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SHARP BOUNDS OF THIRD HANKEL DETERMINANT FOR A CLASS OF STARLIKE FUNCTIONS AND A SUBCLASS OF q-STARLIKE FUNCTIONS

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Abstract. Following the trend of coefficient bound problems in geometric function theory, in the present paper, we obtain the sharp bound of the third Hankel determinant for the classes of starlike functions ($\mathcal{S}^*$) and q-starlike functions related with lemniscate of Bernoulli ($\mathcal{SL}_q^*$). Bound on the functions in the initial class, apart from being sharp, is also an improvement over the known existing bound, and the bound on the latter class generalizes the prior known outcome. Furthermore, the extremal functions of classes $\mathcal{S}^*$ and $\mathcal{SL}_q^*$ are deduced to prove the sharpness of these results.

1. Introduction and preliminaries

Denote the class of analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, defined on the open unit disk $\mathbb{D}$ by $\mathcal{A}$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of the univalent functions. For two analytic functions f and g, we say f is subordinate to g if there exists a Schwarz function $\omega(z)$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $f(z) = g(\omega(z))$. The normalized function f in $\mathcal{S}$ satisfying the inequality

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0, \quad z \in \mathbb{D},$$

which belongs to the class of starlike functions, denoted by $\mathcal{S}^*$. Furthermore, various subclasses of $\mathcal{S}^*$ have been introduced and studied by many authors in

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the past (see [9, 19, 24, 25]). Likewise, Sokół and Stankiewicz [27] introduced the class $\mathcal{SL}^*$, defined as

$$\mathcal{SL}^* := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \sqrt{1+z}, \quad z \in \mathbb{D} \right\}.$$

Since then, enormous work is done for the class $\mathcal{SL}^*$; for ready reference, see [1, 2, 13, 22, 26].

Let $\tilde{q}$, $n \in \mathbb{N}$. For a function $f \in \mathcal{A}$, the $\tilde{q}$th Hankel determinant, is defined by

$$H_{\tilde{q}}(n) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+\tilde{q}-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+\tilde{q}} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+\tilde{q}-1} & a_{n+q} & \cdots & a_{n+2\tilde{q}-2} \end{vmatrix},$$

introduced in [21], and has been studied by several authors. It also plays an important role in the study of singularities (see [6]). Noor [20] studied the rate of growth of $H_{\tilde{q}}(n)$ as $n \to \infty$ for functions in $\mathcal{S}$ with bounded boundary. Different choices of $\tilde{q}$ and n yield various types of Hankel determinants, such as for $\tilde{q} = 2$ and $n = 1$, the famous Fekete–Szegö functional is given by $H_2(1) := a_3 - a_2^2$. Furthermore, the generalized Fekete–Szegö functional is given by $a_3 - \mu a_2^2$, where μ is either real or complex. For $\tilde{q} = n = 2$, we have a second order Hankel determinant $H_2(2) := a_2a_4 - a_3^2$. Also, another type of second order Hankel determinant is obtained by taking $\tilde{q} = 2$ and $n = 3$, mathematically, written as $H_2(3) := a_3a_5 - a_4^2$. The estimations of the sharp bounds for these $H_{\tilde{q}}(n)$ are obtained by many authors for various subclasses of $\mathcal{A}$ (see [5, 23, 29]). Third order Hankel determinant, given by

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2), \tag{1.1}$$

is obtained when $\tilde{q} = 3$ and $n = 1$. A sharp bound of $|H_3(1)|$ was not obtained for any class of analytic functions before 2018. It was achieved by Kowalczyk *et al*. [12], for functions in $\mathcal{A}$ satisfying $\mathrm{Re}(f(z)/z) > \alpha$, $\alpha \in [0, 1)$ and in [11] for convex functions. Following which, Banga and Kumar [4] recently derived a sharp bound of third Hankel determinant as $|H_3(1)| \leq 1/36$ for functions in $\mathcal{SL}^*$, which earlier was calculated to be 43/576 in [23]. Lecko, Sim, and Śmiarowska [16] calculated the sharp bound of the third Hankel determinant to be 1/9 for starlike functions of order 1/2. The credit of initiation of sharp bound of $|H_3(1)|$ goes to Kwon, Lecko, and Sim [14] who deduced p_4 in terms of p_1, where p_i's are the coefficients of the functions in the Carathéodory class $\mathcal{P}$, defined by

$$p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + p_4z^4 + \cdots \quad (z \in \mathbb{D}).$$

Let us recall the q-derivative of a complex valued function defined on a subset of $\mathbb{C}$, defined as below:

$$(D_qf)(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z}, & z \neq 0, \\ f'(0), & z = 0, \end{cases}$$

where $q \in (0,1)$. Whenever f is differentiable on a given subset of $\mathbb{C}$, the above definition of q-derivative implies

$$\lim_{q\to 1^-}(D_q f)(z) = \lim_{q\to 1^-}\frac{f(z)-f(qz)}{(1-q)z} = f'(z).$$

Furthermore, the Taylor series expansion of f yields that

$$(D_q f)(z) = 1 + \sum_{n=2}^{\infty}[n]_q a_n z^{n-1},$$

where

$$[n]_q = \sum_{k=0}^{n-1} q^k = 1 + q + q^2 + \cdots + q^{n-1}, \qquad n \in \mathbb{N}.$$

