(B_1\cup B_2\cup\cdots\cup B_l)}) f((x_\alpha^2)_{\alpha\in I})$$

that $f(e_{I\setminus(B_1\cup B_2\cup\cdots\cup B_l)}(x_\alpha)_{\alpha\in I}) = 0$, for all $(x_\alpha)_{\alpha\in I} \in \mathbb{R}^I$. Hence, $f = k_1 f_1 + k_2 f_2 + \cdots + k_l f_l$. □

Theorem 11.4. *Let I be a nonempty set that has no measurable cardinality. Let $(A_\alpha)_{\alpha\in I}$ be a collection of complex algebras with identities and involutions. Let $A = \prod_{\alpha\in I} A_\alpha$ denote the complex algebra with the natural identity and the natural involution. Let f be a nonzero positive linear functional on A. Then $f = k_1 f_1 + k_2 f_2 + \cdots + k_l f_l$, for some $l \geq 1$, for some $k_i > 0$, $i = 1, 2, \ldots, l$, and for some nonzero positive linear functionals f_i, $i = 1, 2, \ldots, l$, which factor through coordinates, and which satisfy $f_i(e_I) = 1$, for all $i = 1, 2, \ldots, l$.*

Proof. Consider $\mathbb{R}^I$ as a subalgebra of $\prod_{\alpha\in I} A_\alpha$ in terms of identities, and consider the restriction g of f to $\mathbb{R}^I$. Then $g = k_1 g_1 + k_2 g_2 + \cdots + k_l g_l$, for some $l \geq 1$, for some $k_i > 0$, $i = 1, 2, \ldots, l$, and for some distinct nonzero multiplicative linear functionals $g_1, g_2, \ldots, g_l$ on $\mathbb{R}^I$, by Theorem 11.3. Since I has no measurable cardinality, by "(a) implies (b)" part of Theorem 11.1, there is a finite subset $\{\alpha_1, \alpha_2, \ldots, \alpha_l\}$ of I having distinct elements such that $g_i(e_{\{\alpha_j\}}) = 0$ for $i \neq j$, $g_i(e_{\{\alpha_j\}}) = 1$ for $i = j$. Note that

$$\begin{aligned} 0 \leq |f((x_\alpha)_{\alpha\in I} e_{I\setminus\{\alpha_1,\alpha_2,\ldots,\alpha_l\}})|^2 &\leq |f(e_{I\setminus\{\alpha_1,\alpha_2,\ldots,\alpha_l\}})||f((x_\alpha x_\alpha^*)_{\alpha\in I})| \\ &= 0 \times |f((x_\alpha {x_\alpha}^*)_{\alpha\in I})| = 0, \end{aligned}$$

for all $(x_\alpha)_{\alpha\in I} \in A$. So, $f((x_\alpha)_{\alpha\in I}) = f((y_\alpha)_{\alpha\in I})$, where $y_\alpha = 0$, for all $\alpha \in I \setminus \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$ and $y_\alpha = x_\alpha$, for all $\alpha \in \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$, for all $(x_\alpha)_{\alpha\in I} \in A$.

For each $i = 1, 2, \ldots, l$, define $f_i((x_\alpha)_{\alpha\in I}) = f((y_\alpha)_{\alpha\in I})$ with $y_\alpha = \frac{1}{k_i}x_\alpha$ for $\alpha = \alpha_i$, and $y_\alpha = 0$ for $\alpha \neq \alpha_i$, for all i. Then

$$f((x_\alpha)_{\alpha\in I}) = (k_1f_1 + k_2f_2 + \cdots + k_lf_l)((x_\alpha)_{\alpha\in I}), \text{ for all } (x_\alpha)_{\alpha\in I} \in A.$$

