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ON STRONGLY STAR SEMI-COMPACTNESS OF TOPOLOGICAL SPACES

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Abstract. Sabah, Khan, Kočina developed strongly star semi-compactness of a topological space in 2016, which is a variant of star-compactness in which semi-open covers are employed instead of open covers. The goal of this study is to compare the structure of strongly star semi-compactness to that of other topological characteristics with similar structures. Furthermore, the nature of a strongly star semi-compact space's subspace and the features of a strongly star semi-compact subset relative to a space are examined.

1. Introduction

In 1963, Levine [18] introduced the concept of semi-open sets in topological spaces. Since then, numerous mathematicians have stretched different conceptions of topological attributes in new settings of semi-open sets and examined such aspects. A set A in a topological space (X, τ) is semi-open if and only if there exists $U \in \tau$ such that $U \subseteq A \subseteq Cl(U)$, where $Cl(U)$ signifies the closure of the set U. Equivalently, A is semi-open if and only if $A \subseteq Cl(int(A))$ ($int(A)$ represents the interior of the set A). The complement of semi-open set A is called semi-closed [8]. A semi-regular subset of a space (X, τ) is one that is both semi-open and semi-closed. A semi-open set may or may not be open, whereas an open set is always semi-open. The intersection of two semi-open sets may not be semi-open, but the union of any number of semi-open sets is. An open set's intersection with a semi-open set is always semi-open.

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Suppose that M is a subset of a set X and that $\mathcal{U}$ is a collection of subsets of X. Then $(n+1)$-star of M with respect to $\mathcal{U}$ is denoted and defined as

$$St^{n+1}(M,\mathcal{U}) = St(St^n(M,\mathcal{U}),\mathcal{U})$$

where $St^1(M,\mathcal{U}) = St(M,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap M \neq \emptyset\}$ and $n = 1, 2, 3, \ldots$.

A topological space (X, τ) is called a star-compact space (also star finite space with respect to some other literature) if for every open cover $\mathcal{U}$ of X, there exists a finite set $F \subseteq X$ such that $St(F,\mathcal{U}) = X$ [12]. Recent variations of star-compact space and investigations on these properties can be found in [1–7,14–17,21–23,25].

In 2016, Sabah, Khan, and Kočina [20] employed covers by semi-open sets in the concept of star-compact space to discover a variation known as strongly star semi-compactness. Our major goal is to examine the structure of strongly star semi-compactness. While they focused on related selection principles, star s-Menger spaces, and star s-Hurewicz spaces, they did not address the properties of strongly star semi-compactness and its relation to other similar structures. Compactness [13], semi-compactness [11], star compactness [12], and star s-Menger spaces [20] have structures that are closer to that of strongly star semi-compactness, according to the literature. We compare the structure of the above-mentioned topological features in this work. This study also discusses various other topological characteristics relating to special functions and subspaces.

2. Preliminaries

A space X denotes a topological space X with the topology τ throughout the text. Unless otherwise indicated, no separation axioms have been imposed. We use [13] for core concepts and nomenclatures.

Definition 2.1 ([11]). A topological space (X, τ) is called a semi-compact space if every cover of X by semi-open sets has a finite subcover.

Definition 2.2 ([20]). A topological space (X, τ) is called a strongly star semi-compact space if for every cover $\mathcal{U}$ of X by semi-open sets, we can find a finite subset $F \subseteq X$ such that $St(F,\mathcal{U}) = X$.

Definition 2.3 ([20]). A topological space (X, τ) is called star s-Menger if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of covers by semi-open sets, there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}}\{St(V,\mathcal{U}_n) : V \in \mathcal{V}_n\}$ is a cover of X by semi-open sets.

Definition 2.4 ([9]). A function $f : (X, \tau) \to (Y, \sigma)$ is called an irresolute if the inverse image of each semi-open set in Y is a semi-open set in X.

Definition 2.5 ([24]). A subset A of a space (X, τ) is called an α-set if $A \subseteq int(Cl(int(A)))$.

It is evident to mention that every open set of a space is an α-set in that space.

