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GENERALIZED DERIVATIONS ON LIE IDEALS WITH ANNIHILATING ENGEL CONDITIONS

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ABSTRACT. Let $\mathcal{R}$ be a non-commutative prime ring with characteristic different from 2, let $\mathcal{U}$ be the Utumi quotient ring of $\mathcal{R}$, and let $\mathcal{C}$ be the extended centroid of $\mathcal{R}$. Let $\mathcal{G}$ be a generalized derivation on $\mathcal{R}$, let $\mathcal{L}$ be a non-central Lie ideal of $\mathcal{R}$, let $0 \neq c \in \mathcal{R}$, and let n, r, s, t be fixed positive integers. If $cu^s[\mathcal{G}(u^n), u^r]_k u^t = 0$, for all $u \in \mathcal{L}$, then one of the following properties holds:

(1) $\mathcal{R}$ satisfies s_4.
(2) There exists $\lambda \in \mathcal{C}$ such that $\mathcal{G}(\zeta) = \lambda\zeta$ for all $\zeta \in \mathcal{R}$.
(3) If $\mathcal{C}$ is a finite field, then $\mathcal{R} \cong M_l(\mathcal{C})$, an $l \times l$ matrix ring over $\mathcal{C}$ for $l > 2$.

1. Introduction and preliminaries

Throughout this article, unless otherwise stated, $\mathcal{R}$ always refers to a prime ring with center $\mathcal{Z}(\mathcal{R})$. The Utumi quotient ring of $\mathcal{R}$ is denoted by $\mathcal{U}$. The center of $\mathcal{U}$ is known as the extended centroid of $\mathcal{R}$, and it is denoted by $\mathcal{C}$. The axiomatic formulation and definition of the Utumi quotient ring can be found in [4]. The commutator of two elements u and v of $\mathcal{R}$ is denoted by $[u, v]$, and it is defined by $uv-vu$. Define $[u, v]_0 = u$, and for $k \geq 1$, the kth commutator of two elements u and v is given by $[u, v]_k = [[u, v], v]_{k-1} = \sum_{i=0}^{k}(-1)^i\binom{k}{i}v^i u v^{k-i}$. An additive mapping $d : \mathcal{R} \to \mathcal{R}$ is said to be a derivation if $d(\zeta\chi) = d(\zeta)\chi + \zeta d(\chi)$ for all $\zeta, \chi \in \mathcal{R}$. A very obvious example of a derivation on $\mathcal{R}$ is the additive map δ_p, which is defined by $\delta_p(\zeta) = [p, \zeta]$ for all $\zeta \in \mathcal{R}$, and for some fixed $p \in \mathcal{R}$, this type of derivation is known as inner derivation induced by an element p. A derivation is called an outer derivation if it is not inner. An additive mapping

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$\mathcal{F} : \mathcal{R} \to \mathcal{R}$ is said to be a generalized derivation if there exists a derivation d on $\mathcal{R}$ such that $\mathcal{F}(\zeta\chi) = F(\zeta)\chi + \zeta d(\chi)$ for all $\zeta, \chi \in \mathcal{R}$.
A series of research articles have been produced to investigate the relationship between the structure of prime ring $\mathcal{R}$ and the behavior of some specific maps defined on a particular subset of $\mathcal{R}$. The first result in this direction was proved by Posner. Posner [22] proved that if d is a derivation of a prime ring $\mathcal{R}$ such that $[d(\zeta), \zeta] \in \mathcal{Z}(\mathcal{R})$ for all $\zeta \in \mathcal{R}$, then either $d = 0$ or $\mathcal{R}$ is a commutative ring. By demonstrating the Posner's conclusion on the Lie ideal $\mathcal{L}$ of $\mathcal{R}$, Lanski [17] generalized it. Specifically, Lanski proved that that if $[d(\zeta), \zeta]_k \in \mathcal{C}$ for all $\zeta \in \mathcal{L}$ and $k > 0$, then char$(\mathcal{R})$ is different from 2 and $\mathcal{R}$ is contained $M_2(\mathcal{K})$, for some suitable field $\mathcal{K}$; equivalently, $\mathcal{R}$ satisfies s_4, the standard identity of four non-commuting variables. More recently Argaç et al. [2] generalized Lanski's result by replacing the derivation d by the generalized derivation $\mathcal{G}$. More precisely, it is proved that if $[\mathcal{G}(\zeta), \zeta]_k = 0$, for all $\zeta \in \mathcal{L}$, then either $\mathcal{G}(\zeta) = a\zeta$ with $a \in \mathcal{C}$ or $\mathcal{R}$ satisfies the standard identity s_4. The study of generalized derivations on Lie ideals and left ideals are given in [1, 6–10, 21, 23] where further references can be found out. In this article, we continue this line of investigation concerning the identity $cu^s[\mathcal{G}(u^n), u^r]_k u^t = 0$ for all $u \in \mathcal{L}$, where $r, n, s, t, k > 0$ are fixed integers and $0 \neq c \in \mathcal{R}$. We prove the following main result in this article.

