Khayyam J. Math. 9 (2023), no. 1, 30-37
DOI: 10.22034/KJM.2022.367135.2696

Khayyam Journal of Mathematics
emis.de/journals/KJM
kjm-math.org

ANTI-DERIVATIONS ON TRIANGULAR RINGS

HOGER GHAHRAMANI[1*], MOHAMMAD N. GHOSSEIRI[1] AND TAHEREH REZAEI[1]

Communicated by H.R. Ebrahimi Vishki

Abstract. The aim of this paper is to give necessary and sufficient conditions for anti-derivations to be zero on 2-torsion free triangular rings. As an application of our main result, we present sufficient conditions for anti-derivations to be zero on block upper triangular matrix rings.

1. Introduction

Let R be an associative ring. An additive mapping $\delta : R \to R$ is said to be a *derivation* if $\delta(st) = s\delta(t) + \delta(s)t$ for all $s, t \in R$. Also, δ is called an *anti-derivation* if $\delta(st) = \delta(t)s + t\delta(s)$ for all $s, t \in R$. For any fixed element $x \in R$, the mapping $I_x : R \to R$ given by $I_x(s) = xs - sx$ is a derivation, which is said to be an *inner derivation*. A generalization of derivation and anti-derivation is a Jordan derivation, which is defined as follows: An additive mapping $\delta : R \to R$ defined by $\delta(st + ts) = s\delta(t) + \delta(s)t + t\delta(s) + \delta(t)s$ for all $s, t \in R$ is called *Jordan derivation*. Let R be a 2-torsion free ring; then an additive mapping $\delta : R \to R$ is a Jordan derivation if and only if $\delta(s^2) = s\delta(s) + \delta(s)s$ for all $s \in R$. Obviously, any derivation and anti-derivation is a Jordan derivation, but the converse is, in general, not true (see [2]). It is natural to ask whether all Jordan derivations are derivations? In 1975, Herstein [8] proved that every Jordan derivation from a 2-torsion free prime ring into itself is a derivation and that there is no nonzero anti-derivation on a 2-torsion free prime ring. Herstein's conclusion about Jordan derivations has been extended to different rings and algebras in various directions. Brešar [3] showed that every Jordan derivation from a 2-torsion free semiprime ring into itself is a derivation. Zhang and Yu [11]

Date: Received: 23 July 2022; Accepted: 18 September 2022.
*Corresponding Author.
2020 *Mathematics Subject Classification.* Primary 16W25; Secondary 16S99.
Key words and phrases. anti-derivation, triangular ring, block upper triangular matrix ring.

proved that every linear Jordan derivation of triangular algebras is a derivation, which is true for Jordan derivations on triangular rings [5]. Triangular rings are a class of non-semiprime rings. Many authors have studied Jordan derivations on different rings and algebras. We refer the reader to [2, 6, 10] and references therein. The question "when every anti-derivation is zero?" has received less attention. Benkovič [2] obtained some results about anti-derivation on upper triangular matrix algebras. Also, in [1], some results about continuous linear anti-derivations on C^*-algebras have been achieved. In this paper, we obtain necessary and sufficient conditions for anti-derivation to be zero on triangular rings, and then as an application of it, we acquire sufficient conditions for anti-derivations to be zero on block upper triangular matrix rings.

This paper is organized as follows. In Section 2, some necessary definitions and preliminaries are provided. In Section 3, we present the main result. The last section is devoted to the characterization of anti-derivations on block upper triangular matrix rings.

2. Preliminaries

Recall that a *Triangular ring* $Tri(A, M, B)$ is a ring of the form

$$Tri(A, M, B) = \left\{ \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} : a \in A, m \in M, b \in B \right\}$$

under the usual matrix operations, where A and B are unital rings and M is a unital (A, B)-bimodule, which is faithful as a left A-module and also as a right B-module. Basic examples of triangular rings are upper triangular matrix rings, block upper triangular matrix rings, and nest algebras. There have been considerable studies on triangular rings (see [5] and references therein).

Throughout this paper, we assume that A and B are 2-torsion free rings with identities 1_A and 1_B, respectively, and that M is a 2-torsion free unital (A, B)-bimodule, which is faithful as a left A-module and also as a right B-module. Then the triangular ring $Tri(A, M, B)$ is a 2-torsion free ring with identity $\mathbf{I} = \begin{bmatrix} 1_A & 0 \\ 0 & 1_B \end{bmatrix}$.