Lemma 2.6 ([19]). *If A is an α-set of a space X and V is a semi-open set in X, then $V \cap A$ is a semi-open set in the subspace (A, τ_A).*

Lemma 2.7 ([10]). *If Y is an open set in X and $A \subseteq Y$, then A is a semi-open set in X if and only if A is semi-open in the space (Y, τ_Y).*

Lemma 2.8 ([20]). *Let (Y, τ_Y) is a subspace of a topological space (X, τ) and V is a semi-open set in Y. Then there exists a semi-open set U in X such that $V = Y \cap U$.*

3. On strongly star semi-compactness

Proposition 3.1. *Every semi-compact topological space is a strongly star semi-compact space.*

Proof. Let $\mathcal{U}$ be a cover by semi-open sets of a semi-compact space (X, τ). Then there exists a finite subset $\mathcal{U}' = \{U_1, U_2, U_3, \ldots, U_k\} \subseteq \mathcal{U}$ such that $\bigcup \mathcal{U}' = \bigcup_{i=1}^{k} U_i = X$. Now if we take $x_i \in U_i$ for each $i = 1, 2, \ldots, k$ and form a finite set $F = \{x_1, x_2, x_3, \ldots, x_k\}$, then $X = St(F, \mathcal{U}') \subseteq St(F, \mathcal{U}) = X$. Therefore, (X, τ) is a strongly star semi-compact space. □

Example 3.2. Converse of the above proposition may not be true.

Consider the set $X = \mathbb{N}$ and the topology $\tau = \{\{1, 2, 3, \ldots, n\} : n \in \mathbb{N}\} \cup \{X, \emptyset\}$ on X. Any subset $A \subseteq X$ other than $\emptyset$ that does not contain 1 has $int(A) = \emptyset$. Hence $Cl(int(A)) = \emptyset$, which implies $A \nsubseteq Cl(int(A))$. Therefore every semi-open subset of X contains 1. Now consider an arbitrary cover $\mathcal{U}$ of X by semi-open sets. Then for the finite subset $F = \{1\} \subseteq X$, we have $St(F, \mathcal{U}) = \bigcup \mathcal{U} = X$. Hence (X, τ) is a strongly star semi-compact space.

On the other hand, consider the cover $\mathcal{U} = \{U_n = \{1, 2, 3, \ldots, n\} : n \in \mathbb{N}\}$ of (X, τ) by semi-open sets of X. Suppose that it has a finite subcover $\mathcal{U}'$. By the construction of the space, there exists a largest set $U_\alpha \in \mathcal{U}'$, where $\alpha \in \mathbb{N}$. So $\bigcup \mathcal{U} = U_\alpha = \{1, 2, 3, \ldots, \alpha\}$. Thus $\{\alpha + 1, \alpha + 2, \alpha + 3, \ldots\}$ remains uncovered. Thus $\mathcal{U}$ cannot have a finite subcover. Hence (X, τ) is not semi-compact.

Proposition 3.3. *Every strongly star semi-compact space is a star-compact space.*

Proof. Let (X, τ) be a strongly star semi-compact space and let $\mathcal{U}$ be a open cover of X. Since every open set is a semi-open set, $\mathcal{U}$ can be considered as a cover by semi-open sets. Indeed, (X, τ) is a strongly star semi-compact space. Therefore there exists a finite subset $F \subseteq X$ such that $St(F, \mathcal{U}) = X$. Hence (X, τ) is a star-compact space. □

Example 3.4. Converse of the above proposition may not be true.

Let $X = \mathbb{R}$ with the topology $\tau = \{X, \emptyset\} \cup \mathcal{P}(\mathbb{N})$. There does not exist any nontrivial open cover of (X, τ). So (X, τ) is trivially compact, hence star-compact.

Now we consider the cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$, where

$$U_n = \begin{cases} \{1\} \cup (\mathbb{R} \setminus \mathbb{N}) & \text{if } n = 1, \\ \{n\} & \text{otherwise.} \end{cases}$$

Here $\{n\} \in \tau$, therefore each $\{n\}$ is semi-open. So U_n is semi-open for each $n \in \mathbb{N} \setminus \{1\}$.