Theorem 1.1. *[Main Theorem] Let $\mathcal{R}$ be a non-commutative prime ring of characteristic different from 2, let $\mathcal{U}$ be the Utumi quotient ring of $\mathcal{R}$, and let $\mathcal{C}$ be the extended centroid of $\mathcal{R}$. Let $\mathcal{G}$ be a generalized derivation on $\mathcal{R}$ and let $\mathcal{L}$ be a non-central Lie ideal of $\mathcal{R}$. Let n, s, t, r, k are fixed integers such that $cu^s[\mathcal{G}(u^n), u^r]_k u^t = 0$ for all $u \in \mathcal{L}$ and for some $0 \neq c \in \mathcal{R}$. Then one of the following properties holds:*

(1) *$\mathcal{R}$ satisfies s_4.*
(2) *There exists $\lambda \in \mathcal{C}$ such that $\mathcal{G}(\zeta) = \lambda\zeta$ for all $\zeta \in \mathcal{R}$.*
(3) *If $\mathcal{C}$ is a finite field, then $\mathcal{R} \cong M_l(\mathcal{C})$, an $l \times l$ matrix ring over $\mathcal{C}$ for $l > 2$.*

Let $\mathcal{R}$ be a prime ring and let $\mathcal{M}$ denote the collection of all pairs $(\mathcal{I}, f)$, where $\mathcal{I}$ is an ideal of $\mathcal{R}$ and f is a right module homomorphism from $\mathcal{I}$ into $\mathcal{R}$. Define a relation $\sim$ on $\mathcal{M}$ by $(\mathcal{I}, f) \sim (\mathcal{J}, g)$ for $(\mathcal{I}, f), (\mathcal{J}, g)$ in $\mathcal{M}$. If $f = g$ on some ideal $\mathcal{W}$ of $\mathcal{R}$, where $\mathcal{W} \subset \mathcal{I} \cap \mathcal{J}$. It is trivial to see that this relation is an equivalence relation. Let $\mathcal{U}$ denote the set of equivalence classes of $\mathcal{M}$. Denote the equivalence class $(\mathcal{I}, f)$ by $\tilde{f}$. Moreover, $\mathcal{U}$ forms a ring under the operations $\tilde{f} + \tilde{g} = (\mathcal{I} \cap \mathcal{J}, f + g)$ and $\tilde{f} \cdot \tilde{g} = (\mathcal{I}\mathcal{J}, fg)$, where $\tilde{f}$ is the equivalence class of $(\mathcal{I}, f)$ and $\tilde{g}$ is the equivalence class of $(\mathcal{J}, g)$. The ring $\mathcal{U}$ is the Utumi quotient ring of $\mathcal{R}$. Clearly, $\mathcal{R}$ embeds in $\mathcal{U}$.

We recall the following remarks that are useful to prove our main theorem.