As already noted at Introduction, Zhang and Yu [11] showed that every linear Jordan derivation on triangular algebras is a derivation, which is true for Jordan derivations on triangular rings (see [5]). We state this result in the next theorem, which is used to prove the main result of this paper.

Theorem 2.1. *Every Jordan derivation on a triangular ring $Tri(A, M, B)$ is a derivation.*

In [7, Theorem 3.2] (also, see [4, Proposition 2.6]), the structure of a derivation on a triangular ring has been characterized as follows.

Theorem 2.2. *Let $\delta : Tri(A, M, B) \to Tri(A, M, B)$ be a derivation. Then $\delta = \overline{\delta} + I_{\mathfrak{N}}$, where $I_{\mathfrak{N}}$ is an inner derivation, for some $\mathfrak{N} = \begin{bmatrix} 0 & n \\ 0 & 0 \end{bmatrix}$ $(n \in M)$, and*

$\overline{\delta}$ is a derivation on $Tri(A, M, B)$, which is defined by

$$\overline{\delta}\left(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} \alpha(a) & \tau(m) \\ 0 & \beta(b) \end{bmatrix},$$

where $\alpha : A \to A$ and $\beta : B \to B$ are derivations and $\tau : M \to M$ is an additive mapping that satisfies

$$\tau(am) = a\tau(m) + \alpha(a)m$$

and

$$\tau(mb) = \tau(m)b + m\beta(b)$$

for all $a \in A$, $m \in M$, and $b \in B$.

Finally, we note that the *Lie brackets* on a ring R is defined by $[s, t] = st - ts$ for $s, t \in R$, and $[R, R] = \{[s, t] : s, t \in R\}$. The subring of R generated by $[R, R]$, is denoted by $\langle [R, R] \rangle$.

3. **Main results**

In this note, our main result is the following theorem.

Theorem 3.1. *Let $\mathcal{T} = Tri(A, M, B)$ be the triangular ring. Then the following statement are equivalent:*

(i) *Every anti-derivation on $\mathcal{T}$ is zero;*
(ii) *$\mathcal{T}$ satisfies*

$$m \in M,\ [A, A]m = m[B, B] = 0 \Longrightarrow m = 0.$$

Proof. (ii) $\Rightarrow$ (i): Let $\delta : \mathcal{T} \to \mathcal{T}$ be an anti-derivation; then δ is a Jordan derivation. By Theorem 2.1, δ is a derivation. In view of Theorem 2.2, δ is of the form

$$\delta\left(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} \alpha(a) & \tau(m) \\ 0 & \beta(b) \end{bmatrix} + I_{\mathfrak{N}}\left(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\right), \tag{3.1}$$

where $\mathfrak{N} \in \mathcal{T}$, $\mathfrak{N} = \begin{bmatrix} 0 & n \\ 0 & 0 \end{bmatrix}$ for some $n \in M$, and α, β, and τ satisfy the conditions of Theorem 2.2.

Since δ is a derivation and also an anti-derivation, for all $\mathfrak{T}, \mathfrak{S} \in \mathcal{T}$, we have

$$\delta(\mathfrak{T}\mathfrak{S}) = \mathfrak{T}\delta(\mathfrak{S}) + \delta(\mathfrak{T})\mathfrak{S} = \delta(\mathfrak{S}\mathfrak{T}).$$

So

$$\delta([\mathfrak{T}, \mathfrak{S}]) = 0 \tag{3.2}$$

for all $\mathfrak{T}, \mathfrak{S} \in \mathcal{T}$. Let $a \in A$, $m \in M$. Set

$$\mathfrak{T} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \qquad \mathfrak{S} = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}.$$

Then

$$[\mathfrak{T}, \mathfrak{S}] = \mathfrak{T}\mathfrak{S} = \begin{bmatrix} 0 & am \\ 0 & 0 \end{bmatrix}.$$

According to (3.1) and (3.2), we get

$$0=\delta([\mathfrak{T},\mathfrak{S}])=\delta\left(\begin{bmatrix}0 & am\\ 0 & 0\end{bmatrix}\right)=\begin{bmatrix}0 & \tau(am)\\ 0 & 0\end{bmatrix}+I_{\mathfrak{N}}\left(\begin{bmatrix}0 & am\\ 0 & 0\end{bmatrix}\right)=\begin{bmatrix}0 & \tau(am)\\ 0 & 0\end{bmatrix}.$$

So $\tau(am)=0$ for all $a\in A$ and $m\in M$. Let $a=1_A$; then $\tau(m)=0$ for all $m\in M$. By (3.1), we have

$$\tau(am)=a\tau(m)+\alpha(a)m$$

for all $a\in A$ and $m\in M$. Thus

$$\alpha(a)m=0$$

for all $a\in A$ and $m\in M$. Since M is faithful as a left A-module, we conclude that $\alpha(a)=0$ for all $a\in A$.