On the other hand,

$$Cl(int(U_1)) = Cl(int(\{1\}\cup(\mathbb{R}\setminus\mathbb{N}))) = Cl(\{1\}) = \mathbb{R}\setminus(\mathbb{N}\setminus\{1\}) = \{1\}\cup(\mathbb{R}\setminus\mathbb{N}) = U_1.$$

Thus $\mathcal{U}$ is a cover by semi-open sets. Since $U_n \in \mathcal{U}$ are pairwise disjoint and $\mathcal{U}$ is countably infinite, for any finite subset $F \subset X$, there exists $p \in \mathbb{N} \setminus \{1\}$ such that $p \notin F$. So, $p \notin St(F,\mathcal{U})$. Therefore, $St(F,\mathcal{U}) \neq X$. Thus (X,τ) is not strongly star semi-compact.

Proposition 3.5. *Every strongly star semi-compact space is a star s-Menger space.*

Proof. Let (X,τ) be a strongly star semi-compact space and let $\mathcal{U}$ be an arbitrary cover of X by semi-open sets. Then there exists a finite subset $F = \{x_1, x_2, x_3, \dots x_n\}$ of X such that $St(F,\mathcal{U}) = X$. Since $\mathcal{U}$ is a cover of X, there exists a subset $\mathcal{V} = \{U_1, U_2, U_3, \dots, U_n\}$ of $\mathcal{U}$ such that $x_i \in U_i$, for $i = 1, 2, 3, \dots, n$. Therefore $St(\bigcup\mathcal{V},\mathcal{U}) \subseteq St(F,\mathcal{U}) = X$. Hence (X,τ) is a star s-Manger space. □

It follows that we have the following implication diagram in Figure 1:

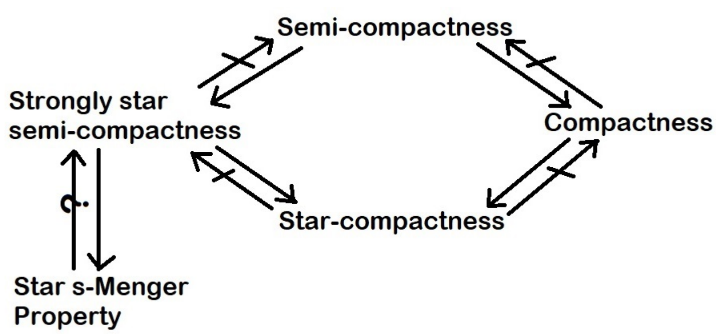

FIGURE 1. Relation diagram

Problem 3.6. Does there exists a topological space that is a star s-Manger space but not strongly star semi-compact space?

4. SUBSPACES OF STRONGLY STAR SEMI-COMPACT SPACES

Example 4.1. An arbitrary subspace of a strongly star semi-compact space may not be strongly star semi-compact.

Let $X = \mathbb{N}$. Then $\mathcal{B} = \{\{1, n\} : n \in \mathbb{N}\} \cup \{\emptyset\}$ is a base for a suitable topology on X. Suppose that τ is the topology generated by $\mathcal{B}$. For a non-trivial subset $A \subset X$, if $1 \notin A$, then $int(A) = \emptyset$ and $Cl(int(A)) = \emptyset$. So every semi-open set must contain 1. Thus for every cover $\mathcal{U}$ of X by semi-open sets, we have a finite subset $F = \{1\}$ such that $St(F,\mathcal{U}) = X$. Hence, (X,τ) is a strongly star semi-compact space.

We consider the subset $M = \mathbb{N} \setminus \{1\} \subset X$. Then (M,τ_M) is a subspace of (X,τ), where $\tau_M = \mathcal{P}(M)$. $\mathcal{V} = \{\{n\} : n \in M\}$ is a cover of M by the semi-open subsets of M. Indeed $\mathcal{V}$ is an infinite collection of pairwise disjoint subsets of M. So for every finite subset $F \subseteq M$, $St(F, \mathcal{V}) \neq M$; that is, (M, τ_M) is not a strongly star semi-compact space. Hence a subspace of a strongly star semi-compact space may not be strongly star semi-compact.

