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ON S-FINITE CONDUCTOR RINGS

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Abstract. Let R be a commutative ring with nonzero identity and let $S \subseteq R$ be a multiplicatively closed subset of R. In this paper, we introduce and study S-finite conductor rings. Moreover, R is said to be an S-finite conductor ring if $(0 : a)$ and $Ra \cap Rb$ are S-finite ideals of R for each $a, b \in R$. Some basic properties of S-finite conductor rings are studied. For instance, we give necessary and sufficient conditions for a ring to be S-finite conductor. Also, we prove that every pre-Schreier S-finite conductor domain is an S-GCD domain and that the converse is true for some particular cases of S. Furthermore, we examine the stability of these rings in localization and study the possible transfer to direct product, trivial ring extension, and amalgamated algebra along an ideal.

1. Introduction

Throughout this paper, all rings are assumed to be commutative with nonzero identity and all modules are nonzero unital. Let R denote such as a ring and S such as a multiplicatively closed subset of R. Also, $Reg(R)$ denotes the set of regular elements of the ring R; $Q(R) := R_{Reg(R)}$, the total quotient ring of R. For an ideal I of R and an element $a \in R$, we denote by $(I : a) := \{x \in R \mid xa \subseteq I\}$ the conductor of Ra into I. Also, for a (fractional) ideal I of R, $I^{-1} := \{x \in Q(R) \mid xI \subseteq R\}$ and $I_v := (I^{-1})^{-1}$. Recall that an R-module M is called a finitely presented R-module if there is an exact sequence of R-modules $F_1 \to F_0 \to M \to 0$ such that both F_0 and F_1 are finitely generated free R-modules. A finitely generated R-module M is said to be a coherent R-module if every finitely generated R-submodule of M is a finitely presented R-module; and a ring R is called a coherent ring if R is coherent as an R-module. An

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excellent summary of work done on coherence up to 1989 can be found in [14]. On the other hand, Zafrullah [21] defined finite conductor domains as a new generalization of coherent domains. Moreover, Glaz [15] extended the definition of a finite conductor domains to rings with zero divisors. A ring R is called a finite conductor ring if $Ra \cap Rb$ and $(0 : a)$ are finitely generated ideals of R for every $a, b \in R$. Anderson and Dumitrescu [1] introduced the concept of S-finite modules as follows: an R-module M is called an S-finite if there exist a finitely generated R-submodule N of M and $s \in S$ such that $sM \subseteq N$. Recently, Bennis and El Hajoui [7] investigated the S-versions of finitely presented modules and coherent modules, which are called, respectively, S-finitely presented modules and S-coherent modules. An R-module M is called an S-finitely presented module for some multiplicatively closed subset S of R if there exists an exact sequence of R-modules $0 \to K \to F \to M \to 0$, where F is a finitely generated free R-module and K is an S-finite R-module. Moreover, an R-module M is said to be S-coherent, if it is finitely generated and every finitely generated submodule of M is S-finitely presented. They showed that the S-coherent rings have a characterization similar to the classical one given by Chase for coherent rings (see [7, Theorem 3.8]). Recall from [16] that a nonzero ideal I of a domain R is said to be S-v-principal if there exist $s \in S$ and $a \in R$ such that $sI \subseteq aR \subseteq I_v$. Also, Hamed and Hizem [16] introduced S-GCD domains. A domain R is called an S-GCD domain if each finitely generated nonzero ideal of R is S-v-principal. Additional information about S-GCD domains can be found in [2].

Some of our results use the $R \propto M$ construction. Let R be a ring and let M be an R-module. Then $R \propto M$, the *trivial* (*ring*) *extension of* R *by* M, is the ring whose additive structure is that of the external direct sum $R \oplus M$ and whose multiplication is defined by $(r_1, m_1)(r_2, m_2) := (r_1r_2, r_1m_2 + r_2m_1)$ for all $r_1, r_2 \in R$ and all $m_1, m_2 \in M$. The basic properties of trivial ring extensions are summarized in the books [14, 17]. Mainly, trivial ring extensions have been useful for solving many open problems and conjectures in both commutative and non-commutative ring theory. See, for instance, [3, 5, 6, 12, 19, 20].