For all $b\in B$ and $m\in M$, we have

$$\tau(mb)=\tau(m)b+m\beta(b),$$

and since $\tau=0$, then $m\beta(b)=0$. The faithfulness of the right B-module M now implies $\beta(b)$=0 for all $b\in B$. Therefore, δ is of the form

$$\delta\left(\begin{bmatrix}a & m\\ 0 & b\end{bmatrix}\right)=I_{\mathfrak{N}}\left(\begin{bmatrix}a & m\\ 0 & b\end{bmatrix}\right), \tag{3.3}$$

where $\mathfrak{N}=\begin{bmatrix}0 & n\\ 0 & 0\end{bmatrix}$.

For all $a_1,a_2\in A$, put

$$\mathfrak{A}_1=\begin{bmatrix}a_1 & 0\\ 0 & 0\end{bmatrix},\qquad \mathfrak{A}_2=\begin{bmatrix}a_2 & 0\\ 0 & 0\end{bmatrix},$$

so

$$[\mathfrak{A}_1,\mathfrak{A}_2]=\begin{bmatrix}[a_1,a_2] & 0\\ 0 & 0\end{bmatrix}.$$

From (3.2) and (3.3), we find

$$\begin{aligned}0=\delta([\mathfrak{A}_1,\mathfrak{A}_2])&=[\mathfrak{A}_1,\mathfrak{A}_2]\mathfrak{N}-\mathfrak{N}[\mathfrak{A}_1,\mathfrak{A}_2]\\ &=\begin{bmatrix}0 & [a_1,a_2]n\\ 0 & 0\end{bmatrix}.\end{aligned}$$

Since $a_1,a_2\in\mathfrak{A}$ are arbitrary, then $[A,A]n=0$.

For all $b_1,b_2\in B$, put $\mathfrak{B}_1=\begin{bmatrix}0 & 0\\ 0 & b_1\end{bmatrix}$, $\mathfrak{B}_2=\begin{bmatrix}0 & 0\\ 0 & b_2\end{bmatrix}$. By using a similar argument as above, we can show that $n[B,B]=0$. According to the assumption (ii), $n=0$, and so $\delta=0$.

(i) $\Rightarrow$ (ii): Let $m\in M$ such that

$$[A,A]m=m[B,B]=0.$$

Set $\mathfrak{W}=\begin{bmatrix}0 & m\\ 0 & 0\end{bmatrix}$, and define the additive mapping $\delta:\mathcal{T}\to\mathcal{T}$ by

$$\delta(\mathfrak{T})=I_{\mathfrak{W}}(\mathfrak{T})$$

for all $\mathfrak{T} \in \mathcal{T}$. Hence, δ is an inner derivation, and for all

$$\begin{bmatrix} a_1 & m_1 \\ 0 & b_1 \end{bmatrix}, \begin{bmatrix} a_2 & m_2 \\ 0 & b_2 \end{bmatrix} \in \mathcal{T},$$

we have

$$\begin{aligned} \delta\Big(\begin{bmatrix} a_1 & m_1 \\ 0 & b_1 \end{bmatrix} \begin{bmatrix} a_2 & m_2 \\ 0 & b_2 \end{bmatrix} \Big) &= \delta\Big(\begin{bmatrix} a_1a_2 & * \\ 0 & b_1b_2 \end{bmatrix} \Big) \\ &= \begin{bmatrix} a_1a_2 & * \\ 0 & b_1b_2 \end{bmatrix} \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_1a_2 & * \\ 0 & b_1b_2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & a_1a_2m - mb_1b_2 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{3.4}$$

On the other hand, in view of the condition $[A, A]m = m[B, B] = 0$, we have

$$\begin{aligned} &\begin{bmatrix} a_2 & m_2 \\ 0 & b_2 \end{bmatrix} \delta\left(\begin{bmatrix} a_1 & m_1 \\ 0 & b_1 \end{bmatrix} \right) + \delta\left(\begin{bmatrix} a_2 & m_2 \\ 0 & b_2 \end{bmatrix} \right) \begin{bmatrix} a_1 & m_1 \\ 0 & b_1 \end{bmatrix} \\ &= \begin{bmatrix} a_2 & m_2 \\ 0 & b_2 \end{bmatrix} \begin{bmatrix} 0 & a_1m - mb_1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & a_2m - mb_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 & m_1 \\ 0 & b_1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & a_2a_1m - a_2mb_1 + a_2mb_1 - mb_2b_1 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & a_1a_2m - mb_1b_2 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{3.5}$$

