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ITERATED FUNCTION SYSTEMS OVER ARBITRARY SHIFT SPACES

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Abstract. The orbit of a point $x \in X$ in a classical iterated function system (IFS) can be defined as $\{f_u(x) = f_{u_n} \circ \cdots \circ f_{u_1}(x) : u = u_1 \cdots u_n$ is a word of a full shift Σ on finite symbols and f_{u_i} is a continuous self map on $X\}$. One also can associate to $\sigma = \sigma_1\sigma_2 \cdots \in \Sigma$ a non-autonomous system $(X,\, f_\sigma)$, where the trajectory of $x \in X$ is defined as $x,\, f_{\sigma_1}(x),\, f_{\sigma_1\sigma_2}(x), \ldots$. Here instead of the full shift, we consider an arbitrary shift space Σ. Then we investigate basic properties related to this IFS and the associated non-autonomous systems. In particular, we look for sufficient conditions that guarantee that in a transitive IFS one may have a transitive $(X,\, f_\sigma)$ for some $\sigma \in \Sigma$ and how abundance are such σ's.

1. Introduction

In a classical dynamical system, here called a *conventional dynamical system*, we have a phase space and a unique map where the trajectories of points are obtained by iterating this map. However, in various problems, including applied ones, one may have some finite sequences of maps in place of a single map acting on the same phase space. As an example, let X be the space of a mixture of some materials that are supposed to be mixed by the application of two robotic arms r_0 and r_1 and only one of them at each unit of time. Due to some technical considerations, two r_1 cannot be applied in a row, though this consideration is not in place for r_0. Thus the application of these arms and hence the dynamics of the system are bound to the golden subshift, that is, the subshift whose forbidden set is $\{11\}$. In fact, there are many natural processes whose evolution evolves

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with discrete time and are involved with two or more interactions. For instance, two or more maps have appeared in Physics [2, 18], in Economy [21], and in Biology [7]. In Mathematics, this has been studied either by non-autonomous systems in literature, such as [14] or as iterated function system (IFS). In fact, IFS was first discovered by Hutchinson in 1981 [12], who was able to obtain some fancy fractal sets by some finite continuous and contracting maps where images of those fractals generated by a computer are now a source of motivation for popularizing Mathematics among the general audience. Later, IFS appeared in many studies [3, 4, 9–11, 20] where they addressed some internal dynamical problems.

What we call here the "classical" IFS, is a general dynamical system, that is, the action of a semigroup on a compact metric space X arising from free combinations of some k continuous maps $\{f_0, \dots, f_{k-1}\}$ on X [9, 20]. In fact, the semigroup is the set of words in the full shift Σ_k, either one-sided or two-sided on k symbols, and its operation is defined by concatenating any two words. We write $f_u = f_{u_n} \circ \cdots \circ f_{u_1}$, where $u = u_1 \cdots u_n$ is a word in Σ_k. Hence no limitation is applied as in our aforesaid example on the robotic arms when their words were forbidden to have 11 as a subword. In this paper, we apply some limitations on the classical IFS by replacing the full shift Σ_k with a subshift $\Sigma \subseteq \Sigma_k$, and we call it just IFS versus the classical IFS when $\Sigma = \Sigma_k$. Thus one may look at X as a phase space, and the subshift Σ as a parameter space showing how the maps must be combined.

A summary of the results in this paper is as follows. In Section 2, we formalize the definitions and notations. Section 3 is mainly devoted to the definitions of transitivity in IFS and the relation between them. In particular, we show that when the shift space is sofic, topological transitivity in the constituent IFS implies the point transitivity along a transitive orbit in the shift space; a fact which is not necessarily satisfied for nonsofics. In Section 4, we like to see how large the set $S = \{\sigma \in \Sigma : \ \exists\, x \in X, \overline{\mathcal{O}_\sigma(x)} = X\}$ can be. In Section 5, mixing and exactness of an IFS versus those properties along orbits through some examples, have been considered.

2. Preliminaries

2.1. **Symbolic dynamics.** A brief recall of the symbolic dynamics is given here. Notations and main ideas are borrowed from [16], and the proofs of the claims can be found there. Let $\mathcal{A}$ be a nonempty finite set and let $\Sigma_{|\mathcal{A}|} = \mathcal{A}^{\mathbb{Z}}$ (resp. $\mathcal{A}^{\mathbb{N}}$) be the collection of all bi-infinite (resp. right-infinite) sequences of symbols from $\mathcal{A}$. The map $\tau : \Sigma_{|\mathcal{A}|} \to \Sigma_{|\mathcal{A}|}$ defined by $\tau(\sigma)_i = \sigma_{i+1}$ is called the *shift map*, and the pair $(\Sigma_{|\mathcal{A}|}, \tau)$ is the *full shift* on k symbols. Any closed invariant subset Σ of $\Sigma_{|\mathcal{A}|}$ is called a *subshift* or a *shift space*. A *word* or *block* over $\mathcal{A}$ is a finite sequence of symbols from $\mathcal{A}$. Denote by $\mathcal{L}_n(\Sigma)$ the set of all admissible n-words and call $\mathcal{L}(\Sigma) := \bigcup_{n=0}^{\infty} \mathcal{L}_n(\Sigma)$ the *language* of Σ. For $u \in \mathcal{L}_k(\Sigma)$, let the cylinder ${}_\ell[u]_{\ell+k-1} ={}_\ell[u_\ell \cdots u_{\ell+k-1}]_{\ell+k-1}$ be the set $\{\sigma = \cdots \sigma_{-1}\sigma_0\sigma_1 \cdots \in \Sigma : \sigma_\ell \cdots \sigma_{\ell+k-1} = u\}$. If $\ell = 0$, then we drop the subscripts and we just write $[u]$.

A shift space Σ is *irreducible* if for every ordered pair of words $u,\, v \in \mathcal{L}(\Sigma)$, there is a word $w \in \mathcal{L}(\Sigma)$ such that $uwv \in \mathcal{L}(\Sigma)$. A point $\sigma \in \Sigma$ is *transitive* if every word in Σ appears in σ infinitely many often. A subshift Σ is irreducible if and only if Σ has a transitive point.

Shift spaces described by a finite set of forbidden blocks are called *shifts of finite type* (SFT) and their factors are called *sofic*. A word $w \in \mathcal{L}(\Sigma)$ is called *synchronizing* if $uwv \in \mathcal{L}(\Sigma)$ whenever $uw, wv \in \mathcal{L}(\Sigma)$. A *synchronized system* is an irreducible shift which has a synchronizing word. Any sofic is synchronized.