In present note, we define S-finite conductor rings as a new generalization of finite conductor rings, S-coherent rings, and S-GCD domains. If R is a ring and S is a multiplicatively closed subset of R, then we say that R is an S-finite conductor ring if $Ra \cap Rb$ and $(0 : a)$ are S-finite ideals of R for any $a, b \in R$. In Theorem 2.5, we give some characterizations of S-finite conductor rings. In addition, we show that every pre-Schreier S-finite conductor domain is an S-finite conductor domain (see Theorem 2.6). Moreover, we prove that if R is an S-finite conductor ring, then R_S is a finite conductor ring, and the converse is not true in general (see Proposition 2.7 and Example 2.8). Also, we characterize finite conductor rings in term of S-finite conductor rings (see Theorem 2.9). Moreover, we study some particular cases of the trivial ring extension and examine conditions under which $R \propto M$ is an $(S \propto M)$-finite conductor ring (see Theorem 2.12 and Proposition 2.13). Finally, we investigate the S-finite conductor property that the amalgamation $A \bowtie^f J$ might inherit from the ring A for some classes of ideals J and homomorphisms f.

2. Main results

Definition 2.1. Let R be a ring and let S be a multiplicatively closed subset of R. We say that R is an S-finite conductor ring if $(0 : a)$ and $Ra \cap Rb$ are S-finite ideals for each $a, b \in R$.

Remark 2.2. Let R be a ring and let S be a multiplicatively closed subset of R.

(1) If R is an S-coherent ring, then R is an S-finite conductor ring.
(2) If R is an S-GCD domain, then R is an S-finite conductor domain.
(3) If R is a finite conductor ring, then R is an S-finite conductor ring. The converse is true if $S \subseteq U(R)$, the set of units of R.

The following example shows that the converse of Remark 2.2(3) is not true, in general.

Example 2.3. Let M be a countable direct sum of copies of $\mathbb{Z}/2\mathbb{Z}$, let $R = \mathbb{Z} \propto M$ and let $S = \{(2,0)^n \mid n \in \mathbb{N}\}$ be a multiplicatively closed subset of R. Then, by Remark [7, Remark 3.4], R is an S-coherent ring (so S-finite conductor ring). However, R is not a finite conductor since $(0 : (2,0)) = 0 \propto M$ is not finitely generated.

For each multiplicatively closed subset $S \subseteq R$, $S^* := \{a \in R \mid \frac{a}{1} \text{ is a unit of } R_S\}$ denotes the saturation of S. Note that S^* is a multiplicatively closed subset containing S.

Proposition 2.4. *Let R be a ring. Then the following statements hold:*

(1) *If $S_1 \subseteq S_2$ are multiplicatively closed subsets of R and R is an S_1-finite conductor ring, then R is an S_2-finite conductor ring.*
(2) *If S is a multiplicatively closed subset of R, then R is an S-finite conductor ring if and only if R is an S^*-finite conductor ring, where S^* is the saturation of S.*

Proof. Clear. □

Let I be an ideal of a ring R. We denote by $\mu(I)$ the cardinality of a minimal set of generators of I. The following theorem gives equivalent conditions for a ring to be an S-finite conductor ring. It is well known that if we take S to be a subset of the group of units of R, then these conditions are all equivalent to R being a finite conductor ring.

Theorem 2.5. *Let R be a ring and let S be a multiplicatively closed subset of R. Then the following statements are equivalent:*

(1) *R is an S-finite conductor ring.*
(2) *Every (fractional) ideal of R with $\mu(I) \leq 2$ is S-finitely presented.*
(3) *$(a : b)$ is an S-finite ideal of R for each $a, b \in Q(R)$.*

In addition, if R is a domain, then the above three conditions are equivalent to

(4) *I^{-1} is S-finite for any (fractional) ideal I of R with $\mu(I) \leq 2$.*

Proof. (1) ⇒ (2) Assume that R is an S-finite conductor ring. Let I be an ideal of R such that $\mu(I) \leq 2$. If $I = Ra$ for some $a \in R$, then the result