Comparing (3.4) and (3.5), we find that δ is an anti-derivation. According to the hypothesis, $\delta = 0$. Hence $\mathfrak{W}\mathfrak{T} = \mathfrak{T}\mathfrak{W}$ for all $\mathfrak{T} \in \mathcal{T}$, so $am = mb$ for all $a \in A$, $m \in M$, and $b \in B$. If we set $a = 1_A$ and $b = 0$, then we conclude $m = 0$, which completes the proof. □

In the next corollary, we will give sufficient conditions for anti-derivation to be zero on triangular rings.

Corollary 3.2. *Let $\mathcal{T} = Tri(A, M, B)$ be the triangular ring such that $\langle [A, A] \rangle = A$ or $\langle [B, B] \rangle = B$. Then every anti-derivation on $\mathcal{T}$ is zero.*

Proof. Let $m \in M$ such that

$$[A, A]m = m[B, B] = 0.$$

Then

$$\langle [A, A] \rangle m = m \langle [B, B] \rangle = 0.$$

If $\langle [A, A] \rangle = A$, then $Am = 0$, and since A is a ring with identity and M is a unital module, we have $m = 0$. According to Theorem 3.1, every anti-derivation on $\mathcal{T}$ is zero. By similar arguments, if $\langle [B, B] \rangle = B$, then we can prove that every anti-derivation on $\mathcal{T}$ is zero. □

4. Derivations on block upper triangular matrix rings

In this section, we present sufficient conditions for anti-derivations to be zero on block upper triangular matrix rings. We first introduce block upper triangular matrix rings.

We denote the ring of all $n \times n$ matrices over a unital ring R, by $M_n(R)$, $n \geq 1$, and the subring of all upper triangular matrices by $T_n(R)$. Suppose that $n \geq 1$ and that $n = n_1 + n_2 + \cdots + n_k$, for a finite sequence of positive integers $n_1, n_2, \ldots, n_k$, where $k \geq 1$. The *block upper triangular matrix ring* $\mathcal{T} = \mathcal{T}(n_1, n_2, \ldots, n_k)$ is a subalgebra of $M_n(R)$ of all matrices of the form

$$\mathfrak{A} = \begin{bmatrix} \mathfrak{A}_{11} & \mathfrak{A}_{12} & \cdots & \mathfrak{A}_{1k} \\ 0 & \mathfrak{A}_{22} & \cdots & \mathfrak{A}_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathfrak{A}_{kk} \end{bmatrix},$$

where $\mathfrak{A}_{ij}$ is an $n_i \times n_j$ matrix over R. Also, we call k the number of *summands* of $\mathcal{T}(n_1, n_2, \ldots, n_k)$. Note that $M_n(R)$ is a special case of block upper triangular matrix rings. In particular, if $k = 1$ with $n_1 = n$, then $\mathcal{T}(n_1, n_2, \ldots, n_k) = M_n(R)$. Also, when $k = n$ and $n_i = 1$ for each $1 \leq i \leq k$, then $\mathcal{T}(n_1, n_2, \ldots, n_k) = T_n(R)$.

Let $\mathcal{T}(n_1, n_2, \ldots, n_k) \subseteq M_n(R)$ be a block upper triangular matrix ring. The identity matrix $\mathbf{I}_n$ is the identity of $\mathcal{T}(n_1, n_2, \ldots, n_k)$, and $\mathfrak{E}_{ij}$ is the matrix unit. Suppose that $\mathfrak{F}_1 = \sum_{i=1}^{n_1} \mathfrak{E}_i$ and $\mathfrak{F}_j = \sum_{i=1}^{n_j} \mathfrak{E}_{i+n_1+\cdots+n_{j-1}}$ for $2 \leq j \leq k$, where $\mathfrak{E}_l = \mathfrak{E}_{ll}$. Then $\{\mathfrak{F}_1, \ldots, \mathfrak{F}_k\}$ is a set of nontrivial idempotents of $\mathcal{T}(n_1, n_2, \ldots, n_k)$ such that $\mathfrak{F}_1 + \cdots + \mathfrak{F}_k = \mathbf{I}_n$ and $\mathfrak{F}_i \mathfrak{F}_j = \mathfrak{F}_j \mathfrak{F}_i = 0$ for $1 \leq i, j \leq k$ with $i \neq j$. Moreover, we have $\mathfrak{F}_j \mathcal{T}(n_1, n_2, \ldots, n_k) \mathfrak{F}_j \cong M_{n_j}(R)$ for any $1 \leq j \leq k$.