This completes the proof. □

Definition 11.5. Let $(A_\alpha)_{\alpha\in I}$ be a collection of vector spaces over the field $\mathbb{R}$ or $\mathbb{C}$. Let $A = \prod_{\alpha\in I} A_\alpha$ be the product of vector spaces. Two elements $(x_\alpha)_{\alpha\in I}$ and $(y_\alpha)_{\alpha\in I}$ in A are said to be disjoint, if $y_\beta = 0$ whenever $x_\beta \neq 0$, and $x_\gamma = 0$ whenever $y_\gamma \neq 0$. A linear functional f on A is said to be sequentially disjointness preserving, if for every sequence $((x_{i\alpha})_{\alpha\in I})_{i=1}^\infty$ of pairwise disjoint elements in A, it is true that $f(\sum_{i=1}^\infty (x_{i\alpha})_{\alpha\in I}) = \sum_{i=1}^\infty f((x_{i\alpha})_{\alpha\in I}) = f((x_{j\alpha})_{\alpha\in I})$, for some j.

Theorem 11.6. *Let I be a nonempty set. Let $A = \prod_{\alpha\in I} A_\alpha$ when each $A_\alpha = B$ is a topological vector space. That is, $A = B^I$. Let $B_e = \{(x_\alpha)_{\alpha\in I} \in A : x_\alpha = x_\beta,$ for all $\alpha, \beta \in I\}$, and suppose that B has no measurable cardinality. Let D be a directed set that has no measurable cardinality. Let f be a linear functional on A such that $(f(z_\delta))_{\delta\in D}$ converges to zero whenever $(z_\delta)_{\delta\in D}$ is a net in B_e, which converges to zero. Let $(x_\delta)_{\delta\in D}$ be a net in A converging to zero. Then $(f(x_\delta))_{\delta\in D}$ converges to zero, if f is sequentially disjointness preserving on A.*

Proof. Suppose that f is sequentially disjointness preserving on A. For each $x_\delta = (x_{\delta\alpha})_{\alpha\in I}$, there is a nonempty subset I_δ of I such that $x_{\delta\alpha} = x_{\delta\beta}$, for all $\alpha, \beta \in I_\delta$ and such that $f(x_\delta) = f(z_\delta)$, when $z_\delta = (z_{\delta\alpha})_{\alpha\in I}$ is defined by $z_{\delta\alpha} = x_{\delta\alpha}$ for $\alpha \in I_\delta$ and $z_{\delta\alpha} = 0$ for $\alpha \notin I_\delta$, because f is sequentially disjointness preserving, and B has no measurable cardinality. (*Verification*: If $f((x_{\delta\alpha})_{\alpha\in I}) = 0$, then I_δ can be any nonempty subset of I satisfying the condition $x_{\delta\alpha} = x_{\delta\beta}$, for all $\alpha, \beta \in I_\delta$. Suppose $f((x_{\delta\alpha})_{\alpha\in I}) \neq 0$. Define $\mu_\delta : \mathbb{P}(I) \to \{0, 1\}$ by $\mu_\delta(J) = 0$ when $f((y_\alpha)_{\alpha\in I}) = 0$, $\mu_\delta(J) = 1$ when $f((y_\alpha)_{\alpha\in I}) = f((x_{\delta\alpha})_{\alpha\in I})$, and when $y_\alpha = x_{\delta\alpha}$, for all $\alpha \in J$, and $y_\alpha = 0$, for all $\alpha \in I \setminus J$. Since f is sequentially disjointness preserving, μ_δ is a countably additive $\{0, 1\}$-measure on I. For each $x \in B$, let $I_x = \{\beta \in I : x_{\delta\beta} = x\}$. Then the set $\{I_x : x \in B\}$ has no measurable cardinality, because B has no measurable cardinality. Then there is unique $x \in B$ such that $\mu_\delta(I_x) = 1$, by $(5')$ given after Theorem 11.1. Let us take I_δ as this I_x.)