A subshift Σ is *specified* or has the *specification property*, if there is $N \in \mathbb{N}$ such that if $u, v \in \mathcal{L}(\Sigma)$, then there is w of length N such that $uwv \in \mathcal{L}(\Sigma)$. A specified system is mixing and synchronized, and any mixing sofic is specified. A *coded system* is the closure of the set of sequences obtained by freely concatenating the words in a list of words. In particular, any synchronized system is coded.

All synchronized systems have an (edge) labeled graph presentation called *cover*. These are directed graphs whose edges are with assigned labels from $\mathcal{A}$ and infinite walk on the graph and recording the labels will represent a point in the subshift. The set of all such points is dense in the subshift.

2.2. **Iterated function systems.** Throughout the paper, X will be a compact metric space. The *classical* iterated function system (IFS) consists of finitely many continuous self maps $\mathcal{F} = \{f_0,\, \dots,\, f_{k-1}\}$ on X. The *forward orbit* of a point $x \in X$, denoted by $\mathcal{O}^+(x)$, is the set of all values of finite possible combinations of f_i's at x. We need the following equivalent statement: Let $\Sigma_{|\mathcal{F}|}$ be the full shift on k symbols and let $\mathcal{L}(\Sigma_{|\mathcal{F}|})$ called the *language of* $\Sigma_{|\mathcal{F}|}$ be the set of words or blocks. Define $f_u(x) : X \to X$ by

$$f_{u_n} \circ \cdots \circ f_{u_1}(x), \quad u = u_1 \cdots u_n \in \mathcal{L}(\Sigma_{|\mathcal{F}|}). \tag{2.1}$$

Then $\mathcal{O}^+(x) = \{f_u(x) : \ u \in \mathcal{L}(\Sigma_{|\mathcal{F}|})\}$. Such iterated function systems, here called *classical IFS*, have been the subject of study for quite a long time.

Here we define an IFS to be

$$\mathfrak{I} = (X,\, \mathcal{F} = \{f_0,\, \dots,\, f_{k-1}\},\, \Sigma). \tag{2.2}$$

where each f_i is continuous and Σ is an arbitrary subshift on k symbols, not necessarily the full shift $\Sigma_{|\mathcal{F}|}$ as in the classical IFS. By this setting, $\Sigma_{|\mathcal{F}|}$ above will be replaced with Σ, and thus $\mathcal{O}^+(x) = \{f_u(x) : \ u \in \mathcal{L}(\Sigma)\}$ is the forward orbit of x. In particular, $f_u(f_v(x)) = f_{vu}(x)$ whenever vu is admissible or equivalently $vu \in \mathcal{L}(\Sigma)$. Let $u = u_1 \cdots u_n \in \mathcal{L}(\Sigma)$, and set $u^{-1} := u_n \cdots u_1$. Then for $A \subseteq X$,

$$\begin{aligned} (f_u)^{-1}(A) &= (f_{u_n} \circ \cdots \circ f_{u_1})^{-1}(A) \\ &= f_{u_1}^{-1} \circ \cdots \circ f_{u_n}^{-1}(A) \\ &= f_{u^{-1}}^{-1}(A), \end{aligned}$$

where for the last equality, we used (2.1). Also,

$$\begin{aligned} f_{u^{-1}}^{-1}(f_{v^{-1}}^{-1}(A)) &= f_{v^{-1}u^{-1}}^{-1}(A) = f_{(uv)^{-1}}^{-1}(A) \\ &= (f_{uv})^{-1}(A). \end{aligned}$$

Thus the backward orbit and the (full) orbit of a point $x \in X$ are $\mathcal{O}_-(x) = \{f^{-1}_{u^{-1}}(x) : u \in \mathcal{L}(\Sigma)\}$ and $\mathcal{O}(x) = \mathcal{O}^+_-(x) = \mathcal{O}^+(x) \cup \mathcal{O}_-(x)$, respectively.

When all f_i's are homeomorphisms, the backward, forward, and full trajectory of x are defined.

We say $\mathcal{F} = \{f_0, \ldots, f_{k-1}\}$ is *surjective*, *injective*, and *homeomorphism* if all f_i's in $\mathcal{F}$ are so.

When $k = 1$ and $\Sigma = \{0^\infty\}$, we simply have the classical dynamical system, here called *conventional dynamical system* denoted either by the pair (X, f) or $\mathfrak{I} = (X, \{f_0\}, \{0^\infty\})$.

3. Transitivity

Two sorts of transitivity are very common in the study of topological dynamical systems: topological transitivity and point transitivity. These two concepts are the same for surjective conventional dynamical systems on the compact metric spaces such as subshifts but not for IFS's and non-autonomous dynamical systems. Hence we say it a transitive point in Σ, but will emphasize point transitivity or topological transitivity in other places.

Definition 3.1. Consider $\mathfrak{I}$ as in (2.2) and let U and V be arbitrary open sets in X. Then $\mathfrak{I}$ is

(1) *"forward" point transitive*, if there is $x \in X$ such that $\{f_u(x) : u \in \mathcal{L}(\Sigma)\}$ is dense in X. We drop "forward" when it is clear from the context. *Backward point transitivity* is likewise defined.
(2) *topological transitive*, if there is $u \in \mathcal{L}_n(\Sigma)$ such that $f_u(U) \cap V \neq \emptyset$.
(3) *mixing*, if there is $M = M(U, V) \in \mathbb{N}$ such that for $n \geq M$, there is $u \in \mathcal{L}_n(\Sigma)$ such that $f_u(U) \cap V \neq \emptyset$.
(4) *exact*, if there is $u(U) \in \mathcal{L}(\Sigma)$ such that for any $uu' \in \mathcal{L}(\Sigma)$, $f_{uu'}(U) = X$.

We have the following implications in any IFS:

$$\text{exactness} \;\Rightarrow\; \text{mixing} \;\Rightarrow\; \text{topological transitivity} \;\Rightarrow \text{point transitivity}. \tag{3.1}$$

The first two implications follow from the definition and the last from the next proposition. Also, since conventional dynamical systems are IFS, they provide examples that the first two implications are not reversible.