The initiation of the above defined q-calculus was done by Jackson [8]. In geometric function theory, subclasses of normalized analytic functions have been studied from different viewpoints. Ismail, Merkes, and Styer [7] generalized the class $\mathcal{S}^*$ of starlike functions by introducing a new class with the usage of q-calculus. This marked the beginning of the introduction of q-version of various classes in geometric function theory. For instance, Srivastava and Bansal [28] studied a certain family of q-Mittag-Leffler functions, and Mahmood *et al*. [18] dealt with q-starlike functions associated with conic domains. Recently, Khan *et al*. [10] used q-derivative operator to define a new subclass of starlike functions related with the lemniscate of Bernoulli, given as

$$\mathcal{SL}_q^* := \left\{ f \in \mathcal{A} : \frac{z(D_q f)(z)}{f(z)} \prec \sqrt{\frac{2(1+z)}{2+(1-q)z}},\ z \in \mathbb{D} \right\},$$

or equivalently, a function $f \in \mathcal{A}$ is in $\mathcal{SL}_q^*$ if it satisfies

$$\left| \left(\frac{z(D_q f)(z)}{f(z)} \right)^2 - \frac{1}{1-q} \right| < \frac{1}{1-q}.$$

This implies that on choosing $\omega = z(Dqf)(z)/f(z)$, the analytic characterization of the class $\mathcal{SL}_q^*$ can be expressed as $|\omega^2 - 1/(1-q)| < 1/(1-q)$, which is the interior of the right loop of the lemniscate of Bernoulli. The specialty of this class lies in the fact that it reduces to a well-known class $\mathcal{SL}^*$, when $q \to 1^-$. The authors in [10] obtained the sharp bounds of Fekete–Szegö functional, $|H_2(2)|$, initial coefficients a_2, a_3, a_4 ,and a_5, and upper bound of third Hankel determinant for functions in $\mathcal{SL}_q^*$.

Our study focuses on the estimation of sharp bound of $|H_3(1)|$ for functions in $\mathcal{SL}_q^*$ and $\mathcal{S}^*$. It was found in [3] that $|H_3(1)| \leq 16$ for functions in $\mathcal{S}^*$, which is improved by Zaprawa [30], wherein he proved $|H_3(1)| \leq 1$. Later in [15], it was further improved to 8/9. Again, in 2021, Zaprawa, Milutin, and Tuneski [31] calculated the same to be 5/9, to which we eventually improve in the present paper to a sharp estimate of 4/9. In addition, we obtain $|H_3(1)| \leq \frac{(1+q)^2}{16q^2(1+q+q^2)^2}$ for functions in $\mathcal{SL^*}_q$. This bound apart from being sharp is an improvement over the bound obtained in [10]. Moreover, for $q \to 1^-$, this bound reduces to earlier known sharp bound for $\mathcal{SL}^*$ [4]. We also give extremal functions to justify our claims.

We state below a lemma for the formulas of p_2, p_3 [17], and p_4 [14] in order to prove our results.

Lemma 1.1. *Let $p \in \mathcal{P}$ and of the form $1 + \sum\limits_{n=1}^{\infty} p_n z^n$. Then*

$$2p_2 = p_1^2 + \lambda(4 - p_1^2),$$

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)\lambda - p_1(4 - p_1^2)\lambda^2 + 2(4 - p_1^2)(1 - |\lambda|^2)\mu,$$

and

$$\begin{aligned} 8p_4 =& p_1^4 + (4 - p_1^2)\lambda(p_1^2(\lambda^2 - 3\lambda + 3) + 4\lambda) \\ & - 4(4 - p_1^2)(1 - |\lambda|^2)(p_1(\lambda - 1)\mu + \overline{\lambda}\mu^2 - (1 - |\mu|^2)\delta), \end{aligned}$$

for some δ, λ, and μ such that $|\delta| \leq 1$, $|\lambda| \leq 1$, and $|\mu| \leq 1$.

2. Main results

This section begins with the following result.

Theorem 2.1. *Let $q \in (0, 1)$ and let $f \in \mathcal{SL}_q^*$ of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then*

$$|H_3(1)| \leq \frac{(1 + q)^2}{16q^2(1 + q + q^2)^2}.$$

Proof. For $f \in \mathcal{SL}_q^*$, we refer the reader to [10] for the expressions of a_2, a_3, and a_4. On the similar lines, we compute

$$\begin{aligned} a_5 = & \frac{1}{32768q^4(1 + q^2)(1 + q + q^2)}\Big(512p_1p_3q^2(2 - 10q - 8q^2 - 9q^3 + 3q^4) + p_1^4(8 \\ & - 140q + 802q^2 - 1435q^3 - 340q^4 - 1193q^5 + 1015q^6 - 320q^7 + 35q^8) \\ & + 32p_1^2p_2q(6 - 68q + 175q^2 + 89q^3 + 148q^4 - 93q^5 + 15q^6) + 256q^2(1 + q \\ & + q^2)(16p_4q + p2^2(2 - 13q + 3q^2))\Big). \end{aligned}$$

Now substituting the values of above a_i's in (1.1) with $p := p_1 \in [0, 2]$, we obtain

$$\begin{aligned} H_3(1) := & \frac{1}{4194304\, q^2(1 + q^2)(1 + q + q^2)^2}\Big(- 8192pp_2p_3(-14 - 28q - 13q^2 \\ & - 28q^3 - 12q^4 + q^5) - 512p_1^3p_3(-14 - 10q - 217q^2 + 23q^3 - 8q^4 + 5q^5 \\ & + q^6) + 16p_1^4p_2(-31 + 1111q - 10148q^2 + 3026q^3 - 594q^4 - 84q^5 - 19q^6 \\ & + 3q^7) + p_1^6(239 - 4972q + 35429q^2 - 13002q^3 + 3964q^4 + 370q^5 + 63q^6 \\ & - 44q^7 + q^8) + 4096(-16p_3^2(1 + q)^2(1 + q^2) + 16p_2p_4(1 + q + q^2)^2 \\ & + p_2^3(-13 + q)(1 + q + q^2)^2) + 256p_1^2(16p_4(-15 + q)(1 + q + q^2)^2 \\ & + p_2^2(-27 - 102q + 670q^2 - 188q^3 + 6q^4 + 14q^5 + 3q^6))\Big). \end{aligned}$$