follows from the standard exact sequence $0 \to (0:a) \to R \to I \to 0$. Now, we suppose that $I = Ra + Rb$, where $a, b \in R$. Fix an exact sequence of R-modules $0 \to Ra \cap Rb \to Ra \bigoplus Rb \to I \to 0$. By the hypothesis, we get that I is an S-finitely presented ideal. Now, we suppose that I is a fractional ideal of R. Then, there is $a \in Reg(R)$ such that $aI \subseteq R$. Since $\mu(I) \leq 2$, we have $I \cong aI$ is S-finitely presented.
(2) $\Rightarrow$ (3) Let $a, b \in R$ and let $0 \to K \to R^2 \to I \to 0$ be the canonical S-finite presentation of $I := Ra + Rb$. Also, we define the R-module epimorphism $u : K \to (a : b)$ by $u(r, s) = -s$ for each $(r, s) \in K$. As K is an S-finite R-module, then $(a : b)$ is an S-finite ideal of R.
(3) $\Rightarrow$ (1) It suffices to prove that $Ra \cap Rb$ is an S-finite ideal of R for each $a, b \in R$. This, in turn, follows from the fact that $Ra \cap Rb = (a : b)b$.
Now, we will prove that (4) $\Leftrightarrow$ (1) under the additional hypothesis that R is a domain. Assume that R is an S-finite conductor domain. Let I be an ideal of R such that $\mu(I) \leq 2$. If $I = Ra$ for some $a \in R$, then $aI^{-1} = R$ and so I^{-1} is finitely generated. Now, we suppose that $I = Ra + Rb$ for some elements $a, b \in R \setminus \{0\}$. One can see that $I^{-1} = R\frac{1}{a} \cap R\frac{1}{b}$. It follows that $abI^{-1} = Ra \cap Rb$ is S-finite, and thus I^{-1} is S-finite. The converse is clear. □

Let R be a domain. Recall that an element x of R is called *primal* if $x \mid ab$ implies that $x = rs$, where $r \mid a$ and $s \mid b$. Also, an element x of R is said to be a *completely primal* if every factor of x is primal. According to [8], a domain R is called a *pre-Schreier domain* if every element of R is primal.

Theorem 2.6. *If R be a pre-Schreier S-finite conductor domain for some multiplicatively closed subset S of R, then R is an S-GCD domain. The converse is true if S is generated by completely primal elements of R.*

Proof. Let $a, b \in R \setminus \{0\}$. Since R is an S-finite conductor domain, then there exist $s \in S$ and a finite subset $\{x_1, \dots, x_n\}$ of R such that $s(Ra \cap Rb) \subseteq (x_1, \dots, x_n) \subseteq Ra \cap Rb$. Also, by [22, Theorem 1.1], there exists $d \in Ra \cap Rb$ such that $d \mid x_i$ for each $i = 1, \dots, n$. It follows that $s(Ra \cap Rb) \subseteq Rd \subseteq Ra \cap Rb$, and hence R is an S-GCD domain. The converse follows from Remark 2.2, [4, Theorem 4.2], and [2, Theorem 2.2]. □

Proposition 2.7. *Assume that R is an S-finite conductor ring for some multiplicatively closed subset S of R. Then R_S is a finite conductor ring.*

Proof. Let $\frac{a}{s} \in R_S$. As $t(0 : a) \subseteq (x_1, \dots, x_n) \subseteq (0 : a)$ for some $t \in S$ and $\{x_1, \dots, x_n\} \subseteq R$, an easy calculation reveals that $\{\frac{x_1}{1}, \dots, \frac{x_n}{1}\}$ is a finite generating set for $(0 : \frac{a}{s})$ as an ideal of R_S. Now, we let $\frac{b}{t}, \frac{b'}{t'} \in R_S$. Since R is an S-finite conductor ring, there exist $u \in S$ and a finite subset $\{x_1, \dots, x_n\} \subseteq R$ such that $u(Rb \cap Rb') \subseteq (x_1, \dots, x_n) \subseteq Rb \cap Rb'$. One can prove that $R_S\frac{b}{t} \cap R_S\frac{b'}{t'}$ is generated by $\{\frac{x_1}{1}, \dots, \frac{x_n}{1}\}$, as needed. □

We next give an example of a domain R and a multiplicatively closed subset S such that R_S is a finite conductor domain, but R is not an S-finite conductor domain.

Example 2.8 ([2, Example 3.1])**.** Let $A = \mathbb{Z}_{(p)} + Y\mathbb{Q}[[Y]]$, where $\mathbb{Z}$ is the ring of integers, p a prime number, $\mathbb{Q}$ the field of rational numbers, and Y an indeterminate over $\mathbb{Q}$. Let $S = \{p^n \mid n \in \mathbb{N}\}$, and note that S is a multiplicatively closed subset of A such that $A_S = \mathbb{Q}[[Y]]$. Now, let $R = A + XA_S[[X]] = \mathbb{Z}_{(p)} + Y\mathbb{Q}[[Y]] + X\mathbb{Q}[[X,Y]]$. So, $R_S = \mathbb{Q}[[X,Y]]$ is a GCD domain (so a finite conductor domain). On the other hand, by [2, Example 3.1], R is a pre-Schreier domain that is not an S-GCD domain, which gives that R is not an S-finite conductor domain by Theorem 2.6.