Theorem 4.2. *A semi-regular subspace of a strongly star semi-compact space is strongly star semi-compact as a subspace.*

Proof. Let M be a semi-regular subspace of a strongly star semi-compact space (X, τ) and let $\mathcal{U} = \{U_\beta : \beta \in \Lambda\}$ be a cover of M by semi-open sets in M. However, we know that semi-open sets in a semi-open subspace are semi-open in the whole space. Therefore U_β is semi-open in X for each $\beta \in \Lambda$. We also have M as semi-closed set in X. Therefore $X \backslash M$ is semi-open in X. Thus $\mathcal{V} = \mathcal{U} \cup \{X \backslash M\}$ is a cover of X by semi-open sets in X. Nevertheless, X is strongly star semi-compact. Therefore there exists a finite subset $F \subseteq X$ such that $St(F, \mathcal{V}) = X$. It follows that $M \cap F$ is a finite subset of M and $St(M \cap F, \mathcal{U}) = M$. Hence M is strongly star-semi compact as a subspace. □

5. Strongly star semi-compact subsets relative to a space

Definition 5.1. In a topological space (X, τ), a subset $G \subseteq X$ is called a strongly star semi-compact subset relative to X if for every semi-open cover $\mathcal{V}$ of G by semi-open sets in X, there exists a finite subset $F \subseteq G$ such that $St(F, \mathcal{V}) \supseteq G$.

Theorem 5.2. *A necessary and sufficient condition for an open set S of a space (X, τ) to be strongly star semi-compact as subspace is that S is semi-compact relative to X.*

Proof. Let (S, τ_S) be a strongly star semi-compact subspace of (X, τ), where $S \in \tau$, and suppose that $\mathcal{U} = \{U_\beta : \beta \in \Lambda\}$ is a cover of S by semi-open sets in X. Since $S \in \tau$, S is an α-set in X and by Lemma 2.6, $U_\beta \cap S$ is semi-open in S for each $\beta \in \Lambda$, that is, $\mathcal{U}_S = \{U_\beta \cap S : \beta \in \Lambda\}$ is a cover of S by semi-open sets in S. However (S, τ_S) is strongly star semi-compact. Therefore there exist a finite subset $F \subset S$ such that $S = St(F, \mathcal{U}_s)$. So $S \subseteq St(F, \mathcal{U})$. Hence S is a semi-compact subset relative to X.

Conversely, let S be strongly star semi-compact relative to X, where $S \in \tau$, and suppose that $\mathcal{U} = \{U_\beta : \beta \in \Lambda\}$ is a cover of S by semi-open sets in S. By Lemma 2.7, U_β is semi-open in X for all $\beta \in \Lambda$, that is, $\mathcal{U}$ is a cover of S by semi-open sets in X. However, S is strongly star semi-compact relative to X. Therefore there exists a finite subset $F \subseteq S$ with $S \subseteq St(F, \mathcal{U})$. So, $St(F, \mathcal{U}) = S$. Hence S is a strongly star semi-compact subspace. □

Theorem 5.3. *Let M and N be subsets of a topological space (X, τ) such that $M \subseteq N$ and N is an α-set in X. Then M is strongly star semi-compact relative to N if and only if M is strongly star semi-compact relative to X.*

Proof. Let M and N be subsets of X in a topological space (X, τ), where $M \subseteq N$ and N is an α-set in X. Suppose that M is strongly star semi-compact relative to the subspace (N, τ_N) and that $\mathcal{U} = \{U_\beta : \beta \in \Lambda\}$ is a cover of M by semi-open subsets in X. By Lemma 2.6, $\mathcal{U}_N = \{U_\beta : \beta \in \Lambda\}$ is a cover of M by semi-open sets in N. However M is strongly star semi-compact relative to N. Therefore there exists a finite subset $F \subseteq M$ such that $M \subseteq St(F,\mathcal{U}_N) \subseteq St(F,\mathcal{U})$. Hence X is strongly star semi-compact relative to X.