Applying Lemma 1.1 in the above equation for the values of p_2, p_3, and p_4 and furthermore reducing it to the simpler form, we arrive at

$$H_3(1) = \frac{\tau_1(p,\lambda) + \tau_2(p,\lambda)\mu + \tau_3(p,\lambda)\mu^2 + \zeta(p,\lambda,\mu)\delta}{4194304\ q^2(1+q^2)(1+q+q^2)^2}, \tag{2.1}$$

whenever δ, μ, $\lambda \in \overline{\mathbb{D}}$ and

$$\begin{aligned}
\tau_1(p,\lambda) := Ap^6 + p^2(4-p^2)\lambda\Big(&8(-15+183q-804q^2+434q^3-242q^4-20q^5\\
&-3q^6+3q^7)p^2+64(45+66q+262q^2+28q^3+62q^4+6q^5+3q^6)\\
&(4-p^2)\lambda-512(7+15q-3q^2+17q^3+5q^4-q^5)(4-p^2)\lambda^2\\
&-2048(7-q)(1+q+q^2)^2\lambda+4096q^2(4-p^2)\lambda^3-512(7-q)(1\\
&+q+q^2)^2p^2\lambda^2+128(22+50q+35q^2+59q^3+20q^4+q^5+q^6)p^2\lambda\Big)\\
&-2048(5-q)(1+q+q^2)^2(4-p^2)^2\lambda^3,\\
\tau_2(p,\lambda) := (4-p^2)(1-|\lambda|^2)\Big(&256(6+2q+41q^2-15q^3-5q^5-q^6)p^3+2048(7\\
&-q)(1+q+q^2)^2p^3\lambda+p(4-p^2)\lambda(2048(6+12q+5q^2+12q^3+4q^4\\
&-q^5)-16384q^2\lambda)\Big),\\
\tau_3(p,\lambda) := (4-p^2)(1-|\lambda|^2)(&2048(7-q)(1+q+q^2)^2p^2\overline{\lambda}-(4-p^2)(16384q^2|\lambda|^2\\
&+16384(1+q^2)(1+q)^2)),\\
\zeta(p,\lambda,\mu) := (4-p^2)(1-|\lambda|^2)&(1-|\mu|^2)(1+q+q^2)^2(-(14336-2048q)p^2\\
&+16384(4-p^2)\lambda),
\end{aligned}$$

where $A := 55-308q+1349q^2-698q^3+620q^4-46q^5-25q^6-20q^7+q^8$. Taking modulus over equation (2.1) and applying triangle inequality, we get

$$|H_3(1)| \leq \frac{|\tau_1(p,\lambda)| + |\tau_2(p,\lambda)|y + |\tau_3(p,\lambda)|y^2 + |\zeta(p,\lambda,\mu)|}{4194304\ q^2(1+q^2)(1+q+q^2)^2} \leq \tilde{T}(p,x,y),$$

where $x := |\lambda|$, $y := |\mu|$, and the fact $|\delta| \leq 1$, and we have

$$\begin{aligned}
\tilde{T}(p,x,y) &:= \frac{t_1(p,x) + t_2(p,x)y + t_3(p,x)y^2 + t_4(p,x)(1-y^2)}{4194304\ q^2(1+q^2)(1+q+q^2)^2}\\
&=: \frac{T(p,x,y)}{4194304\ q^2(1+q^2)(1+q+q^2)^2}
\end{aligned}$$

with

$$\begin{aligned}
t_1(p,x) :=Ap^6 + p^2(4-p^2)x\Big(&8(15-183q+804q^2-434q^3+242q^4+20q^5\\
&+3q^6-3q^7)p^2+64(45+66q+262q^2+28q^3+62q^4+6q^5+3q^6)(4\\
&-p^2)x+512(7+15q-3q^2+17q^3+5q^4-q^5)(4-p^2)x^2+2048(7
\end{aligned}$$

$$- q)(1+q+q^2)^2x + 4096(4-p^2)x^3 + 512(7-q)(1+q+q^2)^2p^2x^2 + 128(22+50q+35q^2+59q^3+20q^4+q^5+q^6)p^2x\Big) + 2048(5-q)(1+q+q^2)^2(4-p^2)^2x^3,$$

$$t_2(p,x) := (4-p^2)(1-x^2)\Big(256(6+2q+41q^2-15q^3-5q^5-q^6)p^3 + 2048(7-q)(1+q+q^2)^2p^3x + p(4-p^2)x(2048(6+12q+5q^2+12q^3+4q^4-q^5)+16384q^2x)\Big),$$

$$t_3(p,x) := (4-p^2)(1-x^2)(2048(7-q)(1+q+q^2)^2p^2x + (4-p^2)(16384q^2x^2 + 16384(1+q^2)(1+q)^2)),$$

$$t_4(p,x) := (4-p^2)(1-x^2)(1+q+q^2)^2((14336-2048q)p^2 + 16384(4-p^2)x).$$

In order to achieve the desired bound, we need to maximize $T(p,x,y)$ in the closed cuboid $\mathfrak{C} : [0,2]\times[0,1]\times[0,1]$. We accomplish this by estimating maximum values in the interior of $\mathfrak{C}$, interior of the six faces, and finally on the twelve edges.