Remark 1.2. Let $\mathcal{K}$ be any field and let $\mathcal{R} = M_m(\mathcal{K})$ be the algebra of all $m \times m$ matrices over $\mathcal{K}$ with $m \geq 2$. Then the unit matrix e_{ij} is an element of $[\mathcal{R}, \mathcal{R}]$ for all $1 \leq i \neq j \leq m$.Moreover, e_{ij} has entry 1 at the (i, j)th place and zero everywhere else.

Remark 1.3 ([3]). Let $\mathcal{R}$ be a prime ring and let $\mathcal{I}$ be a two-sided ideal of $\mathcal{R}$. In $\mathcal{R}$, $\mathcal{I}$, $\mathcal{U}$, if any one of these satisfies a generalized polynomial identity (GPI), then rest two will also satisfy the same polynomial identity.

Remark 1.4 ([19]). Let $\mathcal{R}$ be a prime ring and $\mathcal{I}$ a two-sided ideal of $\mathcal{R}$. In $\mathcal{R}$, $\mathcal{I}$, $\mathcal{U}$ if any one of these satisfies a generalized differential identity then rest two will also satisfy the same differential identity.

Remark 1.5 ([3]). Let $\mathcal{R}$ be a prime ring. Then any derivation δ of $\mathcal{R}$ can be extended uniquely to the derivation of $\mathcal{U}$.

Remark 1.6 ([14, Kharchenko Theorem]). Let $\mathcal{R}$ be a prime ring, let d be a nonzero derivation on $\mathcal{R}$, and let $\mathcal{I}$ be a nonzero ideal of $\mathcal{R}$. If $\mathcal{I}$ satisfies the differential identity,

$$f\big(\zeta_1, \zeta_2, \ldots, x_n, d(\zeta_1), d(\zeta_2), \ldots, d(\zeta\big) = 0$$

for any $\zeta_1, \ldots, \zeta_n \in \mathcal{I}$, then either

- $\mathcal{I}$ satisfies the GPI
$$f(\zeta_1, \zeta_2, \ldots, \zeta_n, \chi_1, \chi_2, \ldots, \chi_n) = 0$$
for all $\chi_1, \ldots, \chi_n \in \mathcal{R}$,
or
- d is $\mathcal{U}$-inner,
$$f\big(\zeta_1, \zeta_2, \ldots, \zeta_n, [p, \zeta_1], [p, \zeta_2], \ldots, [p, \zeta_n]\big) = 0.$$

Remark 1.7. Let $\mathcal{X} = \{\zeta_1, \zeta_2, \ldots\}$ represent a countable set of non-commuting indeterminates $\zeta_1, \zeta_2, \ldots$. Let $\mathcal{C}\{\mathcal{X}\}$ denote the free algebra over $\mathcal{C}$ on the set $\mathcal{X}$ and let $\mathcal{T} = \mathcal{U} *_{\mathcal{C}} \mathcal{C}\{\mathcal{X}\}$, denote the free product of the $\mathcal{C}$-algebras $\mathcal{U}$ and $\mathcal{C}\{\mathcal{X}\}$. The members of $\mathcal{T}$ are known as the generalized polynomials with coefficients in $\mathcal{U}$. Let $\mathcal{B}$ be a set of $\mathcal{C}$-independent vectors of $\mathcal{U}$. Then any $g \in \mathcal{T}$ can be expressed in the form $g = \sum_i \beta_i u_i$, where $\beta_i \in \mathcal{C}$ and u_i are $\mathcal{B}$-monomials of the form $a_0\xi_1 a_1\xi_2 a_2 \cdots \xi_n a_n$, with $a_0, a_1, \ldots, a_n \in \mathcal{B}$ and $\xi_1, \xi_2, \ldots, \xi_n \in \mathcal{X}$. Any generalized polynomial $g = \sum_i \beta_i u_i$ is trivial; that is, g is the zero element in $\mathcal{T}$ if and only if $\beta_i = 0$ for each i. Further details can be found in [5]. If each monomial of a generalized polynomial $f(\zeta_1, \ldots, \zeta_n)$ contains each ζ_i only once for $1 \le i \le n$, then $f(\zeta_1, \ldots, \zeta_n)$ is said to be multilinear polynomial.