Since f is sequentially disjointness preserving and D has no measurable cardinality, the intersection of all I_δ for which $f(x_\delta) \neq 0$ is nonempty. (*Verification*: Define $\mu_D : \mathbb{P}(I) \to \{0, 1\}$ by $\mu(J) = 1$ if $f((y_\alpha)_{\alpha\in I}) \neq 0$, for some $(y_\alpha)_{\alpha\in I} \in A$ for which $y_\alpha = 0$, for all $\alpha \in I \setminus J$; and $\mu(J) = 0$, otherwise. Since f is sequentially disjointness preserving mapping, μ_D is a countably additive $\{0, 1\}$-measure on I. Then I_δ is a member of the corresponding M_1 whenever $f(x_\delta) \neq 0$. Then the intersection of all such I_δ should also be a member of M_1, by $(5')$ given after Theorem 11.1). So, without loss of generality, let us assume that all these I_δ are equal, when they are considered as the intersection of all I_δ for which $f(x_\delta) \neq 0$.

For each $\delta \in D$, when $f(x_\delta) \neq 0$, define $y_\delta = (y_{\delta\alpha})_{\alpha\in I}$ in A by $y_{\delta\alpha} = x_{\delta\alpha}$ if $\alpha \in I_\delta$ and $y_{\delta\alpha} = x_{\delta\beta}$ with $\beta \in I_\delta$ if $\alpha \notin I_\delta$. When $f(x_\delta) = 0$, then let us define $y_\delta = 0$. Then $f(y_\delta) = f(x_\delta)$, for all $\delta \in D$. Since $y_\delta \in B_e$, for all $\delta \in D$ and

$(y_\delta)_{\delta\in D}$ converges to zero, then $(f(x_\delta))_{\delta\in D}$ converges to zero, because $(f(y_\delta))_{\delta\in D}$ converges to zero. $\square$

Remark 11.7. Observe that B_e given in the previous Theorem 11.6 is topologically homeomorphic and linearly isomorphic with B. This remark can be applied to the following Corollary 11.8, along with Theorem 11.3.

Corollary 11.8. *If f is a real positive linear functional on $\mathbb{R}^I$ (In particular, if f is a real multiplicative linear functional on $\mathbb{R}^I$), for some nonempty set I, then $(f(x_\delta))_{\delta\in D}$ converges to zero, whenever $(x_\delta)_{\delta\in D}$ converges to zero and D has no measurable cardinality.*

Theorem 11.9. *Let I be a nonempty set. Let $((A_\alpha,(p_{\alpha i})_{i=1}^{\infty}))_{\alpha\in I}$ be a collection of nonzero commutative Fréchet algebras in which each $p_{\alpha i}$ is a sub-multiplicative seminorm. Let $A=\prod_{\alpha\in I}A_\alpha$. Let f be a nonzero real multiplicative linear functional on A. Let $(x_\delta)_{\delta\in D}$ be a bounded net converging to zero in A, when D has no measurable cardinality. Then $(f(x_\delta))_{\delta\in D}$ converges to zero.*

Proof. Since A is a complete LMC algebra, the real multiplicative linear functional f is sequentially continuous on A, by Corollary 8.2. For each $J\subseteq I$, let us use the notation $\prod_{\alpha\in J}A_\alpha$ even for the subalgebra $\{(x_\alpha)_{\alpha\in I}\in A : x_\alpha=0,\ \text{for all } \alpha\in I\setminus J\}$ of A, when $J=\emptyset$, it is the zero subalgebra. Define $\mu_f:\mathbb{P}(I)\to\{0,1\}$ by $\mu_f(J)=0$ when $f(\prod_{\alpha\in J}A_\alpha)=\{0\}$, and $\mu_f(J)=1$ when $f(\prod_{\alpha\in J}A_\alpha)=\mathbb{R}$. The sequential continuity of f on A implies that μ_f is a countably additive measure on I such that $\mu_f(I)=1$, because f is nonzero on A. (*Verification*: Let us observe that $\mu_f(I\setminus J)=0$ whenever $\mu_f(J)=1$, and that $\mu_f(I\setminus J)=1$ whenever $\mu_f(J)=0$. Let $J_1,J_2,\dots$ be a sequence of pairwise disjoint subsets of I for which $\mu(J_n)=0$, for all n. Let $J=\cup_{n=1}^{\infty}J_n$. Consider a general element $(y_\alpha)_{\alpha\in J}$ that may be considered in the form $\sum_{n=1}^{\infty}(y_\alpha)_{\alpha\in J_n}$ in terms of the symbols for Cartesian products. Then $f((y_\alpha)_{\alpha\in J})=\sum_{n=1}^{\infty}f((y_\alpha)_{\alpha\in J_n})=0$, by the sequential continuity of f. This proves that $\mu_f(J)=0$. This proves that μ_f is countably additive.)