Let P be a prime ideal of a ring R. If R is an $(R \setminus P)$-finite conductor ring, then we say that R is a *P-finite conductor ring*.

Theorem 2.9. *Let R be a ring. Then the following conditions are equivalent:*

(1) *R is a finite conductor ring.*
(2) *R is a P-finite conductor ring for each $P \in Spec(R)$.*
(3) *R is an $\mathfrak{m}$-finite conductor ring for each $\mathfrak{m} \in Max(R)$.*

Proof. (1) $\Rightarrow$ (2) It follows from Remark 2.2.
(2) $\Rightarrow$ (3) Clear.
(3) $\Rightarrow$ (1) Assume that R is an $\mathfrak{m}$-finite conductor ring for all maximal ideals $\mathfrak{m}$ of R. Let $a, b \in R$. So, $(0 : a)$ and $Ra \cap Rb$ are $\mathfrak{m}$-finite ideals for every $\mathfrak{m} \in Max(R)$. Hence, by the proof of [1, Proposition 12], $(0 : a)$ and $Ra \cap Rb$ are finitely generated ideals of R. Thus R is a finite conductor ring. $\square$

Proposition 2.10. *Let $\{R_i \mid 1 \leq i \leq n\}$ be a finite family of rings and let S_i be a multiplicatively closed subset of R_i. Set $R := R_1 \times \cdots \times R_n$ and $S := S_1 \times \cdots \times S_n$. Then R is an S-finite conductor ring if and only if R_i is an S_i-finite conductor ring for each $i = 1, \ldots, n$.*

Proof. It suffices to prove the converse. Let $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in R$. Then $(0 : a) = (0 : a_1) \times \cdots \times (0 : a_n)$ and $Ra \cap Rb = (R_1a_1 \cap R_1b_1) \times \cdots \times (R_na_n \cap R_nb_n)$. As R_i is an S_i-finite conductor ring for each $i = 1, \ldots, n$, we must have $(0 : a)$ and $Ra \cap Rb$ are S-finite ideals of R. Thus R is an S-finite conductor ring. $\square$

Proposition 2.11. *Let R be a ring and let S be a multiplicatively closed subset of R. If R is an S-finite conductor ring, then R_T is an S_T-finite conductor ring for every multiplicatively closed subset T of R.*

Proof. Let T be a multiplicatively closed subset of R and let $a, b \in R_T$. So, there exist $x, y \in R$ such that $R_Ta = R_Tx$ and $R_Tb = R_Ty$. Since R_T is a flat R-module, then $R_Ta \cap R_Tb = R_Tx \cap R_Ty = (Rx \cap Ry)R_T$ and $(0 : a) = (0 : x)R_T$. As R is an S-finite conductor ring, we conclude that $R_Ta \cap R_Tb$ and $(0 : a)$ are S_T-finite ideals of R_T. Thus R_T is an S_T-finite conductor ring, as needed. $\square$

Theorem 2.12. *Let $(R, \mathfrak{m})$ be a local ring, let S be a multiplicatively closed subset of R, and let M be an R-module such that $\mathfrak{m}M = (0)$. Then $R \propto M$ is an $(S \propto M)$-finite conductor ring if and only if R is an S-finite conductor ring, $\mathfrak{m}$ is an S-finite ideal of R, and M is an S-finite R-module.*

Proof. Suppose that $R \propto M$ is an $(S \propto M)$-finite conductor ring, and let $a \in R$. Then $(0 : (a, 0)) = (0 : a) \propto N$ is an $(S \propto M)$-finite ideal of $R \propto M$, where $N := \{m \in M \mid am = 0\}$. This implies that $(0 : a)$ is an S-finite ideal of R. Also, let $a, b \in R$. We will prove that $Ra \cap Rb$ is an S-finite ideal of R. This, in turn, follows from the fact that $(R \propto M)(a, 0) \cap (R \propto M)(b, 0) = (Ra \cap Rb) \propto 0$ is an $(S \propto M)$-finite ideal of $R \propto M$. On the other hand, let $0 \neq m \in M$. So, $(0 : (0, m)) = \mathfrak{m} \propto M$ is an $(S \propto M)$-finite ideal of $R \propto M$, which gives that $\mathfrak{m}$ is an S-finite ideal of R and that M is an S-finite R-module.