Theorem 4.1. *Let R be a 2-torsion free unital ring such that $\langle [R, R] \rangle = R$, and let $\mathcal{T} = \mathcal{T}(n_1, n_2, \ldots, n_k)$ be a block upper triangular ring in $M_n(R)$ with $k \geq 2$. If $n_1 = 1$ or $n_k = 1$, then every anti-derivation on $\mathcal{T}$ is zero.*

Proof. Suppose that $n_1 = 1$. Set $\mathfrak{P} = \mathfrak{F}_1$ and $\mathfrak{Q} = \mathbf{I}_n - \mathfrak{P} = \mathfrak{F}_2 + \cdots + \mathfrak{F}_k$. Then $\mathfrak{P}$ and $\mathfrak{Q}$ are nontrivial idempotents of $\mathcal{T}$ such that $\mathfrak{P}\mathfrak{Q} = \mathfrak{Q}\mathfrak{P} = 0$. Also $\mathfrak{Q}\mathcal{T}\mathfrak{P} = 0$, $\mathfrak{P}\mathcal{T}\mathfrak{P}$, and $\mathfrak{Q}\mathcal{T}\mathfrak{Q}$ are subrings of $\mathcal{T}$ with unity $\mathfrak{P}$ and $\mathfrak{Q}$, respectively, such that $\mathfrak{P}\mathcal{T}\mathfrak{P} \cong M_{n_1}(R) = R$ and $\mathfrak{Q}\mathcal{T}\mathfrak{Q} \cong \mathcal{T}(n_2, n_3, \ldots, n_k) \subseteq M_{n-n_1}(R)$ (ring isomorphisms) is a block upper triangular ring with $k-1$ summands $n_2, \ldots, n_k$. So $\mathfrak{P}\mathcal{T}\mathfrak{P}$ and $\mathfrak{Q}\mathcal{T}\mathfrak{Q}$ are 2-torsion free unital rings, and $\mathfrak{P}\mathcal{T}\mathfrak{Q}$ is a 2-torsion free unital $(\mathfrak{P}\mathcal{T}\mathfrak{P}, \mathfrak{Q}\mathcal{T}\mathfrak{Q})$-bimodule, which is faithful as a left $\mathfrak{P}\mathcal{T}\mathfrak{P}$-module and also as a right $\mathfrak{Q}\mathcal{T}\mathfrak{Q}$-module. It is easy to check that

$$\mathcal{T} \cong Tri(\mathfrak{P}\mathcal{T}\mathfrak{P}, \mathfrak{P}\mathcal{T}\mathfrak{Q}, \mathfrak{Q}\mathcal{T}\mathfrak{Q})$$

as an isomorphism of rings. Since $\langle [R, R] \rangle = R$ and $\mathfrak{P}\mathcal{T}\mathfrak{P} \cong R$, it follows from Corollary 3.2 that every anti-derivation on $\mathcal{T}$ is zero.

If $n_k = 1$, then we set $\mathfrak{P} = \mathfrak{F}_1 + \cdots + \mathfrak{F}_{k-1}$ and $\mathfrak{Q} = \mathbf{I}_n - \mathfrak{P} = \mathfrak{F}_k$. By using similar arguments as above, we can show that every anti-derivation on $\mathcal{T}$ is zero. □

The following result is straightforward.

Corollary 4.2. *Let R be a 2-torsion free unital ring such that $\langle [R,R] \rangle = R$, and let $T_n(R)$ with $n \geq 2$ be an upper triangular matrix ring over R. Then every anti-derivation on $\mathcal{T}$ is zero.*

To prove the next result, we need the following lemma, which was shown in [9, Corollary in p. 9].

Lemma 4.3. *If R is a simple non-commutative ring, then $\langle [R,R] \rangle = R$.*

Note that if R is a simple ring, then $M_n(R)$ with $n \geq 2$ is a simple non-commutative ring.