I. We begin with interior points of $\mathfrak{C}$, which means taking $(p,x,y) \in (0,2)\times(0,1)\times(0,1)$.
For this, we calculate

$$\frac{\partial T}{\partial y} = (4-p^2)(1-x^2)(2y(16384(4-p^2)((1+q^2)(1+q)^2+q^2x^2) + 2048p^2x(7-q)(1+q+q^2)^2 - ((14336-2048q)p^2 + 16384(4-p^2)x)(1+q+q^2)^2) + 256(6+2q+41q^2-15q^3-5q^5-q^6)p^3 + 2048(7-q)(1+q+q^2)^2p^3x + 2048px(4-p^2)(6+12q+5q^2+12q^3+4q^4-q^5) + 16384pq^2(4-p^2)x^2).$$

On solving $\partial T/\partial y = 0$, we obtain $y = y_0$, given as

$$y_0 := \frac{\tilde{A}}{2048(1-x)\left((7-q)(1+q+q^2)^2p^2 - 8(4-p^2)\left(\frac{1+2q+2q^2+2q^3+q^4}{q^2} - x\right)\right)},$$

where $\tilde{A} := p^3(128(6+2q+41q^2-15q^3-5q^5-q^6) + 1024(7-q)(1+q+q^2)^2x) + 1024px(4-p^2)(6+12q+5q^2+12q^3+4q^4-q^5+8q^2x)$. For $y_0 \in (0,1)$, we must have

$$(7-q)(1+q+q^2)^2p^2 > 8(4-p^2)\left(\frac{1+2q+2q^2+2q^3+q^4}{q^2} - x\right)$$

and

$$\tilde{A} + 16384(4-p^2)(1-x)\left(\frac{1+2q+2q^2+2q^3+q^4}{q^2} - x\right) < 2048(1-x)(7-q)(1+q+q^2)^2p^2. \tag{2.2}$$

Let us assume $p \to 2$. Then there exists $x \in (0, \frac{101}{216})$ for every $q \in (0,1)$ such that (2.2) holds. Moreover when we consider $x \in [\frac{101}{216}, 1)$, then there exists no $p \in (0,2)$

for all $q \in (0,1)$ such that (2.2) holds. Assuming $x \to 0$, we compute (2.2) that holds for $p \geq 1.48855$ for every $q \in (0,1)$. In fact whenever $p \in (0, 1.48855)$, there exists no $x \in (0,1)$ for all $q \in (0,1)$ such that (2.2) holds. Thus we conclude a possible solution existing in $[1.48855, 1) \times (0, \frac{101}{216})$ for inequality (2.2). A computation shows

$$\left.\frac{\partial T}{\partial p}\right|_{y=y_0} \neq 0,$$

in this interval. Therefore, there exists no critical point in the interior of $\mathfrak{C}$.

II. Now we compute the maximum value of T in the interior of all the six faces of $\mathfrak{C}$.

On the face $p = 0$, $T(p, x, y)$ reduces to

$$\begin{aligned} T(0, x, y) = 262144(1 - x^2)((1 + q^2)(1 + q)^2 + q^2x^2 - (1 + q + q^2)^2x)y^2 \\ + 32768x(1 + q + q^2)^2(x^2(5 - q) + 8(1 - x^2)), \end{aligned} \tag{2.3}$$

which in turn differentiating with respect to y becomes

$$\frac{\partial T}{\partial y} = 524288y(1 - x^2)(x - 1)\left(x - \frac{1 + 2q + 2q^2 + 2q^3 + q^4}{q^2}\right) \neq 0 \quad x, y \in (0, 1).$$

This clearly shows there exists no critical point for $T(0, x, y)$ in $(0, 1) \times (0, 1)$.

On the face $p = 2$,

$$\tilde{T}(p, x, y) = \tilde{T}(2, x, y) = \frac{A}{65536q^2(1 + q^2)(1 + q + q^2)^2} \leq \frac{(1 + q)^2}{16q^2(1 + q + q^2)^2},$$

$x, y \in (0, 1)$, as we have $-4041 - 8500q - 6843q^2 - 8890q^3 - 3476q^4 - 46q^5 - 25q^6 - 20q^7 + q^8 \leq 0$ for $q \in (0, 1)$.

On the face $x = 0$, $T(p, x, y)$ becomes

$$\begin{aligned} T(p, 0, y) = {} & p^6(55 - 308q + 1349q^2 - 698q^3 + 620q^4 - 46q^5 - 25q^6 - 20q^7 + q^8) \\ & + 256(4 - p^2)\big(- p^3(-6 - 2q - 41q^2 + 15q^3 + 5q^5 + q^6)y \\ & + 64(4 - p^2)(1 + q)^2(1 + q^2)y^2 + 8p^2(-7 + q)(1 + q + q^2)^2(-1 + y^2)\big) \\ := {} & h_1(p, y). \end{aligned} \tag{2.4}$$

On solving $\frac{\partial h_1}{\partial y} = 0$, we get

$$y =: y_1 = \frac{p^3(-6 - 2q - 41q^2 + 15q^3 + 5q^5 + q^6)}{16(32(1 + q)^2(1 + q^2) + p^2(-15 - 29q - 35q^2 - 27q^3 - 13q^4 + q^5))}. \tag{2.5}$$

For $0 < p \leq 1.46$, we have $y_1 \leq 0$ for every $q \in (0, 1)$. There exists some $q \in (0, 1)$ whenever $p \in (1.46, 2)$ such that $y_1 > 0$. On substituting (2.5) in $\frac{\partial h_1}{\partial p}$ and simplifying further, we get $\frac{\partial h_1}{\partial p} \neq 0$, where $p \in (1.46, 2)$, $q \in (0, 1)$. Thus $h_1(p, y)$ has no critical point in $(0, 2) \times (0, 1)$.