Remark 1.8 ([13]). For $l \ge 2$, Let $M_l(\mathcal{K})$ be a $l \times l$ matrix algebra over infinite field $\mathcal{K}$. If $\mathcal{B}_1, \ldots, \mathcal{B}_k$ are matrices in $M_l(\mathcal{K})$, which are non-scalar, then there exists an invertible matrix $\mathcal{B} \in M_m(\mathcal{K})$ such that matrices $\mathcal{B}\mathcal{B}_1\mathcal{B}^{-1}, \ldots, \mathcal{B}\mathcal{B}_k\mathcal{B}^{-1}$ have all nonzero entries.

2. Inner case

Proposition 2.1. *Let $\mathcal{R}$ be a non-commutative prime ring with extended centroid $\mathcal{C}$, Utumi quotient ring $\mathcal{U}$, and char$(\mathcal{R}) \neq 2$. If*

$$c\big[u^s a u^{n+t} + u^{n+s} b u^t, u^r\big]_k = 0 \tag{2.1}$$

for all $u \in [\mathcal{R}, \mathcal{R}]$, where $n, s, t, k, r > 0$ are fixed positive integers and $0 \neq c \in \mathcal{R}$, then one of the following properties holds:

(1) *$\mathcal{R}$ satisfies s_4.*
(2) *$a, b \in \mathcal{C}$.*
(3) *If $\mathcal{C}$ is finite, then $\mathcal{R} \cong M_l(C)$ for $l > 2$.*

We use the following lemmas in what follows to prove the above proposition.

Lemma 2.2. *For $l \geq 3$, let $\mathcal{R} = M_l(\mathcal{K})$ be an $l \times l$ matrix algebra over an infinite field $\mathcal{K}$ and let char$(\mathcal{R}) \neq 2$. If*

$$c\big[u^s a u^{n+t} + u^{n+s} b u^t, u^r\big]_k = 0 \tag{2.2}$$

for all $u \in [\mathcal{R}, \mathcal{R}]$, where s, t, k, n, r are fixed positive integers and $0 \neq c \in \mathcal{R}$, then $a, b \in \mathcal{K} \cdot I_l$.

Proof. From the hypothesis,

$$\begin{aligned} 0 &= c\big[u^s a u^{n+t} + u^{n+s} b u^t, u^r\big]_k \\ &= c\sum_{i=0}^{k}(-1)^i \binom{k}{i} u^{ri}(u^s a u^{n+t} + u^{n+s} b u^t)u^{r(k-i)} \end{aligned} \tag{2.3}$$

for all $u \in [\mathcal{R}, \mathcal{R}]$. Suppose that both a and b are not central elements. Denote $a = \sum_{i,j}^{l} a_{ij}e_{ij}$, $b = \sum_{i,j}^{l} b_{ij}e_{ij}$ and $c = \sum_{i,j}^{l} c_{ij}e_{ij}$ where $a_{ij}, b_{ij}, c_{ij} \in \mathcal{K} \cdot I_l$. Since equation (2.3) is invariant under the action of any automorphism of $\mathcal{R}$ thus from Remark 1.2 all the entries of a and b are nonzero. Note that if we left multiply c by an appropriate e_{1j}, then we may assume that $c = e_{11} + \sum_{j=2}^{l} c_{1j}e_{1j}$. Assume that ϕ_i is an inner automorphism of $\mathcal{R}$ which is defined by $\phi_i(y) = (1 + c_{1i}e_{1i})y(1 - c_{1i}e_{1i})$ for $2 \leq i \leq l$. Then $\phi_{1+1}(c) = e_{11} + \sum_{j=3}^{l} c_{1j}e_{1j}$, $\phi_3\phi_2(c) = e_{11} + \sum_{j=4}^{l} c_{1j}e_{1j}, \ldots, \phi_l \ldots \phi_3\phi_2(c) = e_{11}$. Replacing a, b, c by $\phi(a), \phi(b), \phi(c)$, respectively, we may assume that $c = e_{11}$. Thus $\mathcal{R}$ satisfies the following condition:

$$\begin{aligned} 0 &= e_{11}\big[u^s a u^{n+t} + u^{n+s} b u^t, u^r\big]_k \\ &= e_{11}\sum_{i=0}^{k}(-1)^i \binom{k}{i} u^{ri}(u^s a u^{n+t} + u^{n+s} b u^t)u^{r(k-i)} \end{aligned} \tag{2.4}$$

for all $u \in [\mathcal{R}, \mathcal{R}]$.
It is clear that $n + s > s$. Since all the entries of b are nonzero, assume without loss of generality that $b_{13} \neq 0$. Let $u = \beta(e_{11} - e_{22}) + (e_{33} - e_{ll})$ for some $\beta \in \mathcal{K}$. Then, $u^j = \beta^j(e_{11} + (-1)^j e_{22}) + (e_{33} + (-1)^j e_{ll})$, $e_{11}u^j = \beta^j e_{11}$ and $u^j e_{33} = e_{33}$. Choosing $u = \beta(e_{11} - e_{22}) + (e_{33} - e_{ll})$ in equation (2.4) and right multiplying by

e_{33}, we get

$$\begin{aligned} 0 &= e_{11}\sum_{i=0}^{k}(-1)^i\binom{k}{i}u^{ri}(u^s a u^{n+t}+u^{n+s}bu^t)u^{r(k-i)}e_{33} \\ &= \sum_{i=0}^{k}(-1)^i\binom{k}{i}(\beta^{ri+s}e_{11}ae_{33}+\beta^{ri+n+s}e_{11}be_{33} \\ &= \big(\beta^{rk+s}(-1)^k a_{13}+\beta^{rk+s+n}b_{13}+\sum_{i=0}^{k-1}\beta^{ri+s}(-1)^i\binom{k}{i}a_{13} \\ &\quad +\sum_{i=0}^{k-1}\beta^{ri+s+n}(-1)^i\binom{k}{i}b_{13}\big)e_{13}, \end{aligned}$$

which implies

$$\begin{aligned} &\big(\beta^{rk+s}(-1)^k a_{13}+\beta^{rk+s+n}b_{13}+\sum_{i=0}^{k-1}\beta^{ri+s}(-1)^i\binom{k}{i}a_{13} \\ &+\sum_{i=0}^{k-1}\beta^{ri+s+n}(-1)^i\binom{k}{i}b_{13}\big)=0 \end{aligned} \tag{2.5}$$

for all $\beta\in\mathcal{K}$. Since $n+s>s$ and $\mathcal{K}$ is an infinite field, then using the Vandermonde determinant argument in equation (2.5), we obtain $b_{13}=0$, a contradiction. Thus $b\in\mathcal{K}$. Thus equation (2.2) reduces to $c[u^s a u^{n+t},u^r]_k=0$. Again using similar arguments as above, we can show that $a\in\mathcal{K}$. □

Lemma 2.3. *Let $\mathcal{R}$ be a non-commutative prime ring with Utumi quotient ring $\mathcal{U}$, extended centroid $\mathcal{C}$, and $char(\mathcal{R})\neq 2$ such that*

$$c\big[u^s a u^{n+t}+u^{n+s}bu^t,u^r\big]_k=0 \tag{2.6}$$

for all $u\in[\mathcal{R},\mathcal{R}]$, where s,t,n,r,k are fixed positive integers and $0\neq c\in\mathcal{R}$. If $\mathcal{R}$ does not satisfy any nontrivial GPI, then $a,b\in\mathcal{C}$.