Conversely, let $(a, m) \in R \propto M$. If a is invertible in R, then (a, m) is invertible in $R \propto M$. Then, without loss of generality, we may assume that $a \in \mathfrak{m}$. Hence $(0 : (a, m)) = \{(b, m') \in \mathfrak{m} \propto M \mid ab = 0\}$. Moreover, we have $(0 : (a, m)) = \mathfrak{m} \propto M$ if $a = 0$ and $(0 : (a, m)) = (0 : a) \propto M$ if $a \neq 0$. In the both cases, we conclude that $(0 : (a, m))$ is an S-finite ideal. Now, let $(a, m), (b, m') \in R \propto M$, where $a, b \in \mathfrak{m}$, and set $J = (R \propto M)(a, m) \cap (R \propto M)(b, m')$. Assume that $J \subsetneqq (R \propto M)(a, m)$ and $J \subsetneqq (R \propto M)(b, m')$. Let $(c, f) \in J$. So, there are $(a_1, e_1), (b_1, f_1) \in \mathfrak{m} \propto M$ such that $(c, f) = (a_1, e_1)(a, m) = (b_1, f_1)(b, m')$. Hence $(c, f) = (a_1 a, 0) = (b_1 b, 0)$. It follows that $J = (Ra \propto 0) \cap (Rb \propto 0) = (Ra \cap Rb) \propto 0$ is an $(S \propto M)$-finite ideal of $R \propto M$. Thus $R \propto M$ is an $(S \propto M)$-finite conductor ring. □

Next, we explore a different context, namely, the trivial ring extension of a domain by its quotient field.

Proposition 2.13. *Let R be a domain that is not a field and let $K = Q(R)$. Then $R \propto K$ is never an $(S \propto K)$-finite conductor ring for every multiplicatively closed subset S of R.*

Proof. The result follows from $(0 : (0, x)) = 0 \propto K$ is not an $(S \propto K)$-finite ideal for each $x \in K \setminus \{0\}$. □

Let A and B be two rings, let J be an ideal of B, and let $f : A \to B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f B = \{(a, f(a) + j) \mid a \in A \text{ and } j \in J\},$$

called the amalgamation of A with B along J with respect to f. This construction has been first introduced and studied by D'Anna, Finocchiaro, and Fontana [9, 10, 13]. In particular, if I is an ideal of A and $id_A : A \to A$ is the identity homomorphism on A, then $A \bowtie I = A \bowtie^{id_A} I = \{(a, a + i) \mid a \in R \text{ and } i \in I\}$ is the amalgamated duplication of A along J (introduced and studied by D'Anna and Fontana in [11]).

Theorem 2.14. *Let $(A, \mathfrak{m})$ be a local ring, let $f : A \to B$ be a ring homomorphism, let S be a multiplicatively closed subset of A such that $S \cap \ker(f) = \emptyset$, and let J be a proper ideal of B. Let $R = A \bowtie^f J$ and let $S' = \{(s, f(s)) \mid s \in S\}$.*

(1) *If R is an S'-finite conductor ring, then A is an S-finite conductor ring.*
(2) *Assume that $f(\mathfrak{m})J = (0)$, in which $J \subseteq Rad(B)$, the Jacobson radical of B. Then the following assertions are equivalent:*
   (a) *$A \bowtie^f J$ is an S'-finite conductor ring.*

(b) *A is an S-finite conductor ring, $\mathfrak{m}$ and $\mathfrak{m}a \cap \mathfrak{m}b$ are S-finite ideals of A for all $a, b \in \mathfrak{m}$, and J, $Jk \cap Jl$, and $(0 : k) \cap J$ are $f(S)$-finite of $f(A) + J$ for all $k, l \in J$.*

(c) *$\mathfrak{m}$, $(0 : a)$, and $\mathfrak{m}a \cap \mathfrak{m}b$ are S-finite ideals of A for all $a, b \in \mathfrak{m}$, and J, $Jk \cap Jl$, and $(0 : k) \cap J$ are $f(S)$-finite ideals of $f(A) + J$ for all $k, l \in J$.*

In order to prove the theorem, we start by giving some lemmas that prepares the way.