On the face $x = 1$, $T(p, x, y)$ reduces to

$$\begin{aligned} T(p, 1, y) = {} & -32768(-5 + q)(1 + q + q^2)^2 + p^6(1 + q)^3(-1 + 7q + 19q^2 + 9q^3 \\ & + q^4 + q^5) + 1024p^2(77 + 146q + 246q^2 + 140q^3 + 94q^4 + 6q^5 + 3q^6) \\ & - 32p^4(929 + 1783q + 2636q^2 + 1666q^3 + 878q^4 - 4q^5 + 29q^6 + 3q^7) \end{aligned}$$

$$=: h_2(p). \tag{2.6}$$

On differentiating h_2 with respect to p, we obtain

$$\begin{aligned}\frac{\partial h_2}{\partial p} &= 6p^5(1+q)^3(-1+7q+19q^2+9q^3+q^4+q^5)+2048p(77+146q+246q^2\\ &\quad +140q^3+94q^4+6q^5+3q^6)-128p^3(929+1783q+2636q^2+1666q^3\\ &\quad +878q^4-4q^5+29q^6+3q^7),\end{aligned}$$

further which becomes 0 at $p=0$ and $p=p_0$, given by

$$p_0 := \sqrt{\frac{32(929+1783q+2636q^2+1666q^3+878q^4-4q^5+29q^6+3q^7)}{3(1+q)^3(-1+7q+19q^2+9q^3+q^4+q^5)}-\tilde{A}},$$

where

$$\tilde{A} = \frac{64\sqrt{2}A_0}{3(-1+4q+37q^2+86q^3+92q^4+50q^5+15q^6+4q^7+q^8)},$$

and

$$\begin{aligned}A_0 = (&107909+414041q+1008402q^2+1557100q^3+1804144q^4+1471838q^5\\ &+913014q^6+363176q^7+107408q^8+9900q^9+6570q^{10}+346q^{11}\\ &+41q^{12}+15q^{13})^{(1/2)}.\end{aligned}$$

A calculation yields that $p=0$ is a point of minima, that p_0 is a point of maxima, and that the maximum value is given by a huge mathematical expression in q, which is computed to be less than $262144(1+q)^2(1+q^2)$.

On the face $y=0$, we have $T(p,x,0) =: h_3(p,x)$, given by

$$\begin{aligned}h_3(p,x) :=&p^6(55-308q+1349q^2-698q^3+620q^4-46q^5-25q^6-20q^7+q^8)\\ &+2048(4-p^2)(1+q+q^2)^2(-1+x^2)(-32x+p^2(-7+q+8x))\\ &+8(4-p^2)x\Big(-1024(-5+q)(1+q+q^2)^2x^2+32p^2x(101-2q^5\\ &+3q^6+16x+2q^4(51+8x)+4q^3(29+20x)+2q(85+24x)\\ &+2q^2(207-64x+32x^2))-p^4\big(-15+3q^7+8x+22q^4(-11\\ &+8x)+4q^5(-5+8x)+q^6(-3+8x)+q(183-272x+128x^2)\\ &+q^3(434-720x+384x^2)+4q^2(-201+384x-352x^2+128x^3)\big)\Big).\end{aligned}$$

A calculation yields that there is no common solution to the system of equations $\frac{\partial h_3}{\partial x}=0$ and $\frac{\partial h_3}{\partial p}=0$ in $(0,2)\times(0,1)$. Similarly we can show that there exists no critical point for $T(p,x,1)$.

III. Finally, we estimate the maximum value on the edges of the cuboid $\mathfrak{C}$. Start with $T(p,0,0) =: h_4(p)$, given by

$$\begin{aligned}h_4(p) =&(1+q+q^2)^2(4p^2(14336-2048q)-p^4(14336-2048q))+(55\\ &-308q+1349q^2-698q^3+620q^4-46q^5-25q^6-20q^7+q^8)p^6,\end{aligned}$$

obtained from (2.4). On solving $\frac{\partial h_4}{\partial p} = 0$, we get either $p = 0$ or $p =: p_0$, given by

$$p_0 := \frac{1}{\sqrt{3A}}\Bigg(2048(7-q)(1+q+q^2)^2 - 64\sqrt{2}\Bigg(23933 + 97507q + 203268q^2 + 309564q^3 + 313752q^4 + 250248q^5 + 114774q^6 + 34938q^7 - 6156q^8 - 644q^9 + 1352q^{10} + 192q^{11} - 75q^{12} + 3q^{13}\Bigg)^{1/2}\Bigg)^{1/2}.$$

We compute that the function $h_4(0) = 0$ is a minimum value of $h_4(p)$ and that $h_4(p_0)$ is a huge mathematical expression in q, which is also a maximum value of $h_4(p)$. Furthermore, we have $\tilde{T}(p_0, 0, 0) \leq (1+q)^2/16q^2(1+q+q^2)^2$. Substituting $y = 1$ in (2.4), we obtain

$$T(p,0,1) = h_5(p) = (4-p^2)\Big(-256(-6-2q-41q^2+15q^3+5q^5+q^6)p^3 + (4-p^2)\big(16384(1+q^2)(1+q)^2\big)\Big) + Ap^6.$$

The function $h_5(p)$ is a decreasing function of p for all q. Thus

$$\max_{p\in[0,2]} \tilde{T}(p,0,1) = \tilde{T}(0,0,1) = \frac{(1+q)^2}{16q^2(1+q+q^2)^2}.$$