Proof. Suppose that a,b are not central elements. From the hypothesis, $\mathcal{R}$ satisfies the following condition:

$$h(\zeta_1,\zeta_2)=c\big[[\zeta_1,\zeta_2]^s a[\zeta_1,\zeta_2]^{n+t}+[\zeta_1,\zeta_2]^{n+s}b[\zeta_1,\zeta_2]^t,[\zeta_1,\zeta_2]^r\big] \tag{2.7}$$

for all $\zeta_1,\zeta_2\in\mathcal{R}$. We know from Remark 1.3 that $\mathcal{R}$ and $\mathcal{U}$ satisfy the same GPIs. Therefore $\mathcal{U}$ satisfies equation (2.7). Suppose that $h(\zeta_1,\zeta_2)$ is a trivial GPI for $\mathcal{U}$. Let $\mathcal{T}=\mathcal{U}*_{\mathcal{C}}\mathcal{C}\{\zeta_1,\zeta_2\}$ be the free product of $\mathcal{U}$ and $\mathcal{C}\{\zeta_1,\zeta_2\}$, the free $\mathcal{C}$- algebra in two indeterminates, which is non-commuting. Clearly, $h(\zeta_1,\zeta_2)$ is a zero element of $\mathcal{T}$. Since $a\notin\mathcal{C}$ then $\{1,a\}$ will be linearly $\mathcal{C}$ independent and therefore equation (2.7) will be a nontrivial polynomial identity of $\mathcal{T}$ because it has a nontrivial monomial $c[\zeta_1,\zeta_2]^{s+rk}a[\zeta_1,\zeta_2]^{nt}=0$, which is a contradiction. Thus $a\in\mathcal{C}$. By similar arguments, we can show that $b\in\mathcal{C}$. □

Proof of Proposition 2.1. Throughout the proof, we assume that $\mathcal{R}$ does not satisfy s_4. If $\mathcal{R}$ does not satisfy any nontrivial GPI, then by Lemma 2.3, $a,b\in\mathcal{C}$

and we are done. Thus we may assume that equation (2.1) is a nontrivial GPI for $\mathcal{R}$. We know from Remark 1.3 that $\mathcal{R}$ and $\mathcal{U}$ satisfy the same polynomial identity. Thus equation (2.1) is also an identity for $\mathcal{U}$. Let $\mathcal{F}$ be the algebraic closure of $\mathcal{C}$ if $\mathcal{C}$ is infinite, and set $\mathcal{F} = \mathcal{C}$ if $\mathcal{C}$ is finite. Clearly, $\mathcal{U} \subseteq \mathcal{U} \otimes_{\mathcal{C}} \mathcal{F}$, and $\mathcal{U}$ is embedded in $\mathcal{U} \otimes_{\mathcal{C}} \mathcal{F}$ via the map: $x \to x \otimes 1 \in \mathcal{U} \otimes_{\mathcal{C}} \mathcal{F}$. Thus $\mathcal{U}$ is a subring of $\mathcal{U} \otimes_{\mathcal{C}} \mathcal{F}$. From [18], equation (2.1) is a GPI for $\mathcal{U} \otimes_{\mathcal{C}} \mathcal{F}$. Moreover, in the light of [11], $\mathcal{U} \otimes_{\mathcal{C}} \mathcal{F}$ is a prime ring with extended centroid $\mathcal{F}$. Hence $\mathcal{U} \otimes_{\mathcal{C}} \mathcal{F}$ satisfies equation (2.1), and it is a prime ring with extended centroid $\mathcal{F}$, which is either finite or algebraically closed. In the view of the Martindale theorem [20], $\mathcal{U} \otimes_{\mathcal{C}} \mathcal{F}$ is a prime ring with nonzero socle and $\mathcal{F}$ as its associated division ring. By [16, p. 75], $\mathcal{U} \otimes_{\mathcal{C}} \mathcal{F}$ is a dense subring of $End(\mathcal{V}_{\mathcal{F}})$, the ring of $\mathcal{F}$-linear transformations on the vector space $\mathcal{V}$ over $\mathcal{F}$. Since $\mathcal{R}$ is non-commutative therefore $dim_{\mathcal{F}}\mathcal{V} \geq 2$. If $dim_{\mathcal{F}}\mathcal{V} = l < \infty$, then $\mathcal{U} \otimes_{\mathcal{C}} \mathcal{F} \cong M_l(\mathcal{F})$. If $\mathcal{F}$ is infinite, then $a, b \in \mathcal{F}$, consequently $a, b \in \mathcal{C}$. Again if $\mathcal{F}$ is finite, then $\mathcal{F} = \mathcal{C}$ and $\mathcal{R} = \mathcal{U} = \mathcal{U} \otimes_{\mathcal{C}} \mathcal{F} = End(\mathcal{V}_{\mathcal{F}}) = M_l(\mathcal{C})$.
Next, we assume the case when $dim_{\mathcal{F}}\mathcal{V} = \infty$. Assume that both a, b are not central elements. By Martindale's theorem for any idempotent $p^2 = p \in soc(\mathcal{U} \otimes_{\mathcal{C}} \mathcal{F})$, we have $p(\mathcal{U} \otimes_{\mathcal{C}} \mathcal{F})p = M_{l'}(\mathcal{F})$, where $dim_{\mathcal{F}}\mathcal{V} = l'$. Since $a, b \notin \mathcal{F}$ there exist $h_1, h_2 \in soc(\mathcal{U} \otimes_{\mathcal{C}} \mathcal{F})$ such that $[a, h_1] \neq 0$ and $[b, h_2] \neq 0$. By Littof's theorem [**?**], there exists idempotent $p \in soc(\mathcal{U} \otimes_{\mathcal{C}} \mathcal{F})$ such that $ah_1, h_1a, bh_2, h_2b, h_1, h_2 \in p(\mathcal{U} \otimes_{\mathcal{C}} \mathcal{F})p$. Since $\mathcal{U} \otimes_{\mathcal{C}} \mathcal{F}$ satisfies the identity