Proposition 3.2. *Let $\mathfrak{I} = (X, \mathcal{F}, \Sigma)$ be an IFS. If for arbitrary nonempty open sets U, V, there is $u \in \mathcal{L}(\Sigma)$ such that $(f_u)^{-1}(U) \cap V \neq \emptyset$, then $\mathfrak{I}$ is point transitive.*

Proof. Let $\mathcal{B} = \{U_n : n \in \mathbb{N}\}$ be a countable base for X. Fix $n \in \mathbb{N}$ and set

$$G_n := \cup_{u \in \mathcal{L}(\Sigma)} (f_u)^{-1}(U_n). \tag{3.2}$$

By the assumption for an arbitrary open set V, $G_n \cap V \neq \emptyset$ and so the open set G_n is dense. As a result, $\cap_{n\in\mathbb{N}} G_n$ is residual. Hence for $x \in \cap_{n\in\mathbb{N}} G_n$ and any $n \in \mathbb{N}$, there is $u \in \mathcal{L}(\Sigma)$ such that $x \in (f_u)^{-1}(U_n)$. This means $f_u(x) \in U_n$ and so x is a transitive point. □

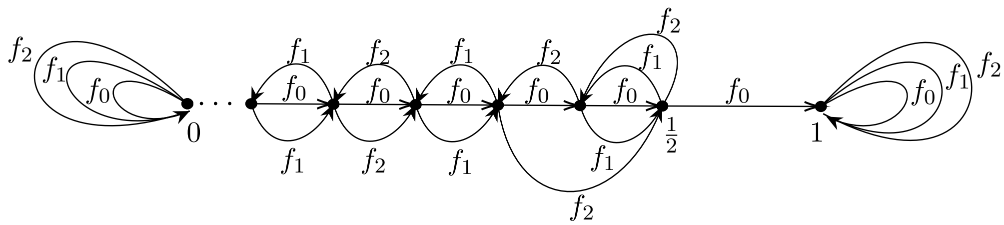

FIGURE 1. Nodes represent the points in X. The farthest node on the right is 1, the second $\frac{1}{2}$ and so on.

In the above proposition, we did not assume that our IFS is surjective; though unlike a surjective conventional dynamical system, even applying surjectivity, the last implication in (3.1) is not reversible. This fact was noted (for classical IFS) in some literature [15, 17]; however, we did not find any proof, so we bring our own.

Proposition 3.3. *In a surjective IFS, topological and point transitivity are not equivalent.*

Proof. We construct an example of a classical IFS which is point transitive but not topological transitive.

Let $\mathfrak{I} = (X, \{f_0, f_1, f_2\}, \Sigma_3)$, where $X = \{1/n : n \in \mathbb{N}\} \cup \{0\} \subset \mathbb{R}$ is equipped with the subspace topology.

Our maps are defined as follows (see Figure 1). For all i, $f_i(0) = 0$ and $f_i(1) = 1$.

$$\begin{aligned} f_0(\frac{1}{n+1}) &= \frac{1}{n}, \quad n \geq 1, \\ f_1(\frac{1}{2n+1}) &= \frac{1}{2n}, \quad n \geq 1, \\ f_1(\frac{1}{2n}) &= \frac{1}{2n+1}, \quad n \geq 1 \quad \text{and} \quad f_1(\frac{1}{2}) = \frac{1}{3}. \end{aligned}$$

Also, $f_2(\frac{1}{4}) = \frac{1}{2}$, $f_2(\frac{1}{2}) = \frac{1}{3}$ and

$$\begin{aligned} f_2(\frac{1}{2n+1}) &= \frac{1}{2n+2}, \quad n \geq 1, \\ f_2(\frac{1}{2n+2}) &= \frac{1}{2n+1}, \quad n \geq 2. \end{aligned}$$

All maps are continuous, open, and surjective. Both f_1 and f_2 are homeomorphisms, but f_0 is not injective: $f_0(\frac{1}{2}) = f_0(1) = 1$.

Observe that any point $x = \frac{1}{n}$, $n \geq 2$ is transitive. However, the system is not topological transitive. Because, for any u, $f_u(\{1\}) \cap \{\frac{1}{2}\} = \emptyset$. □

It is worth mentioning that if $\mathcal{F}$ was homeomorphism in Definition 3.1, then topological and point transitivity were equivalent [6].

3.1. **Dynamics along an orbit as a non-autonomous dynamical system.** Let X be a topological space and let $f_n : X \to X$ be a continuous map for $n \in \mathbb{N}$. Then the sequence $\{f_n\}_1^\infty$ denoted by $f_{1,\infty}$ defines a non-autonomous discrete dynamical system $(X,\, f_{1,\infty})$ [14]. In an IFS, dynamics along a σ also defines a non-autonomous system, which we show it by $(X,\, f_\sigma)$ or $f_\sigma := \{f_{\sigma_i}\}_{i=1}^\infty$ (resp. $f_\sigma := \{f_{\sigma_i}\}_{i=-\infty}^{+\infty}$) when Σ is one-sided (resp. two-sided). If Σ is over a finite alphabet, then clearly f_σ is defined only by finitely many different f_i's.

Let $\sigma = \sigma_1\sigma_2\cdots \in \Sigma$. Then the sequence $x,\, f_{\sigma_1}(x),\, f_{\sigma_1\sigma_2}(x), \ldots$ is the *trajectory of x along σ*, and $\mathcal{O}_\sigma^+(x)$ the set of points in this trajectory is the *(forward) orbit of x along σ*. The backward orbit and backward trajectory may be defined similarly for the case where Σ is a two-sided subshift. Hence one may say that $\mathfrak{I} = (X,\, \mathcal{F},\, \Sigma_{|\mathcal{F}|})$ has property P along $\sigma \in \Sigma$ if the respective non-autonomous system $(X,\, f_\sigma)$ has property P. By this convention, the following definition may sound abundance, though we bring it for the sake of completeness.

Definition 3.4. Let $\mathfrak{I} = (X,\, \mathcal{F},\, \Sigma_{|\mathcal{F}|})$ be an IFS and let U and V be arbitrary non-empty open sets in X. Then $\mathfrak{I}$ is called

(1) *forward point transitive along an orbit* $\sigma \in \Sigma$, if there is a point $x \in X$, called the transitive point, such that $\overline{\mathcal{O}_\sigma^+(x)} = X$.
(2) *topological transitive along an orbit* $\sigma \in \Sigma$, if there is $n \in \mathbb{N}$ such that $f_{\sigma_1\cdots\sigma_n}(U) \cap V \neq \emptyset$.
(3) *mixing* (resp. *exact*) *along an orbit* $\sigma \in \Sigma$, if there is $N \in \mathbb{N}$ such that for $n \geq N$, $f_{\sigma_1\cdots\sigma_n}(U) \cap V \neq \emptyset$ (resp. $f_{\sigma_1\cdots\sigma_n}(U) = X$).

Note that if one of the properties given in the above definition holds along some $\sigma \in \Sigma$ for an IFS, then the IFS posses that property as well. However, the converse is not true. For instance, the example given in Proposition 3.3 is point transitive but not point transitive along any orbit.