Form (2.6), which is independent of y, we get $\tilde{T}(p,1,0) = \tilde{T}(p,1,1) = \tilde{T}(p,1,y)$. Thus $\tilde{T}(p,1,0) = \tilde{T}(p,1,1) \leq \frac{(1+q)^2}{16q^2(1+q+q^2)^2}$. Substituting $x = 0$ in (2.3), we obtain $\tilde{T}(0,0,y) = y^2(1+q)^2/16q^2(1+q+q^2)^2$, which is clearly an increasing function of y for all q, and we have

$$\tilde{T}(0,0,y) \leq \tilde{T}(0,0,1) = \frac{(1+q)^2}{16q^2(1+q+q^2)^2}.$$

Evaluating (2.6) at $p = 0$, we get

$$\tilde{T}(0,1,y) = \frac{5-q}{128q^2(1+q^2)}.$$

The value of $\tilde{T}(p,x,y)$ on the edges $p = 2$, $x = 1$; $p = 2$, $x = 0$; $p = 2$, $y = 0$; and $p = 2$, $y = 1$ is, respectively, equal to $\tilde{T}(2,1,y) = \tilde{T}(2,0,y) = \tilde{T}(2,x,0) = \tilde{T}(2,x,1) = \tilde{T}(2,x,y)$ as $\tilde{T}(2,x,y)$ is independent of both x and y, which further equals to

$$\frac{A}{65536q^2(1+q^2)(1+q+q^2)^2} \leq \frac{(1+q)^2}{16q^2(1+q+q^2)^2}.$$

Evaluating (2.3) at $y = 0$, we deduce

$$T(0,x,0) = h_6(x) = 32768(1+q+q^2)^2 x(8-(3+q)x^2).$$

On solving $h_6'(x) = 0$, we get

$$x = x_0 := \frac{512(1+q+q^2)}{\sqrt{294912 + 688128q + 1081344q^2 + 884736q^3 + 491520q^4 + 98304q^5}}.$$

A computation shows that x_0 is a point of maxima and that the maximum value is given by

$$\max_{x\in[0,1]} h_6(x) = h_6(x_0) = \frac{\sqrt{2}(1+q+q^2)}{12\sqrt{3(3+q)}} \qquad (0<q<1).$$

Also, we have

$$\max_{0\le x\le 1} \tilde{T}(0,x,0) \le \frac{(1+q)^2}{16q^2(1+q+q^2)^2}.$$

Now evaluating (2.3) at $y=1$, we obtain

$$T(0,x,1) = 262144(1-x^2)((1+q^2)(1+q)^2+q^2x^2)+32768x^3(5-q)(1+q+q^2)^2,$$

which is clearly a decreasing function of x and attains the maximum value at $x=0$, given by $\frac{(1+q)^2}{16q^2(1+q+q^2)^2}$.

Altogether I–III yield $|H_3(1)| \le \frac{(1+q)^2}{16q^2(1+q+q^2)^2}$. The result is sharp as equality occurs for the function $\tilde{f}:\mathbb{D}\to\mathbb{C}$, satisfying the following equation:

$$\frac{z(D_q\tilde{f})(z)}{\tilde{f}(z)} = \sqrt{\frac{2(1+z^3)}{2+(1-q)z^3}}.$$

□

Let $q\to 1^-$ in the above theorem. Then it reduces to the following result obtained by Banga and Kumar [4].

Corollary 2.2. *Let $f\in\mathcal{SL}^*$. Then $|H_3(1)|\le 1/36$.*

Moreover, extremal functions also coincide in the case of $q\to 1^-$.

Theorem 2.3. *Let $f\in\mathcal{S}^*$ of the form $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$. Then the sharp bound for the third order Hankel determinant for such functions is given by*

$$|H_3(1)|\le 4/9. \tag{2.7}$$

Proof. For $f\in\mathcal{S}^*$, we have

$$\frac{zf'(z)}{f(z)} = \frac{1+\omega(z)}{1-\omega(z)}, \tag{2.8}$$

for some Schwarz function $\omega(z)$. Define a function $p(z)=\frac{1+\omega(z)}{1-\omega(z)}$. Then evidently $p\in\mathcal{P}$. Equation (2.8) now reduces to

$$\frac{zf'(z)}{f(z)} = p(z) = 1+p_1z+p_2z^2+p_3z^3+\cdots.$$

The Taylor series of which yield

$$a_2=p_1,\quad a_3=\frac{p_2+p_1^2}{2},\quad a_4=\frac{p_1^3+3p_1p_2+2p_3}{6},$$

and

$$a_5=\frac{p_1^4+6p_1^2p_2+3p_2^2+8p_1p_3+6p_4}{24}.$$

Here, we assume that $p_1 =: p$ lies in the interval $[0,2]$ due to the invariant property of class $\mathcal{P}$ under rotation. Equation (1.1), together with the above expressions of a_i's, yields

$$H_3(1) = \frac{-p^6 + 3p^4p_2 + 8p^3p_3 + 24pp_2p_3 - 9p^2p_2^2 - 18p^2p_4 - 9p_2^3 - 16p_3^2 + 18p_2p_4}{144}.$$

Applying Lemma 1.1 in the above equation for the values of p_2, p_3, and p_4 and further reducing it to the simpler form, we arrive at

$$H_3(1) = \frac{1}{1152}\bigg(\tau_1(p,\lambda) + \tau_2(p,\lambda)\mu + \tau_3(p,\lambda)\mu^2 + \varsigma(p,\lambda,\mu)\delta\bigg),$$

where δ, μ, $\lambda \in \overline{\mathbb{D}}$, and

$$\begin{aligned}\tau_1(p,\lambda) &:= -2p^2\lambda^2(4-p^2)^2 - 10p^2\lambda^3(4-p^2)^2 + p^2\lambda^4(4-p^2)^2\\ &\qquad + 3p^4\lambda(4-p^2) + 3p^4\lambda^2(4-p^2) - 36p^2\lambda^2(4-p^2) - 9p^4\lambda^3(4-p^2),\\ \tau_2(p,\lambda) &:= (4-p^2)(1-|\lambda|^2)\left(12p^3 + 36p^3\lambda + p\lambda(4-p^2)(20-4\lambda)\right),\\ \tau_3(p,\lambda) &:= (4-p^2)(1-|\lambda|^2)\left(36p^2\overline{\lambda} - 4(4-p^2)(|\lambda|^2+8)\right),\\ \varsigma(p,\lambda,\mu) &:= (4-p^2)(1-|\lambda|^2)(1-|\mu|^2)\left(-36p^2 + 36\lambda(4-p^2)\right).\end{aligned}$$