Lemma 2.15. *Let $f : A \longrightarrow B$ be a ring homomorphism, let S be a multiplicatively closed subset of A, and let J be an ideal of B. Let I and K be two ideals of A and B, respectively, such that $K \subseteq J$.*

(1) *Assume that $f(I)J \subseteq K$. If I is an S-finite ideal of A and K is an $f(S)$-finite ideal of $f(A)+J$, then $I \bowtie^f K$ is an S'-finite ideal of $A \bowtie^f J$.*

(2) *Assume that $(A, \mathfrak{m})$ is a local ring such that $f(\mathfrak{m})J = (0)$. Then $I \bowtie^f K$ is an S'-finite ideal of $A \bowtie^f J$ if and only if I is an S-finite ideal of A and K is an $f(S)$-finite ideal of $f(A) + J$.*

Proof. (1) Since I is an S-finite ideal of A, there exist $s_1 \in S$ and $a_1, \dots, a_n \in I$ such that $sI \subseteq (a_1, \dots, a_n) \subseteq I$. Since K is an $f(S)$-finite ideal of $f(A)+J$, there exist $s_2 \in S$ and $k_1, \dots, k_m \in K$ such that $f(s_2)K \subseteq (k_1, \dots, k_m)f(A) + J \subseteq K$. Put $s = s_1 s_2$. Then for each $a \in I$, there exist $\alpha_1, \dots, \alpha_n \in A$ such that $sa = \sum_{i=1}^{n} \alpha_i a_i$, and for each $k \in K$, we can find $\beta_1, \dots, \beta_m \in A$ and $l_1, \dots, l_m \in J$ such that $f(s)k = \sum_{j=1}^{m} (f(\beta_j) + l_j)k_j$. Hence for each $(a, f(a) + k) \in I \bowtie^f K$, we have

$$\begin{aligned}(s, f(s))(a, f(a) + k) &= (sa, f(sa) + f(s)k) \\ &= (\sum_{i=1}^{n} \alpha_i a_i, \sum_{i=1}^{n} f(\alpha_i) f(a_i) + \sum_{j=1}^{m} (f(\beta_j) + l_j)k_j) \\ &= \sum_{i=1}^{n} (\alpha_i, f(\alpha_i))(a_i, f(a_i)) + \sum_{j=1}^{m} (\beta_j, f(\beta_j) + l_j)(0, k_j) \\ &\in \sum_{i=1}^{n} (A \bowtie^f J)(a_i, f(a_i)) + \sum_{j=1}^{m} (A \bowtie^f J)(0, k_j).\end{aligned}$$

Therefore we obtain

$$(s, f(s))I \bowtie^f K \subseteq (\{((a_i, f(a_i)), (0, k_j) \mid 1 \leq i \leq n, 1 \leq j \leq m\}) \subseteq I \bowtie^f K,$$

which implies that $I \bowtie^f K$ is S'-finite.

(2) By (1) it remains to show that if $I \bowtie^f J$ is S'-finite, then I (resp., K) is S-finite (resp., $f(S)$-finite) ideal of A (resp., $f(A) + J$). Since $I \bowtie^f J$ is S'-finite, then there exist $s \in S$ and $(a_1, f(a_1) + j_1), \dots, (a_n, f(a_n) + j_n)$ such that $(s, f(s))I \bowtie^f K \subseteq ((a_1, f(a_1) + k_1), \dots, (a_n, f(a_n) + k_n)) \subseteq I \bowtie^f K$. We get easily that $sI \subseteq (a_1, \dots, a_n) \subseteq I$. Let $k \in K$. Then $(0, k) \in I \bowtie^f K$. So there

exist $(\alpha_1, f(\alpha_1)+l_1), \dots, (\alpha_n, f(\alpha_n)+l_n) \in A \bowtie^f J$ such that

$$\begin{aligned}(s, f(s))(0,k) &= \sum_{i=1}^{n}(\alpha_i, f(\alpha_i)+l_i)(a_i, f(a_i)+k_i) \\ &= \sum_{i=1}^{n}(\alpha_i a_i, f(\alpha_i)f(a_i)+f(a_i)l_i+(f(\alpha_i)+l_i)k_i.\end{aligned}$$

So, $f(s)k = \sum_{i=1}^{n}(f(\alpha_i)+l_i)k_i$ since $a_i \in \mathfrak{m}$ for each $i = 1, \dots, n$ and $f(\mathfrak{m})J = (0)$. Therefore $f(s)k \in \sum_{i=1}^{n}(f(A)+J)k_i$, as desired. □

Lemma 2.16. *Let $f : A \longrightarrow B$ be a ring homomorphism, let S be a multiplicatively closed subset of A, let J be a proper ideal of B, and let $a, b \in A$. If $A \bowtie^f J(a, f(a)) \cap A \bowtie^f J(b, f(b))$ is an S'-finite ideal of $A \bowtie^f J$, then $Aa \cap Ab$ is an S-finite ideal of A.*