Similar implications as in (3.1) hold here as well. So we have the following result.

Proposition 3.5 ([19, Proposition 4.6])**.** *If an IFS has topological transitivity along σ, then it is point transitive along σ.*

Proof. The proof is similar to the proof of Proposition 3.2 by replacing (3.2) with $G_n = \cup_{\ell\in\mathbb{N}}(f_{\sigma_1\sigma_2\cdots\sigma_\ell})^{-1}(U_n)$ and applying the same reasoning. □

The converse of the above proposition is not necessarily true as the next example shows. This example also shows that point transitivity along an orbit does not imply that the transitive points in X are residual along that orbit.

Example 3.6. Let $X = [0,\, 1]$ and let $\mathfrak{I} = (X,\, \{f_0,\, f_1\},\, \Sigma_{|\mathcal{F}|})$, where

$$f_0(x) = \begin{cases} 2x, & 0 \leq x \leq \dfrac{1}{2}, \\ 1 & \dfrac{1}{2} \leq x \leq 1, \end{cases} \quad \text{and} \quad f_1(x) = \begin{cases} 0, & 0 \leq x \leq \dfrac{1}{2}, \\ 2x-1, & \dfrac{1}{2} \leq x \leq 1. \end{cases}$$

Also let $f : [0,\, 1] \to [0,\, 1]$ be defined as $f(x) = 2x \mod 1$, that is,

$$f(x) = \begin{cases} f_0(x), & 0 \le x \le \dfrac{1}{2}, \\ f_1(x), & \dfrac{1}{2} \le x \le 1. \end{cases}$$

Let $z \in (0,\, 1/2)$ be a transitive point of f and set $\sigma := \sigma_1\sigma_2 \cdots \in \Sigma_{|\mathcal{F}|}$, where $\sigma_1 = 0$ and for $i > 1$, $\sigma_i = 0$ (resp. $\sigma_i = 1$) whenever $f_{\sigma_1 \cdots \sigma_{i-1}}(z) \in (0,\, 1/2)$ (resp. $f_{\sigma_1 \cdots \sigma_{i-1}}(z) \in (1/2,\, 1)$). By this settings, z is a transitive point and so the non-autonomous system $([0,\, 1],\, f_\sigma)$ is point transitive, but not topological transitive. Because for $U = (1/2,\, 1)$, $V = (0,\, 1/2)$ and for any $n \in \mathbb{N}$, $f_{\sigma_1 \cdots \sigma_n}(U) \cap V = \emptyset$.

3.2. **Transitivity in IFS vs transitivity in the subshift.** In general, there is no meaningful relation between the dynamical properties of $(\Sigma,\, \tau)$ and that of $\mathfrak{I}$. For instance, consider $\mathfrak{I} = (X,\, \mathcal{F},\, \Sigma)$, where $X = [0,\, 1]$ and $\mathcal{F} = \{f_0(x) \equiv 0,\, f_1(x) = 2x \mod 1\}$. Let Σ be the golden mean shift; that is, a subshift of $\Sigma_2 = \{0,\, 1\}^{\mathbb{N}}$ whose forbidden set is $\{11\}$. Then, Σ has rich dynamical properties; however, this is not true for $\mathfrak{I}$. The situation is different when our maps in $\mathcal{F}$ are surjective and Σ has some certain properties.

In this section, we like to address the following questions.

(1) Does transitivities given in Definition 3.1 imply some sort of transitivity given in Definition 3.4?
(2) If the answer to the above question is affirmative, in which situation there is a transitive $t \in \Sigma$ such that for some $x \in X$, $\overline{O_t^+(x)} = X$?

The following example shows, as one expects, that transitivity depends on the subshift.

Example 3.7. Let $\mathfrak{I} = (X,\, \mathcal{F},\, \Sigma)$, where Σ is an SFT generated by $\mathcal{W} = \{01,\, 10\}$ and f_0 is the shift map on the two-sided full shift $X = \{0,\, 1\}^{\mathbb{Z}}$ and $f_1 = f_0^{-1}$. Clearly this system is not point transitive. Moreover, if $W ={}_{-1}[000]_1$ and $V ={}_{-1}[111]_1$ are two open central cylinders in X and if w is any word in Σ, then $f_w^{-1} W \cap V = \emptyset$ and so $\mathfrak{I}$ is not topological transitive either. However, if Σ were generated by $\mathcal{W} \cup \{0\}$, then the constituent IFS was both topological and point transitive, showing that transitivity depends on our subshift.

Later the sets X and $\mathcal{F} = \{f_0,\, f_1\}$ introduced in the following example will be used in several occasions such as Examples 3.13 and 4.6 and Proposition 4.5.

Example 3.8. Let $\{x_n\}_{n\in\mathbb{Z}}$ be an increasing sequence ($x_{n+1} > x_n$) in $[0,\, 1]$ such that $\lim_{n\to+\infty} x_n = 1$ and $\lim_{n\to-\infty} x_n = 0$. Let X be the set of points of this sequence together with 0 and 1 and equip X with the subset topology of $[0,\, 1]$.

(1) Let $\mathfrak{I}_1 = (X,\, \{f_0\},\, \{0^{\mathbb{Z}}\})$ be the conventional dynamical system, where

$$f_0(x) = \begin{cases} x & \text{if} \quad x \in \{0,\, 1\}, \\ x_{n+1} & \text{if} \quad x = x_n. \end{cases} \tag{3.3}$$

This system is point transitive but not forward point transitive. In fact, any point in $X \setminus \{0,\, 1\}$ is a transitive point and 0 and 1 are fixed points.

(2) Let $\mathfrak{I}_2 = (X, \{f_0, f_1\}, \Sigma)$ and let f_0 be as in (3.3) but f_1 be defined as

$$f_1(x) = \begin{cases} x & \text{if} \quad x \in \{0, 1\}, \\ x_{n-1} & \text{if} \quad x = x_n. \end{cases} \tag{3.4}$$

Also, let $\Sigma \subseteq \{0, 1\}^{\mathbb{N}_0}$ be generated by $\mathcal{W} = \{w_0, w_1, w_3, \ldots\}$ so that there are two words in $\mathcal{W}$, say w_0 and w_1 such that $|w_0| = |w_1|$ and $\frac{0_{w_0}}{|w_0|} = \frac{1_{w_1}}{|w_1|} > \frac{1}{2}$, where i_{w_j} is the number of i's appearing in w_j. We show that $\mathfrak{I}_2$ is point transitive along some orbits. To see this, let

$$\sigma^0 = w_0 w_1 w_1 w_0 w_0 w_0 w_1 w_1 w_1 w_1 \cdots w_0^n w_1^{n+1} w_0^{n+2} w_1^{n+3} \cdots \tag{3.5}$$

and let $x \in X \setminus \{0, 1\}$. Then, $\overline{\mathcal{O}^+_{\sigma^0}(x)} = X$. One example is when $\Sigma = \Sigma_{|\mathcal{F}|}$ and $\mathcal{W} = \{w_0 = 0, w_1 = 1\}$, where $\frac{0_{w_0}}{|w_0|} = \frac{1_{w_1}}{|w_1|} = 1$.