Let M and N be subsets of X in a topological space (X,τ), where $M \subseteq N$ and N is an α-set in X. Suppose that M is strongly star semi-compact relative to X and that $\mathcal{U} = \{U_\beta : \beta \in \Lambda\}$ is a cover of M by semi-open subsets in (N, τ_N). Then by Lemma 2.8, for each $\beta \in \Lambda$, there exists V_β, semi-open in X such that $U_\beta = V_\beta \cap N$. Now, $\mathcal{V} = \{V_\beta : \beta \in \Lambda\}$ is a cover of M by semi-open sets in X. However, M is strongly star semi-compact relative to (X,τ). Therefore there exists a finite subset $F \subseteq M$ such that $M \subseteq St(F,\mathcal{V})$.

Now suppose that $\mathcal{W}$ is the collection of all $V_\beta \in \mathcal{V}$ such that $F \cap V_\beta \neq \emptyset$. Therefore $M \subseteq St(F,\mathcal{V}) = \cup\mathcal{W}$, which implies $M \subseteq M \cap (\cup\mathcal{W}) = \bigcup_{V_\beta \in \mathcal{W}}(M \cap V_\beta) = \bigcup_{V_\beta \in \mathcal{W}} U_\beta$.

Now for all $V_\beta \in \mathcal{W}$, we have $F \cap V_\beta \neq \emptyset$.

Since $F \subseteq M$, so $F \cap (V_\beta \cap M) \neq \emptyset$ for all $V_\beta \in \mathcal{W}$. Therefore $F \cap U_\beta \neq \emptyset$ for all $V_\beta \in \mathcal{W}$, which implies $M \subseteq \bigcup_{V_\beta} U_\beta \subseteq St(F,\mathcal{U})$.

Hence M is strongly star semi-compact relative to (N, τ_N). □

Proposition 5.4. *In a topological space (X,τ), let $\{A_n : n = 1, 2, 3, \ldots, n\}$ be a finite collection of strongly star semi-compact subsets relative to X. Then $\bigcup_{i=1}^{n} A_n$ is also a strongly star semi-compact subset relative to X.*

Proof. The proof follows directly, hence omitted. □

Theorem 5.5. *If a function $f : (X,\tau) \to (Y,\sigma)$ is an irresolute surjection and if X is a strongly star semi-compact space, then $f(X)$ is strongly star semi-compact relative to Y.*

Proof. Let $\{V_\beta : \beta \in \Lambda\}$ be a cover of $f(X)$ by semi-open sets in Y. Then the collection $\mathcal{U} = \{U_\beta = f^{-1}(V_\beta) : \beta \in \Lambda\}$ is a cover of X by semi-open sets in X. However, X is a strongly star semi-compact space. Therefore there exist a finite subset $F \subseteq X$ such that $St(F,\mathcal{U}) = X$. By surjection, $f(F)$ is a finite subset of $f(X)$. Suppose $f(X) \nsubseteq St(f(F),\mathcal{V})$. Therefore there exists $y \in f(X)$ such that $y \notin St(f(F),\mathcal{V})$.

Therefore, for all $f(F) \cap V_\beta \neq \emptyset$, $y \notin V_\beta$ for all $\beta \in \Lambda$.

Hence, for all $f^{-1}(f(F) \cap V_\beta) \neq \emptyset$, $x \notin f(V_\beta)$ for all $\beta \in \Lambda$ and $f(x) = y$.

Thus, for all $f^{-1}(f(F)) \cap f^{-1}(V_\beta) \neq \emptyset$, $x \notin f(V_\beta)$ for all $\beta \in \Lambda$ and $f(x) = y$.

Finally, for all $F \cap U_\beta \neq \emptyset$, $x \notin U_\beta$ for all $\beta \in \Lambda$ and $f(x) = y$.

As a result, $x \notin St(F,\mathcal{U})$. Therefore $St(F,\mathcal{U}) \neq X$, which is a contradiction. Hence $f(X) \subseteq St(f(F),\mathcal{U})$. That is, $f(X)$ is strongly star semi-compact relative to Y. □

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