Proof. Assume that $A \bowtie^f J(a, f(a)) \cap A \bowtie^f J(b, f(b))$ is an S'-finite ideal of $A \bowtie^f J$. Then there exist $s \in S$ and $(a_1, f(a_1)+k_1), \dots, (a_n, f(a_n)+k_n) \in A \bowtie^f J$ such that $(s, f(s))A \bowtie^f J(a, f(a)) \cap A \bowtie^f J(b, f(b)) \subseteq ((a_1, f(a_1)+k_1), \dots, (a_n, f(a_n)+k_n)) \subseteq A \bowtie^f J(a, f(a)) \cap A \bowtie^f J(b, f(b))$. Let $x \in Aa \cap Ab$. Then $x = \alpha a = \beta b$, where $\alpha, \beta \in A$. So $(s, f(s))(x, f(x)) = (s, f(s))(\alpha, f(\alpha))(a, f(a)) \in (s, f(s))A \bowtie^f J(a, f(a))$. Also, we have

$$(s, f(s))(x, f(x)) = (s, f(s))(\beta, f(\beta))(b, f(b)) \in (s, f(s))A \bowtie^f J(b, f(b)).$$

Hence $(s, f(s))(x, f(x)) \in (s, f(s))A \bowtie^f J(a, f(a)) \cap A \bowtie^f J(b, f(b))$. So $(s, f(s))(x, f(x)) \in \sum_{i=1}^{n} A \bowtie^f J(a_i, f(a_i)+k_i)$. Thus, we get easily that $sx \in \sum_{i=1}^{n} Aa_i$. Hence, we obtain $s(Aa \cap Ab) \subseteq (a_1, \dots, a_n) \subseteq Aa \cap Ab$, which says that $Aa \cap Ab$ is an S-finite ideal of A. □

It was shown that if $(A, \mathfrak{m})$ is a local ring, $f : A \to B$ is a ring homomorphism, and J an ideal of B such that $J \subseteq Rad(B)$, then $U(A \bowtie^f J) = (A \setminus \mathfrak{m}) \bowtie^f J$ (see [18, Lemma 2.5]).

Lemma 2.17. *Let $(A, \mathfrak{m})$ be a local ring, let S be a multiplicatively closed subset of A, let $f : A \longrightarrow B$ be a ring homomorphism, and let J be a proper ideal of B.*

1. *If $(0 : c)$ is an S'-finite of $A \bowtie^f J$ for each $c \in A \bowtie^f J$, then $(0 : a)$ is an S-finite ideal of A for each $a \in A$.*
2. *Assume that $J \subset Rad(B)$ and that $f(\mathfrak{m})J = (0)$. Then $(0 : c)$ is an S'-finite ideal of $A \bowtie^f J$ for each $c \in A \bowtie^f J$ if and only if $\mathfrak{m}$ and $(0 : a)$ are S-finite ideals of A for each $a \in A$, and J and $(0 : k) \cap J$ are $f(S)$-finite ideals of $f(A)+J$ for each $k \in J$.*

Proof. (1) Assume that $(0 : c)$ is an S'-finite ideal of $A \bowtie^f J$. Let $a \in A$. First, assume that $a \notin \mathfrak{m}$. We get $(0 : a) = 0$. Then there is nothing to prove, so assume that $a \in \mathfrak{m}$. Set $c = (a, f(a)) \in A \bowtie^f J$. We can easily show that $(0 : c) = (0 : a) \bowtie^f ((0 : f(a)) \cap J)$. So, by Lemma 2.15(1), $(0 : a)$ is an S-finite ideal of A.

(2) Assume that $(0 : c)$ is an S'-finite ideal of $A \bowtie^f J$. By (1), $(0 : a)$ is S-finite for $a \in A$. Let $k \in J$, and set $c = (0 : k) \in A \bowtie^f J$. We verify easily that $(0 : c) = \mathfrak{m} \bowtie^f ((0 : k) \cap J)$. So $\mathfrak{m}$ is an S-finite ideal of A and $(0 : k) \cap J$

is an $f(S)$-finite ideal of $f(A)+J$ by Lemma 2.15(2). Also, let $a \in A$. Set $c_1 = ((a, f(a)) \in A \bowtie^f J$. Clearly $(0 : c_1) = (0 : a) \bowtie^f J$. So J is an $f(S)$-finite of $f(A)+J$ by Lemma 2.15(2). Conversely, let $(0,0) \neq c = (a, f(a)+j) \in A \bowtie^f J$. Without loss of generality, we may assume that $a \in \mathfrak{m}$. Three cases are then possible:

Case 1: If $a = 0$, then $(0 : c) = \mathfrak{m} \bowtie^f ((0 : j) \cap J)$ is an S'-finite since $\mathfrak{m}$ is S-finite and $(0 : j) \cap J$ is an $f(S)$-finite ideal of $f(A)+J$ (see Lemma 2.15).