Now we set up to show that when Σ is an irreducible sofic, functions are semi-open, that is, the interior of image of any open set is nonempty, and when the respective IFS is topological transitive, then for some transitive $t \in \Sigma$, one has point transitivity along t. This will give an answer to questions 1 and 2 on the beginning of this section for special cases where Σ is an irreducible sofic. First, we recall a classical result.

Lemma 3.9 (Boyle [5])**.** *Let Σ and Σ' be irreducible SFT with $h(\Sigma) > h(\Sigma')$. Then, there is a factor code from Σ onto Σ' if and only if $P(\Sigma) \searrow P(\Sigma')$.*

Proposition 3.10. *Let $\mathfrak{I} = (X, \mathcal{F}, \Sigma)$ be a surjective and topological transitive IFS and maps in $\mathcal{F}$ semi-open. Also let Σ be an irreducible sofic. Then there is a forward transitive $t \in \Sigma$ such that the non-autonomous system (X, f_t) is point transitive.*

Proof. Let $\mathcal{A} = \{0, \ldots, k-1\}$ be the set of characters of Σ. If Σ does not have a fixed point, then replace $\mathcal{A}$ with $\mathcal{A}' = \mathcal{A} \cup \{k\}$ and replace $\mathcal{F}$ with $\{f_0, \ldots, f_{k-1}\} \cup \{f_k\}$, where f_k is the identity map, and set Σ' to be the corresponding subshift whose set of forbidden set is the same as Σ. Observe that $k^{\mathbb{N}_0}$ is a fixed point of Σ'. If $t' \in \Sigma'$ is a transitive point, then t obtained from t' by forgetting the entries whose value is k is transitive in Σ. Thus without loss of generality, we may assume that Σ has a fixed point.

So let $\mathfrak{I}$ be topological transitive, and set $\mathfrak{I}' := (X, \mathcal{F}, \Sigma_{|\mathcal{F}|})$, and let $\mathcal{B} := \{W_m : m \in \mathbb{N}\}$ be a base for the topology on X. First we construct a transitive point $t \in \Sigma_{|\mathcal{F}|}$ such that $\overline{O^+_t(x)} = X$.

Let U_m be an open set such that $\overline{U_m} \subseteq W_m$. Pick $v_1 \in \mathcal{L}(\Sigma_{|\mathcal{F}|})$ such that $f_{v_1}(U_1) \cap U_2 \neq \emptyset$, and consider $f_{v_1 v_1'}$, where v_1' is the concatenation of all characters or words of length 1. Then by the fact that f_i's are semi-open and that our system is topological transitive, there is v_2 such that $f_{v_1 v_1' v_2}(U_1) \cap U_3 \neq \emptyset$. By the same reasoning and induction argument, there is v_k such that for $u_k := v_1 v_1' v_2 \cdots v_i v_i' v_{i+1} \cdots v_{k-1} v_{k-1}' v_k$, we have

$$f_{u_k}(U_1) \cap U_{k+1} \neq \emptyset. \tag{3.6}$$

Here v_i' is the concatenation of all words of length i. Let $C_k = \overline{U_1} \cap (f_{u_k})^{-1}(\overline{U_{k+1}})$ be the compact set in W_1, and note that $C_{k+1} \subseteq C_k$; in particular, $\cap_k C_k$ is a nonempty compact set in W_1. Thus if $x \in \cap_k C_k$, then $f_{u_k}(x) \in W_k$. This means that our system is point transitive along the transitive $t = v_1 v_1' v_2 \cdots \in \Sigma_{|\mathcal{F}|}$. So the problem is set when Σ is a full shift.

Now, let Σ be SFT and recall that we are assuming that it has a fixed point. This means $P(\Sigma_{|\mathcal{F}|}) \searrow P(\Sigma)$, and so by Lemma 3.9, there is a factor code ϕ from $\Sigma_{|\mathcal{F}|}$ onto Σ. In particular, there exists a transitive point $\phi(t) \in \Sigma$ with $\overline{O^+_{\phi(t)}(x)} = X$. It remains to prove the case when Σ is sofic. Indeed, any sofic is a factor of an SFT and transitivity is preserved by factor codes and take this factor code to be a 1-block factor code. By an argument as above, we may extend this SFT to have a fixed point and the new character, if any, will map to a new added character in character set of Σ by the block factor map whose associated map in $\mathfrak{I}$ is identity. As a result, a transitive $t \in \Sigma$ and $x \in X$ exist as required. □

In the above proposition, the same conclusion holds if we are sure that for any k, there is u_k such that as in (3.6), then the intersection has a nonempty interior. In fact, we conjecture that this is the case, that is, if IFS is topological transitive, $\mathcal{F}$ surjective, then for any nonempty open sets U and V, there is $u \in \Sigma_{|\mathcal{F}|}$ such that $f_u(U) \cap V$ has nonempty interior. In that case, we do not require semi-openness in the hypothesis.

Next we bring examples showing that none of the other conditions on the hypothesis of the above proposition can be ignored.

Example 3.11. The alphabet defining our subshift in the above proposition was finite; the conclusion is not valid for an infinite case. Authors in [17, Example 2.1] claimed that in that situation, even when the subshift is a full shift, the topological transitivity does not imply topological transitivity along any orbit.

Example 3.12. The topological transitivity of the IFS in Proposition 3.10 cannot be replaced with point transitivity. For instance, the system given in Proposition 3.3 had all the conditions on the hypothesis of the proposition (subshift was the full shift, and so sofic and all maps were open) except topological transitivity. There we had point transitivity of the IFS, but yet we did not have point transitivity along any orbit.

Now we show that the sofic property cannot be omitted in the hypothesis of the above proposition. Moreover, this example shows that, in general, the topological transitivity of an IFS does not necessarily imply the point transitivity along any $\sigma \in \Sigma$.