Assuming $x := |\lambda|$ and $y := |\mu|$ and using the fact $|\delta| \le 1$, we have

$$|H_3(1)| \le \frac{|\tau_1(p,\lambda)| + |\tau_2(p,\lambda)|y + |\tau_3(p,\lambda)|y^2 + |\varsigma(p,\lambda,\mu)|}{1152} \le S(p,x,y),$$

where

$$S(p,x,y) := \frac{1}{1152}\bigg(s_1(p,x) + s_2(p,x)y + s_3(p,x)y^2 + s_4(p,x)(1-y^2)\bigg) \tag{2.9}$$

with

$$\begin{aligned}s_1(p,x) &:= 2p^2x^2(4-p^2)^2 + 10p^2x^3(4-p^2)^2 + p^2x^4(4-p^2)^2 + 3p^4x(4-p^2)\\ &\qquad + 3p^4x^2(4-p^2) + 36p^2x^2(4-p^2) + 9p^4x^3(4-p^2),\\ s_2(p,x) &:= (4-p^2)(1-x^2)(12p^3 + px(4-p^2)(20+4x) + 36p^3x),\\ s_3(p,x) &:= (4-p^2)(1-x^2)(32(4-p^2) + 4x^2(4-p^2) + 36p^2x),\\ s_4(p,x) &:= (4-p^2)(1-x^2)(36p^2 + 36x(4-p^2)).\end{aligned}$$

Our aim is to maximize $S(p,x,y)$ in the closed cuboid $\mathfrak{C} : [0,2]\times[0,1]\times[0,1]$. We accomplish this by obtaining the maximum values in the interior of $\mathfrak{C}$, in the interior of the six faces and on the twelve edges.

I. First we consider the interior points of $\mathfrak{C}$. Let $(p,x,y) \in (0,2)\times(0,1)\times(0,1)$. In order to achieve the maximum value in the interior of $\mathfrak{C}$, we partially differentiate (2.9) with respect to y and further reduce it to a simpler expression as

$$\begin{aligned}\frac{\partial S}{\partial y} = \frac{1}{1152}(4-p^2)(1-x^2)\Big(&8y(x-1)((4-p^2)(x-8) + 9p^2)\\ &+ 4p\big(x(4-p^2)(5+x) + p^2(3+9x)\big)\Big).\end{aligned}$$

Now $\frac{\partial S}{\partial y} = 0$ yields

$$y =: y_0 = \frac{2p(x(4-p^2)(5+x)+p^2(3+9x))}{(1-x)((4-p^2)(x-8)+9p^2)}.$$

In order to find the critical points, we first ensure $0 < y_0 < 1$, which holds only if

$$p^3(6+18x)+2px(4-p^2)(5+x)+(1-x)(8-x)(4-p^2) < 9p^2(1-x) \quad (2.10)$$

and

$$9p^2 > (4-p^2)(8-x). \quad (2.11)$$

We determine the common solutions for the above inequalities. A computation shows that inequality (2.11) holds for all $x \in (0,1)$ whenever $p > 1.37199$, but inequality (2.10) does not hold in $(0,2) \times (0,1)$. Therefore the function S has no critical point in the given domain of values.

II. Below we calculate the maximum value on the six faces of the cuboid $\mathfrak{C}$. On the face $p = 0$, $S(p,x,y)$ becomes

$$h_1(x,y) := S(0,x,y) = \frac{(1-x^2)(y^2(x-1)(x-8)+9x)}{18}, \quad (2.12)$$

where $x, y \in (0,1)$. We calculate

$$\frac{\partial h_1}{\partial y} = \frac{(1-x^2)y}{9}((x-1)(x-8)) \neq 0, \quad x, y \in (0,1).$$

Clearly, we can infer from above that h_1 has no critical point in $(0,1) \times (0,1)$.

On the face $p = 2$, $S(p,x,y)$ becomes

$$S(2,x,y) = 0, \quad x, y \in (0,1). \quad (2.13)$$

On the face $x = 0$, $S(p,x,y)$ becomes

$$S(p,0,y) =: h_2(p,y) = \frac{(4-p^2)}{288}\left(3p^3y + y^2(8(4-p^2)-9p^2)+9p^2\right), \quad (2.14)$$

for $y \in (0,1)$ and $p \in (0,2)$. Now, we differentiate $h_2(p,y)$ partially with respect to y and obtain

$$\frac{\partial h_2}{\partial y} = \frac{(4-p^2)}{288}\left(3p^3 + 2y(8(4-p^2)-9p^2)\right), \quad p \in (0,2) \text{ and } y \in (0,1).$$

On solving $\partial h_2/\partial y = 0$, we get

$$y = \frac{3p^3}{2(17p^2-32)}, \quad (2.15)$$

which belongs to $(0,1)$ only when $p > p_0 \approx 1.47073$. Upon substituting the value of y from (2.15) in $\partial h_2/\partial p = 0$, we arrive at

$$\frac{p(16384 - 25600p^2 + 12944p^4 - 2048p^6 - 51p^8}{64(32-17p^2)^2} = 0,$$

for $p = 1.20671$ in $(0,2)$. Thus there exists no critical point of h_2 in $(0,2) \times (0,1)$.