Case 2: If $j = 0$, then $(0 : c) = (0 : a) \bowtie^f J$ is an S'-finite ideal of $A \bowtie^f J$ since $(0 : a)$ is an S-finite ideal of A and J is an $f(S)$-finite ideal of $f(A)+J$ (see Lemma 2.15).

Case 3: Assume that $a \neq 0$ and $j \neq 0$. Then we get easily that $(0 : c) = (0 : a) \bowtie^f ((0 : j) \cap J)$ is an S'-finite ideal of $A \bowtie^f J$ since $(0 : a)$ is an S-finite ideal of A and $(0 : j) \cap J$ is an $f(S)$-finite ideal of $f(A)+J$ by Lemma 2.15, as desired. □

Proof of Theorem 2.14 . (1) Assume that R is an S'-finite conductor ring. We will prove that A is an S-finite conductor ring. This, in turn, follows immediately from Lemmas 2.16 and 2.17 (1).

(2) $(a) \Rightarrow (b)$ By (1), A is an S-finite conductor ring. Let $a, b \in A$ and let $k, l \in J$. By [18, Lemma 2.6 (1)], $A \bowtie^f J(a, f(a)+k) \cap A \bowtie^f J(b, f(b)+l) = (\mathfrak{m}a \cap \mathfrak{m}b) \bowtie^f (Jk \cap Jl)$. Then the result follows immediately from Lemmas 2.15(2) and 2.17.

$(b) \Rightarrow (c)$ Clear.

$(c) \Rightarrow (a)$ This follows immediately from Lemmas 2.15(2) and 2.17 and the fact that $(A \bowtie^f J)(a, f(a)+k) \cap (A \bowtie^f J)(b, f(b)+l) = (\mathfrak{m}a \cap \mathfrak{m}b) \bowtie^f (Jk \cap Jl)$ for each $a, b \in A$ and $k, l \in J$. □

Applying Theorem 2.14 to the case when S consists of units elements, we can recover the first two assertions of [18, Theorem 2.1].

Corollary 2.18. *Let $(A, \mathfrak{m})$ be a local ring, let $f : A \longrightarrow B$ be a ring homomorphism, and let J be a proper ideal of B.*

1. *If $A \bowtie^f J$ is a finite conductor ring, then so is A.*
2. *Assume that $f(\mathfrak{m})J = (0)$ and $J \subseteq Rad(B)$. Then the following conditions are equivalent:*
   - (a) *$A \bowtie^f J$ is a finite conductor ring.*
   - (b) *A is a finite conductor ring, $\mathfrak{m}$ and $\mathfrak{m}a \cap \mathfrak{m}b$ are finitely generated ideals of A for all $a, b \in \mathfrak{m}$, and J, $Jk \cap Jl$ and $(0 : k) \cap J$ are finitely generated ideals of $f(A)+J$ for all $k, l \in J$.*
   - (c) *$\mathfrak{m}$, $(0 : a)$, and $\mathfrak{m}a \cap \mathfrak{m}b$ are finitely generated ideals of A for all $a, b \in \mathfrak{m}$, and J, $Jk \cap Jl$, and $(0 : k) \cap J$ are finitely generated ideals of $f(A)+J$ for all $k, l \in J$.*

Note that if S is a multiplicatively closed subset of A, then the set $T = \{(s, s) | \, s \in S\}$ is a multiplicatively closed subset of $A \bowtie I$. As a consequence of Theorem 2.14, we have the following corollary.

Corollary 2.19. *Let $(A, \mathfrak{m})$ be a local ring and let I be a proper ideal of A.*

(1) *If $A \bowtie I$ is a T-finite conductor ring, then A is an S-finite conductor ring.*

(2) *Assume that $\mathfrak{m}I = (0)$. Then the following assertions are equivalent:*
   (*a*) *$A \bowtie I$ is a T-finite conductor ring.*
   (*b*) *A is an S-finite conductor ring, $\mathfrak{m}$, I, and $\mathfrak{m}a \cap \mathfrak{m}b$ are S-finite ideals of A.*
   (*c*) *$\mathfrak{m}$, I, $(0 : a)$, and $\mathfrak{m}a \cap \mathfrak{m}b$ are S-finite ideals of A for all $a, b \in \mathfrak{m}$.*

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