$$pc\big[[\zeta_1,\zeta_2]^s a[\zeta_1,\zeta_2]^{n+t} + [\zeta_1,\zeta_2]^{n+s} b[\zeta_1,\zeta_2]^t, [\zeta_1,\zeta_2]^r\big]_k p = 0 \tag{2.8}$$

for all $\zeta_1, \zeta_2 \in \mathcal{R}$. Thus $p(\mathcal{U} \otimes_{\mathcal{C}} \mathcal{F})p$ satisfies

$$pcp\big[[\zeta_1,\zeta_2]^s pap[\zeta_1,\zeta_2]^{n+t} + [\zeta_1,\zeta_2]^{n+s} pbp[\zeta_1,\zeta_2]^t, [\zeta_1,\zeta_2]^r\big]_k = 0.$$

Thus by Lemma 2.3, $pap, pbp \in \mathcal{F}$. Hence $ah_1 = eaeh_1 = h_1eae = ah_1$, a contradiction. Thus $a \in \mathcal{F}$, which implies $a \in \mathcal{C}$. Similarly $b \in \mathcal{C}$. □

3. Proof of the main theorem 1.1

Since $\mathcal{L}$ is a non-central Lie ideal and char$(\mathcal{R}) \neq 2$, by [15, pp. 4-5], there exists a nonzero two-sided ideal $\mathcal{I}$ of $\mathcal{R}$ such that $\mathcal{I} \subseteq \mathcal{L}$ and $0 \neq [\mathcal{I}, \mathcal{R}] \subseteq \mathcal{L}$. Therefore we have

$$c[\zeta_1,\zeta_2]^s\big[\mathcal{G}([[\zeta_1,\zeta_2]^n), [\zeta_1,\zeta_2]^r\big]_k[\zeta_1,\zeta_2]^t = 0, \tag{3.1}$$

for all $[\zeta_1, \zeta_2] \in [\mathcal{I}, \mathcal{I}]$. Since $\mathcal{R}$ and $\mathcal{I}$ satisfy the same GPI (by Remark 1.3), therefore $\mathcal{R}$ satisfies equation (3.1) for all $\zeta_1, \zeta_2 \in \mathcal{R}$. By Remark 1.4, the generalized derivation $\mathcal{G}$ has the form $\mathcal{G}(\zeta) = a\zeta + d(\zeta)$ for some $a \in \mathcal{U}$ and a derivation d on $\mathcal{U}$.