Example 3.13. Let f_0 and f_1 be homeomorphisms defined in Example 3.8, and let $\mathfrak{I} = (X, \{f_0, f_1\}, \Sigma)$, where $\Sigma \subseteq \{0, 1\}^{\mathbb{N}}$ is the non-sofic shift generated by $\mathcal{W} = \{0^n 1^n :\ n \in \mathbb{N}\}$. Then any $\sigma \in \Sigma$ consists of concatenation of words in $\mathcal{W}$ and their shifts together with points in the closure of them. Thus since $f_{0^n 1^n} \equiv \mathrm{id}$ for $n \in \mathbb{N}$, any $\sigma \in \Sigma$ is either concatenation of words in $\mathcal{W}$ or terminating at 0^∞ or 1^∞. Therefore, $\overline{\mathcal{O}^+_\sigma(x)} \neq X$ for any $\sigma \in \Sigma$ and $x \in X$.

On the other hand, any point $x_0 \in X \setminus \{0, 1\}$ has dense orbit. Because, since $0^{\mathbb{N}_0}$ and $1^{\mathbb{N}_0}$ are points of Σ, so x_0 can travel left and right as far as required by f_0 and f_1, respectively. As a result, $\mathfrak{I}$ is topological transitive and yet not point transitive along any $\sigma \in \Sigma$.

4. The abundance of point transitive non-autonomous systems in an IFS

When a dynamical property such as transitivity, mixing, and exactness occur along a $\sigma \in \Sigma$, then the IFS will possess that property as well, though the converse is not necessarily true. In fact, it may not even hold along just a single orbit. In this section, we investigate transitivity in this respect.

Let

$$S = S(\mathfrak{I}) := \{\sigma \in \Sigma : \exists x \in X \text{ s.t. } \overline{\mathcal{O}_\sigma^+(x)} = X\}. \tag{4.1}$$

In general, except in few cases, a definite structure cannot be given for S, though its largeness can be understood in some cases. Let us demonstrate how different S can be.

Example 4.1.
(1) S may be all of Σ. For an example, let $f_0(x) = 2x \mod 1$ and let $f_1(x) = 3x \mod 1$, and consider $\mathfrak{I} = ([0, 1], \{f_0, f_1\}, \Sigma_2)$.
(2) S may be an empty set. This is the case when we have an IFS, which is not point transitive. Though even for a topological transitive IFS, S still may be empty (see Example 3.13).
(3) S may be residual and yet not all of Σ. The IFS in Example 3.6 has such property.
(4) S may be dense and uncountable, yet not a residual subset; see Example 4.6.

Now we give sufficient conditions for S being dense in Σ; first a weaker version of specification property for subshifts:

Definition 4.2. A subshift Σ is called a *subshift of variable gap length* or SVGL, if there exists $M \in \mathbb{N}$ such that for u and v in $\mathcal{L}(\Sigma)$, there is w with $|w| \leq M$ and $uwv \in \mathcal{L}(\Sigma)$.

When Σ is mixing and SVGL, then Σ has *specification property* and in this situation, there exists $M \in \mathbb{N}$ such that for u and v in $\mathcal{L}(\Sigma)$ there is w with $|w| = M$ and $uwv \in \mathcal{L}(\Sigma)$. Clearly an SVGL is irreducible. Moreover, all sofics are SVGL; however, there are SVGL's that are not sofic. The SVGL is called almost specification property in [13].

Proposition 4.3. *Let $\mathfrak{I} = (X, \mathcal{F}, \Sigma)$ be point transitive along some $\sigma \in \Sigma$, $\mathcal{F}$ surjective and Σ an SVGL. Then, S defined in (4.1) is dense in Σ. If $S \neq \Sigma$, then $\Sigma \setminus S$ is also dense in Σ.*

Proof. We prove the first part; the other part has a similar proof.

Choose any $\sigma = \sigma_1\sigma_2 \cdots \in \Sigma$ such that $\overline{\mathcal{O}_\sigma^+(x)} = X$. Let $[u]$ be a cylinder in Σ and use the SVGL property of Σ to pick $w_n \in \mathcal{L}(\Sigma)$ such that $uw_n\sigma_1\sigma_2 \cdots \sigma_n \in \mathcal{L}(\Sigma)$ with $|w_n| \leq M$, where M is provided by the definition of SVGL. Since

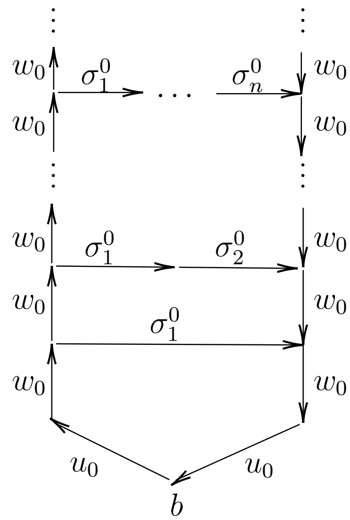

FIGURE 2. Any word in $\mathcal{W}$ starts and terminates at b.

$\{w_n \in \mathcal{L}(\Sigma) :\ |w_n| \leq M\}$ is finite, there is w in $\mathcal{L}(\Sigma)$ and an infinite subsequence n_i such that for all i, $w_{n_i} = w$. Let $\sigma' = uw\sigma_1\sigma_2\cdots$ be the unique point in $\cap_{i\in\mathbb{N}}[uw\sigma_1\sigma_2\cdots\sigma_{n_i}]$, and observe that for $x' \in (f_{uw})^{-1}(x)$, $\overline{\mathcal{O}^+_{\sigma'}(x')} = X$. This implies $\sigma' \in [u] \cap S$ and since $[u]$ was arbitrary, we are done. $\square$

Remark 4.4. Assume the hypothesis of Proposition 4.3, and let for some $\sigma \in \Sigma$, $\omega_\sigma(x)$ be the ω *limit set of* x *along* σ, that is the limit set of $\mathcal{O}^+_\sigma(x) = \{f_{\sigma_1\sigma_2\cdots\sigma_n}(x) :\ n \in \mathbb{N}\}$. The proof of Proposition 4.3 shows that

$$\{\sigma' \in \Sigma :\ \exists x' \in X \text{ s.t. } \omega_{\sigma'}(x') = \omega_\sigma(x)\}$$

is dense in Σ.

The following implications hold for irreducible shifts.

full shift $\Rightarrow$ SFT $\Rightarrow$ sofic $\Rightarrow$ SVGL $\Rightarrow$ synchronized $\Rightarrow$ coded.

Now we show that the SVGL property is a necessity in the hypothesis of Proposition 4.3. This in turn shows that the transitive non-autonomous systems in an IFS whose subshift is synchronized or beyond may be scarce.