For each $\delta\in D$, let $x_\delta=(x_{\delta\alpha})_{\alpha\in I}$. For each fixed $\alpha\in I$, and $i=1,2,3,\dots,$ let us define $M_{\alpha i}=\sup\{p_{\alpha i}(x_{\delta\alpha}) : \delta\in D\}$. For every sequence $(M_{\alpha i})_{i=1}^{\infty}$, let $I_\alpha=\{\beta\in I : (M_{\beta i})_{i=1}^{\infty}=(M_{\alpha i})_{i=1}^{\infty}\}$. Then the set $\{I_\beta:\beta\in I\}$ has no measurable cardinality, because the cardinality of this set is less than or equal to the cardinality of $\mathbb{R}^{\mathbb{N}}$, and $\mathbb{R}^{\mathbb{N}}$ has no measurable cardinality. Then there is at most one I_α such that $\mu_f(I_\alpha)=1$, by(5′) given after Theorem 11.1. By considering the restriction of f to $\prod_{\beta\in I_\alpha}A_\beta$, let us assume without loss of generality that $(M_{\beta i})_{i=1}^{\infty}$ are equal for all $\beta\in I$, in view of Theorem 11.1 and Corollary 4.3.

For each $\alpha\in I$, let us consider $((p_{\alpha i}(x_{\delta\alpha}))_{\delta\in D})_{i=1}^{\infty}$ and let us define $I_\alpha=\{\beta\in I : ((p_{\beta i}(x_{\delta\beta}))_{\delta\in D})_{i=1}^{\infty}=((p_{\alpha i}(x_{\delta\alpha}))_{\delta\in D})_{i=1}^{\infty}\}$. Since D has no measurable cardinality, the set $\{I_\beta:\beta\in I\}$ has no measurable cardinality, because the cardinality of the latter set is less than or equal to the cardinality of $(\mathbb{R})^{(D^{\mathbb{N}})}$. Then there is at most one I_α such that $\mu_f(I_\alpha)=1$. By considering the restriction of f to $\prod_{\beta\in I_\alpha}A_\beta$, let us assume without loss of generality that $((p_{\beta i}(x_{\delta\beta}))_{\delta\in D})_{i=1}^{\infty}$ are equal for all $\beta\in I$.

For every $i = 1, 2, 3, \ldots$, let us define $p_i(x) = \sup_{\alpha \in I} p_{\alpha i}(x_\alpha)$, for all $x = (x_\alpha)_{\alpha \in D} \in A$. Let us now define $Z = \{x \in A : p_i(x) < \infty, \text{ for all } i = 1, 2, 3, \ldots\}$. Then $(Z, (p_i)_{i=1}^{\infty})$ is a commutative Fréchet algebra, f restricted to this algebra is continuous (by Corollary 4.3), and $(x_\delta)_{\delta \in D}$ is a net converging to zero in this algebra. So, $(f(x_\delta))_{\delta \in D}$ converges to zero. $\square$

Theorem 11.10. *Let I be a nonempty set. Let $((A_\alpha, (p_{\alpha i})_{i=1}^{\infty}))_{\alpha \in I}$ be a collection of nonzero complex commutative Fréchet algebras with continuous involutions in which each $p_{\alpha i}$ is a sub-multiplicative seminorm. Let $A = \prod_{\alpha \in I} A_\alpha$ with the natural involution defined by the involutions of the algebras A_α. Let f be a nonzero complex multiplicative linear functional on A such that $\overline{f(x)} = f(x^*)$, for all $x \in A$. Let $(x_\delta)_{\delta \in D}$ be a bounded net converging to zero in A, when D has no measurable cardinality. Then $(f(x_\delta))_{\delta \in D}$ converges to zero.*