Theorem 4.4. *Let R be a 2-torsion free unital simple ring, and let $\mathcal{T} = \mathcal{T}(n_1, n_2, \ldots, n_k)$ be a block upper triangular ring in $M_n(R)$ with $k \geq 2$. If $n_1 \geq 2$ or $n_k \geq 2$, then every anti-derivation on $\mathcal{T}$ is zero.*

Proof. Suppose that $n_1 \geq 2$. Set $\mathfrak{P} = \mathfrak{F}_1$ and $\mathfrak{Q} = \mathbf{I}_n - \mathfrak{P} = \mathfrak{F}_2 + \cdots + \mathfrak{F}_k$. As the proof of Theorem 4.1, we have

$$\mathcal{T} \cong Tri(\mathfrak{P}\mathcal{T}\mathfrak{P}, \mathfrak{P}\mathcal{T}\mathfrak{Q}, \mathfrak{Q}\mathcal{T}\mathfrak{Q}),$$

where $\mathfrak{P}\mathcal{T}\mathfrak{P}$ and $\mathfrak{Q}\mathcal{T}\mathfrak{Q}$ are 2-torsion free unital rings, and $\mathfrak{P}\mathcal{T}\mathfrak{Q}$ is a 2-torsion free unital $(\mathfrak{P}\mathcal{T}\mathfrak{P}, \mathfrak{Q}\mathcal{T}\mathfrak{Q})$-bimodule, which is faithful as a left $\mathfrak{P}\mathcal{T}\mathfrak{P}$-module and also as a right $\mathfrak{Q}\mathcal{T}\mathfrak{Q}$-module. Also, $\mathfrak{P}\mathcal{T}\mathfrak{P} \cong M_{n_1}(R)$. Since $n_1 \geq 2$, $\mathfrak{P}\mathcal{T}\mathfrak{P}$ is a simple non-commutative ring, it follows from Lemma 4.3 that $\langle [\mathfrak{P}\mathcal{T}\mathfrak{P}, \mathfrak{P}\mathcal{T}\mathfrak{P}] \rangle = \mathfrak{P}\mathcal{T}\mathfrak{P}$. By Corollary 3.2, every anti-derivation on $\mathcal{T}$ is zero.

If $n_k \geq 2$, then we set $\mathfrak{P} = \mathfrak{F}_1 + \cdots + \mathfrak{F}_{k-1}$ and $\mathfrak{Q} = \mathbf{I}_n - \mathfrak{P} = \mathfrak{F}_k$. By using similar arguments as above, we can prove that every anti-derivation on $\mathcal{T}$ is zero. □

Finally, we note that it is interesting to study the necessary or sufficient conditions on generalized matrix algebras where any anti-derivation is zero.

References

1. D.A. Abulhamil, F.B. Jamjoom and A.M. Peralta, *Linear maps which are anti-derivable at zero*, Bull. Malays. Math. Sci. Soc. **43** (2020) 4315–4334.
2. D. Benkovič, *Jordan derivations and antiderivations on triangular matrices*, Linear Algebra Appl. **397** (2005) 235–244.
3. M. Brešar, *Jordan derivation on semiprime rings*, Proc. Amer. Math. Soc. **104** (1988) 1003–1006.
4. H. Ghahramani, *Skew polynomial rings of formal triangular matrix rings*, J. Algebra **349** (2012) 201–216.
5. H. Ghahramani, *Characterizing Jordan maps on triangular rings through commutative zero products*, Mediterr. J. Math. **15** (2018), no. 2, 1–10.
6. H. Ghahramani, M.N. Ghosseiri and L. Heidarizadeh, *Linear maps on block upper triangular matrix algebras behaving like Jordan derivations through commutative zero products*, Oper. Matrices, **14** (2020) 189–205.
7. H. Ghahramani and A. Moussavi, *Differential polynomial rings of triangular matrix rings*, Bull. Iranian Math. Soc. **34** (2008), no. 2, 71–96.
8. I.N. Herstein, *Jordan derivations of prime rings*, Proc. Amer. Math. Soc. **8** (1957) 1104–1110.
9. I.N. Herstein, *Topics in Ring Theory*, University of Chicago Press, Chicago, 1969.

10. W. Huang, J. Li and J. He, *Characterizations of Jordan mappings on some rings and algebras through zero products*, Linear Multilinear Algebra **66** (2018) 334–346.
11. J.H. Zhang and W.Y. Yu, *Jordan derivations of triangular algebras*, Linear Algebra Appl. **419** (2006) 251–255.

[1]Department of Mathematics, Faculty of Science, University of Kurdistan, P.O. Box 416, Sanandaj, Kurdistan, Iran.

Email address: h.ghahramani@uok.ac.ir; mnghosseiri@uok.ac.ir; t.rezaei@sci.uok.ac.ir