Case I. If d is an inner derivation induced by an element $w \in \mathcal{U}$, then from equation (3.1), we have

$$c[\zeta_1,\zeta_2]^s\big[(a+w)[\zeta_1,\zeta_2]^n - [\zeta_1,\zeta_2]^n w, [\zeta_1,\zeta_2]^r\big]_k[\zeta_1,\zeta_2]^t = 0,$$

which implies

$$c\big[[\zeta_1,\zeta_2]^s(a+w)[\zeta_1,\zeta_2]^{n+t} - [\zeta_1,\zeta_2]^{n+s} w[\zeta_1,\zeta_2]^t, [\zeta_1,\zeta_2]^r\big]_k = 0$$

for all $\zeta_1, \zeta_2 \in \mathcal{R}$. Thus by Proposition 2.1, we have one of the following conditions:

(1) $\mathcal{R}$ satisfies s_4.
(2) $a + w, w \in \mathcal{C}$; that is, $a, w \in C$ so that $d = 0$ and $\mathcal{G}(\zeta) = a\zeta$,
(3) If $\mathcal{C}$ is finite, then $\mathcal{R} \cong M_l(\mathcal{C})$, $l \times l$ matrix rings and $l > 2$.

Case II: If d is an outer derivation, then equation (3.1) becomes

$$c[\zeta_1,\zeta_2]^s\Big[a[\zeta_1,\zeta_2]^n + \sum_{i+j=n-1}[\zeta_1,\zeta_2]^i([d(\zeta_1),\zeta_2] \\ +[\zeta_1,d(\zeta_2)])[\zeta_1,\zeta_2]^j, [\zeta_1,\zeta_2]^r\Big]_k[\zeta_1,\zeta_2]^t = 0, \tag{3.2}$$

for all $\zeta_1, \zeta_2 \in \mathcal{U}$. By Kharchenko's theorem, expression (3.2) can be written as

$$c[\zeta_1,\zeta_2]^s\Big[a[\zeta_1,\zeta_2]^n + \sum_{i+j=n-1}[\zeta_1,\zeta_2]^i([s_1,\zeta_2] \\ +[\zeta_1,s_2])[\zeta_1,\zeta_2]^j, [\zeta_1,\zeta_2]^r\Big]_k[\zeta_1,\zeta_2]^t = 0,$$

for all $\zeta_1, \zeta_2, s_1, s_2 \in \mathcal{R}$. In particular, $\mathcal{R}$ satisfies the blended component

$$c[\zeta_1,\zeta_2]^s\Big[\sum_{i+j=n-1}[\zeta_1,\zeta_2]^i([s_1,\zeta_2]+[\zeta_1,s_2])[\zeta_1,\zeta_2]^j, [\zeta_1,\zeta_2]^r\Big]_k[\zeta_1,\zeta_2]^t = 0.$$

By Remark 1.6, for $i = 1, 2$, we can replace s_i by $[p, \zeta_i]$, where $p \in \mathcal{U} \setminus \mathcal{C}$. Thus $\mathcal{R}$ satisfies

$$c[\zeta_1,\zeta_2]^s\Big[\sum_{i+j=n-1}[\zeta_1,\zeta_2]^i([[p,\zeta_1],\zeta_2]+[\zeta_1,[p,\zeta_2]][\zeta_1,\zeta_2]^j, [\zeta_1,\zeta_2]^r\Big]_k[\zeta_1,\zeta_2]^t = 0.$$

The above relation implies

$$c[\zeta_1,\zeta_2]^s\Big[[p,[\zeta_1,\zeta_2]^n], [\zeta_1,\zeta_2]^r\Big]_k[\zeta_1,\zeta_2]^t = 0.$$

Thus from Proposition 2.1, $p \in C$, which is a contradiction.

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