Proof. By the previous Theorem 11.9, and by considering the restriction of f to $\{x \in A : x = x^*\}$, it can be concluded that $(f(y_\delta))_{\delta \in D}$ converges to zero, and $(f(z_\delta))_{\delta \in D}$ converges to zero, when $y_\delta = \frac{x_\delta + x_\delta^*}{2}$ and $z_\delta = \frac{x_\delta - x_\delta^*}{2i}$. Hence $(f(x_\delta))_{\delta \in D}$ converges to zero. $\square$

Theorem 11.11. *Let I be a nonempty set. For each $\alpha \in I$, let $((A_\alpha, (p_{\alpha i})_{i=1}^{\infty}))$ be a B_0 algebra with an identity e_α in which $(p_{\alpha i})_{i=1}^{\infty}$ is a family of seminorms such that $p_{\alpha i}(xy) \leq p_{\alpha(i+1)}(x) p_{\alpha(i+1)}(y)$ and $p_{\alpha i}(x) \leq p_{\alpha(i+1)}(x)$, for all $x, y \in A_\alpha$ and for all $i = 1, 2, 3, \ldots$ Let $A = \prod_{\alpha \in I} A_\alpha$. Let f be a nonzero real multiplicative linear functional on A. Let $(x_\delta)_{\delta \in D}$ be a bounded net converging to zero in A, when D has no measurable cardinality. Then $(f(x_\delta))_{\delta \in D}$ converges to zero.*

Proof. Let us consider $\mathbb{R}^I$ as a subalgebra of A through the identities e_α. Since this topological subalgebra is the usual complete LMC algebra, f restricted to this subalgebra is sequentially continuous, by Corollary 8.2. Define $\mu_f : \mathbb{P}(I) \to \{0, 1\}$ by $\mu_f(J) = 0$ when $f(e_J) = \{0\}$, and $\mu_f(J) = 1$ when $f(e_J) = 1$. The sequential continuity of f on $\mathbb{R}^I$ implies that μ_f is a countably additive measure on I such that $\mu_f(I) = 1$, because f is nonzero on A. (*Verification*: Let us observe that $\mu_f(I \setminus J) = 0$ whenever $\mu_f(J) = 1$, and that $\mu_f(I \setminus J) = 1$ whenever $\mu_f(J) = 0$. Let $J_1, J_2, \ldots$ be a sequence of pairwise disjoint subsets of I for which $\mu(J_n) = 0$, for all n. Let $J = \cup_{n=1}^{\infty} J_n$. Then $f(e_J) = \sum_{n=1}^{\infty} f(e_{J_n}) = 0$, because $e_J = \sum_{n=1}^{\infty} e_{J_n}$ and because of the sequential continuity of f on $\mathbb{R}^I$. This proves that $\mu_f(J) = 0$. This proves that μ_f is countably additive.)

For each $\delta \in D$, let $x_\delta = (x_{\delta\alpha})_{\alpha \in I}$. For each fixed $\alpha \in I$, and $i = 1, 2, 3, \ldots$, let us define $M_{\alpha i} = \sup\{p_{\alpha i}(x_{\delta\alpha}) : \delta \in D\}$. Then it is possible to assume without loss of generality that $(M_{\beta i})_{i=1}^{\infty}$ are equal for all $\beta \in I$. It is also possible to assume without loss of generality that $((p_{\beta i}(x_{\delta\beta}))_{\delta \in D})_{i=1}^{\infty}$ are equal for all $\beta \in I$.