Proposition 4.5. *There is* $\mathfrak{I}$ *satisfying in the conditions of the above proposition except that* Σ *is synchronized and for which* S *is not dense in* Σ.

Proof. Let X, $w_0 = 010$, $w_1 = 101$, f_0 and f_1 be as in Example 3.8, and set $u_0 = 000$. Let $\sigma^0 = \sigma_1^0\sigma_2^0\cdots\sigma_\ell^0\cdots = w_0w_1\cdots$ be defined as in (3.5) and let $\mathfrak{I} = (X, \{f_0, f_1\}, \Sigma)$, where Σ is generated by

$$\mathcal{W} = \{u_0w_0^\ell\sigma_1^0\sigma_2^0\cdots\sigma_\ell^0w_0^\ell u_0 :\ \ell \in \mathbb{N}\}.$$

See a presentation for Σ in Figure 2. Note that u_0u_0 is a synchronizing word and so Σ is synchronized.

If $v = v_1 \cdots v_{|v|} \in \mathcal{W}$, then 0 and 1 are fixed by f_v and for any other $x_i \in X \setminus \{0, 1\}$, $f_{v_1 \cdots v_\ell}(x_i) = x_j$, where $1 \leq \ell \leq |v|$ and $j > i$.

Observe that $\sigma^0 \in \Sigma$ does not have u_0 as a subword and also, by the same reasoning for $\mathfrak{I}_2$ in Example 3.8 , $\overline{O^+_{\sigma^0}(x)} = X$ for $x \in X \setminus \{0, 1\}$. On the other hand, if u_0 appears in $\sigma \in \Sigma$ infinitely (resp. finitely) many times, then by our construction where any u_0 appears only on the beginning or ending of members of $\mathcal{W}$, this σ must start with a terminal subword of a $w \in \mathcal{W}$, may be empty, and afterwards has some infinite concatenation of the members of $\mathcal{W}$ (resp. eventually will terminate at w_0^∞). This in turn implies that $\overline{O^+_\sigma(x)} \neq X$ for any $x \in X$. In fact, then 0 and 1 are fixed by the orbit along σ and any other x marches to 1 along that orbit with some relatively minor fluctuations. Hence, if σ is transitive, then for any x, $\overline{O^+_\sigma(x)} \neq X$. In particular, if $\sigma \in [u_0]$, then $\sigma \notin S$ and consequently S is not dense in Σ. $\square$

Observe that by Proposition 3.10, the conclusion of Proposition 4.3 is immediate when Σ is sofic; that is because the orbit of a transitive σ, attained by Proposition 4.3, is again in S and is dense in Σ. However, still we cannot guarantee that S is residual as the next example shows, even for a case where Σ is a mixing SFT.

Example 4.6. Let X and $\mathcal{F} = \{f_0, f_1\}$ be as in Example 3.8, and consider $\mathfrak{I}_\Sigma = (X, \mathcal{F}, \Sigma)$.

(1) First let $\Sigma = \Sigma_{|\mathcal{F}|}$ and let w_i be a word consisting of the concatenation of all words of length $i \in \mathbb{N}$ in $\mathcal{L}(\Sigma_{|\mathcal{F}|})$, and note that $\frac{0_{w_i}}{|w_i|} = \frac{1}{2}$. As a result, if $u = 1^{|w_i|} w_i$, then $f_u(x)$ moves $x \notin \{0, 1\}$ at least $\frac{|w_i|}{2}$ to left. Therefore, for the transitive
$$t = 1^{|w_1|} w_1 1^{|w_2|} w_2 1^{|w_3|} w_3 \cdots \in \Sigma_{|\mathcal{F}|},$$
$1 \notin \overline{\mathcal{O}^+_t(x)}$ and so $\overline{\mathcal{O}^+_t(x)} \neq X$. In particular, this shows that the conclusion of Proposition 3.10 is not necessarily valid for all transitive points in an irreducible sofic shift. Clearly $S(\mathfrak{I}_{\Sigma_{|\mathcal{F}|}})$, although dense, it is not closed and hence it is not a subshift.

(2) To complete our collection of the possible various cases of $S(\mathfrak{I})$, we construct an example, where $S(\mathfrak{I})$ is a dense uncountable but not residual subset of the subshift. To do this let $\Sigma_{\mathcal{W}}$ be the SFT generated by $\mathcal{W} = \{w_0 = 100, w_1 = 011, w_2 = 000\}$, and call the associated IFS $\mathfrak{I}_{\mathcal{W}}$.

We have $\frac{0_{w_0}}{|w_0|} = \frac{1_{w_1}}{|w_1|} = \frac{2}{3}$. Hence if σ^0 is chosen as in (3.5), then $\overline{\mathcal{O}^+_{\sigma^0}(x)} = X$ for $x \in X \setminus \{0, 1\}$. However, $\mathcal{O}^+_{0^\infty}(x)$ is not dense for any $x \in X$, and hence $S(\mathfrak{I}_{\mathcal{W}})$ is not closed and again not a subshift. Also, observe that the subshift $\Sigma_{\mathcal{W}'}$ generated by $\mathcal{W}' = \cup_{k \in \mathbb{N}} \{w_0^k w_1^k, w_1^k w_0^k\}$ is a subsystem of $\Sigma_{\mathcal{W}}$ and any transitive point of that lies in $S(\mathfrak{I}_{\mathcal{W}})$. The latter follows from the fact that $f_u(x) = x$ for $u = w_0^k w_1^k$ or $u = w_1^k w_0^k$, and the fact that $w_0^k w_1^k w_1^k w_0^k$ is a subword for a transitive point in $\Sigma_{\mathcal{W}'}$ for any $k \in \mathbb{N}$. Thus any $x \in X \setminus \{0, 1\}$ moves left and right as far as possible. This implies that $S(\mathfrak{I}_{\mathcal{W}'}) \subset S(\mathfrak{I}_{\mathcal{W}})$ has uncountable points.

Now we show that in this example, $S(\mathfrak{I}_{\mathcal{W}})$ is not a residual subset of $\Sigma_{\mathcal{W}}$. It is an easy consequence of the Birkhoff's ergodic theorem that the frequency of $w_i \in \mathcal{W}$ is $\frac{1}{3}$ for almost all $\sigma \in \Sigma_{\mathcal{W}}$ (we consider the Markov measure on $\Sigma_{\mathcal{W}}$: A unique ergodic Borel measure μ, which is positive on open sets and has the maximum metric entropy among all other measures). This means that the occurrence of 0 is as twice as that of 1 for almost all σ. Thus for $x \in X$ and almost all σ, $\overline{O_\sigma^+(x)} \neq X$. This in turn implies that $\mu(S(\mathfrak{I}_{\mathcal{W}})) = 0$. Now if $S(\mathfrak{I}_{\mathcal{W}})$ was residual in $\Sigma_{\mathcal{W}}$, then $S(\mathfrak{I}_{\mathcal{W}})$ would be measurable and since it is shift invariant it must have full measure, which is impossible for this example.