On the face $x = 1$, $S(p, x, y)$ becomes

$$S(p, 1, y) =: h_3(p) = \frac{p^2}{576}\left(176 - 40p^2 - p^4\right). \tag{2.16}$$

To find the maximum value of h_3, we solve $\partial h_3/\partial p = 0$, which implies $p =: p_0 \approx 1.42948$ in $(0, 2)$. A further calculation reveals $h_3''(p_0) < 0$, indicating that p_0 is the point of maxima and that

$$S(p, 1, y) \leq S(p_0, 1, y) \approx 0.319595, \qquad p \in (0, 2) \text{ and } y \in (0, 1).$$

On the face $y = 0$, $S(p, x, y)$ becomes

$$\begin{aligned} S(p, x, 0) =: h_4(p, x) =& (4 - p^2)\big(144x(1 - x^2) + p^4 x(3 + x - x^2 - x^3) \\ &+ 4p^2(9 - 9x + 2x^2 + 19x^3 + x^4)\big). \end{aligned}$$

A computation yields

$$\begin{aligned} \frac{\partial h_4}{\partial p} =& 2p\big(3p^4 x(-3 - x + x^2 + x^3) + 16(9 - 18x + 2x^2 + 28x^3 + x^4) \\ &- 8p^2(9 - 12x + x^2 + 20x^3 + 2x^4)\big) \end{aligned}$$

and

$$\frac{\partial h_4}{\partial x} = (4-p^2)(144(1-3x^2) + p^4(3+2x-3x^2-4x^3) + 4p^2(-9+4x+57x^2+4x^3)).$$

We observe that there is no common solution for the equations $\frac{\partial h_4}{\partial p} = 0$ and $\frac{\partial h_4}{\partial x} = 0$, which indicates there exists no critical point of $h_4(p, x)$ in $(0, 2) \times (0, 1)$.

On the face $y = 1$, $S(p, x, y)$ becomes $S(p, x, 1)$, given as

$$\begin{aligned} h_5(p, x) := \frac{1}{1152}\Bigg(& (1 - x^2)(512 + 64x^2 + 16px(20 + 4x) + p^2(176x^2 + 160x^3 + 16x^4 \\ &- 256 + 144x - 32x^2) + p^4(12x - 40x^2 - 44x^3 - 8x^4 + 32 - 36x + 4x^2) \\ &+ p^3(48 + 144x - 8x(20 + 4x)) + p^5(-12 - 36x + x(20 + 4x)) \\ &+ p^6(-3x - x^2 + x^3 + x^4))\Bigg). \end{aligned}$$

On solving $\frac{\partial h_5(p,x)}{\partial x} = 0$ and $\frac{\partial h_5(p,x)}{\partial p} = 0$, we observe that there is no common solution to these equations. Hence there exists no critical point of h_5 in $(0, 2) \times (0, 1)$.

III. Finally, we find the maximum values attained by $S(p, x, y)$ on the edges of the cuboid $\mathfrak{C}$. Equations (2.12), (2.13), (2.14), and (2.16) are appropriately used to evaluate $S(p, x, y)$ below for particular values of p, x, and y.

(i) $S(p, 0, 0) = p^2(4 - p^2)/32 =: l_1(p)$. Now, $l_1'(p) = 0$ for $p = 0$ and $p =: \gamma_0 = \sqrt{2}$. Simply by the second derivative test, we obtain that $p = 0$ is the point of minima and that the maximum value $1/8$ is attained at γ_0. So, we have

$$S(p, 0, 0) \leq \frac{1}{8}, \quad p \in [0, 2].$$

(ii) $S(p,0,1) = (4-p^2)(32-8p^2+3p^3)/288$, which is a decreasing function of p in the given range of p. Thus maximum value is obtained at $p=0$ and
$$S(p,0,1) \le S(0,0,1) = \frac{4}{9}, \qquad p \in [0,2].$$
(iii) Since $S(p,1,y)$ is independent of y, we obtain $S(p,1,0) = S(p,1,1) = p^2(176-40p^2-p^4)/576 = h_3(p)$, given in (2.16). Thus
$$S(p,1,0) = S(p,1,1) \le 0.319595, \qquad p \in [0,2].$$
(iv) $S(0,0,y) = 4y^2/9$, clearly which attains the maximum value $4/9$ at $y=1$. So
$$S(0,0,y) \le \frac{4}{9}, \qquad y \in [0,1].$$
(v) $S(0,1,y) = S(2,0,y) = S(2,1,y) = 0$, $y \in [0,1]$.

(vi) $S(0,x,0) = x(1-x^2)/2 =: l_3(x)$. Now $l_3'(x) = (1-3x^2)/2 = 0$ gives $x = \gamma_1 := 1/\sqrt{3}$ in the interval $[0,1]$. Furthermore, the second derivative of $l_3(x)$ is negative at γ_1. Thus γ_1 is the point of maxima and
$$S(0,x,0) \le \frac{1}{3\sqrt{3}} = 0.19245,\ x \in [0,1].$$
(vii) $S(0,x,1) = (1-x^2)(x^2+8)/18$, which is a decreasing function of x in $[0,1]$. So clearly the maximum value is attained at $x=0$, and we have
$$S(0,x,1) \le \frac{4}{9}, \qquad x \in [0,1].$$
(viii) $S(2,x,0) = S(2,x,1) = 0$, $x \in [0,1]$.

Considering I–III cases altogether, the inequality (2.7) is proved. Define the function $\tilde{f} : \mathbb{D} \to \mathbb{C}$ as follows:
$$\tilde{f}(z) = z \exp\left(\int_0^z \frac{\left(\frac{1+t^3}{1-t^3}\right) - 1}{t} dt \right) = z + \frac{2z^4}{3} + \cdots,$$
which clearly belongs to $\mathcal{S}^*$ and for which, we have $a_2 = a_3 = a_5 = 0$ and $a_4 = 2/3$. This shows that the bound $|H_3(1)|$ is sharp as (1.1) yields $|H_3(1)| = 4/9$ for this function. □

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