For every $i = 1, 2, 3, \ldots$, let us define $p_i(x) = \sup_{\alpha \in I} p_{\alpha i}(x_\alpha)$, for all $x = (x_\alpha)_{\alpha \in D} \in A$. Let us now define $Z = \{x \in A : p_i(x) < \infty, \text{ for all } i = 1, 2, 3, \ldots\}$. Then $(Z, (p_i)_{i=1}^{\infty})$ is a B_0 algebra such that $p_i(x) \leq p_{i+1}(x)$ and $p_i(xy) \leq p_{i+1}(x) p_{i+1}(y)$, for every $i = 1, 2, \ldots$, and for all $x, y \in Z$. Then f restricted to $(Z, (p_i)_{i=1}^{\infty})$ is continuous(by Corollary 4.3), and $(x_\delta)_{\delta \in D}$ is a net converging to zero in this algebra. So, $(f(x_\delta))_{\delta \in D}$ converges to zero. $\square$

Theorem 11.12. *Let I be a nonempty set. For each $\alpha \in I$, let $((A_\alpha, (p_{\alpha i})_{i=1}^{\infty}))$ be a B_0 algebra with an identity e_α, and with a continuous involution, in which $(p_{\alpha i})_{i=1}^{\infty}$ is a family of seminorms such that $p_{\alpha i}(xy) \leq p_{\alpha(i+1)}(x)p_{\alpha(i+1)}(y)$ and $p_{\alpha i}(x) \leq p_{\alpha(i+1)}(x)$, for all $x, y \in A_\alpha$ and for all $i = 1, 2, 3, \ldots$ Let $A = \prod_{\alpha \in I} A_\alpha$ with the natural involution defined by the involutions of the algebras A_α. Let f be a nonzero complex multiplicative linear functional on A such that $\overline{f(x)} = f(x^*)$, for all $x \in A$. Let $(x_\delta)_{\delta \in D}$ be a bounded net converging to zero in A, when D has no measurable cardinality. Then $(f(x_\delta))_{\delta \in D}$ converges to zero.*

Proof. It is similar to that of the proof of Theorem 11.10, but by using Theorem 11.11. □

Theorem 11.13. *Let I be a nonempty set. Let $((A_\alpha, ||\cdot||_\alpha))_{\alpha \in I}$ be a collection of nonzero complex Banach algebras with identity elements $e_\alpha, \alpha \in I$. Let $A = \prod_{\alpha \in I} A_\alpha$. Let f be a nonzero complex multiplicative linear functional on A. Let $(x_\delta)_{\delta \in D}$ be a bounded net converging to zero in A, when D has no measurable cardinality. Then $(f(x_\delta))_{\delta \in D}$ converges to zero.*

Proof. Consider $\mathbb{C}^I$ as a complex subalgebra of A through identity elements and consider $\mathbb{R}^I$ as its real subalgebra. Then let us define a countably additive $\{0, 1\}$-measure on I, as it was done in the proof of Theorem 11.11.

For each $\delta \in D$, let $x_\delta = (x_{\delta\alpha})_{\alpha \in I}$. For each fixed $\alpha \in I$, let us define $M_\alpha = \sup\{||x_{\delta\alpha}||_\alpha : \delta \in D\}$. Then it is possible to assume without loss of generality that M_β are equal for all $\beta \in I$. It is also possible to assume without loss of generality that $(||x_{\delta\beta}||_\beta)_{\delta \in D}$ are equal for all $\beta \in I$.

For every $x = (x_\alpha)_{\alpha \in I} \in A$, let us define $||x|| = \sup_{\alpha \in I} ||x_\alpha||_\alpha$. Let us now define $Z = \{x \in A : ||x|| < \infty$, for all $i = 1, 2, 3, \ldots\}$. Then $(Z, ||\cdot||)$ is a complex Banach algebra, f restricted to this algebra is continuous, and $(x_\delta)_{\delta \in D}$ is a net converging to zero in this algebra. So, $(f(x_\delta))_{\delta \in D}$ converges to zero. □

12. CONCLUSION

One used to consider the continuity of certain linear functions in the error analysis of numerical methods. Once continuity happens automatically, then there would be no need to verify continuity in such cases. In this way, one used to find practical applications for automatic continuity. Every technique is important to solve very old problems in automatic continuity, as well as to get applications in numerical analysis.

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