If one chooses w_2 in $\mathcal{W}$ to be 0000, then $\gcd\{|w_i| \ : \ w_i \in \mathcal{W}, 0 \leq i \leq 2\} = 1$, which implies that $\Sigma_{\mathcal{W}}$ is a mixing SFT ([1,8]). So either mixing or non-mixing, there are examples that S, in spite of being invariant and having a transitive point under the shift map, is not residual.

5. Mixing and exactness in an IFS

Clearly mixing along an orbit given in Definition 3.4 implies mixing defined in Definition 3.1 and the converse is not true as the next example shows. This example also shows that if the IFS is mixing, then we may not have mixing along an orbit.

Example 5.1. Let $\mathfrak{I} = (X = \{0,1\}^{\mathbb{N}}, \mathcal{F} = \{f_0, f_1\}, \Sigma_2)$, and for $\xi = \xi_1\xi_2 \cdots \in X$, define

$$f_0(\xi) = 0\xi = 0\xi_1\xi_2\cdots,$$
$$f_1(\xi) = 1\xi = 1\xi_1\xi_2\cdots.$$

For $w = w_1 \cdots w_{n-1}w_n$, set $w^{-1} := w_n w_{n-1} \cdots w_1$, and observe that $f_w(\xi) = w^{-1}\xi$. Now let $[u]$ and $[v]$ be any cylinder, and set $M := |v|$. Then for $m \geq M$ and $w \in \mathcal{L}_m(\Sigma)$, $f_w([u]) \cap [v] \neq \emptyset$ if and only if w is a word terminating at v^{-1} and hence $\mathfrak{I}$ is mixing. On the other hand, assume $\sigma \in \Sigma$, $v = 100$ and u any word. Now for $m \geq 2$, if $f_{\sigma_1\ldots\sigma_m}([u]) \cap [v] \neq \emptyset$, then $w = \sigma_1 \cdots \sigma_m$ terminates at v^{-1} but neither $w0$ nor $w1$ terminates at v^{-1}. This implies that both $f_{w0}([u]) \cap [v]$ and $f_{w1}([u]) \cap [v]$ are empty sets. Thus $\mathfrak{I}$ is not mixing along any orbit σ.

The next result shows that simple dynamics in the individual maps in an IFS may raise rich dynamics in the IFS. Intuitively, if we have two maps in an IFS where one flows all the points in a definite direction and the other on the opposite direction, then the arbitrary combination of these maps can create complicated dynamics. Example 5.1 had this property, but the IFS was not as rich as the following.

Example 5.2. Here we give an example such that none of the maps of the IFS, considering as a conventional dynamical system is transitive but the IFS itself is exact and thus mixing and topological transitive.

Let $\mathfrak{I} = (X = \{0, 1\}^{\mathbb{N}}, \{f_0, f_1\}, \Sigma_2 = \{0, 1\}^{\mathbb{N}})$ be an IFS, where for $\zeta = \zeta_1\zeta_2 \cdots \in \{0, 1\}^{\mathbb{N}}$,

$$f_i(\zeta) = \begin{cases} i\zeta_1\zeta_2 \cdots & \text{if} \quad \zeta_1 = i, \\ \zeta_2\zeta_3 \cdots & \text{if} \quad \zeta_1 \neq i. \end{cases} \tag{5.1}$$

We have the following observations:

(1) f_i is a finite to 1 surjective open map with 0^∞ and 1^∞ its only fixed points.

(2) f_0 (resp. f_1) attracts all points in $X \setminus \{1^\infty\}$ (resp. $X \setminus \{0^\infty\}$) and leaves the point 1^∞ (resp. 0^∞) fixed. Thus f_i is not transitive and has a very simple dynamics.

(3) Any $\zeta = \zeta_1\zeta_2 \cdots \in X$ is periodic of any given even period $p = 2q \in \mathbb{N}$ along $\sigma \in \Sigma$. To see this, set

$$\sigma = (\zeta_1^q \zeta_1^{*q})^\infty = \left(\overbrace{\zeta_1\zeta_1 \cdots \zeta_1}^{q \text{ times}} \overbrace{\zeta_1^*\zeta_1^* \cdots \zeta_1^*}^{q \text{ times}} \right)^\infty,$$

where for $a \in \mathcal{A} = \{0, 1\}$,

$$a^* = \begin{cases} 1, & a = 0, \\ 0, & a = 1. \end{cases} \tag{5.2}$$

Also, any transitive $\zeta = \zeta_1\zeta_2 \cdots \in X$ is transitive along the transitive point $\zeta^* = \zeta_1^*\zeta_2^* \cdots \in \Sigma$. A point such as $\zeta = (\zeta_1\zeta_2 \cdots \zeta_p)^\infty \in X$ is the periodic of period p along the periodic point $\zeta^* = (\zeta_1^*\zeta_1^* \cdots \zeta_p^*)^\infty \in \Sigma$.

(4) $\mathfrak{I}$ is exact along a transitive point.

Proof. Fix an open set $U \subseteq X$, and pick $w \in \mathcal{L}_k(\Sigma_{|\mathcal{F}|})$ such that $[w] \subseteq U$. Set $w^* := w_0^* \cdots w_k^*$, w_i^* defined as in (5.2), and note that $X = f_{w^*v}([w])$, where w^*v is any word whose initial segment is w^*.

The set $\mathcal{L}_m(\Sigma_{|\mathcal{F}|})$ has 2^m words. Set $\mathcal{P}_m(\mathcal{L}_m(\Sigma_{|\mathcal{F}|})) = \{v_1^m, \ldots, v_{2^m!}^m\} \subseteq \mathcal{L}_{m2^m}(\Sigma_{|\mathcal{F}|})$ to be the set of $2^m!$ words constructed from the permutation of words in $\mathcal{L}_m(\Sigma_{|\mathcal{F}|})$ and for $n > m$, let

$$t = v_1^1 v_2^1 \cdots v_1^m \cdots v_{2^m!}^m \cdots v_1^n \cdots v_{2^n!}^n \cdots = u_1 u_2 \cdots \in \Sigma_{|\mathcal{F}|},$$

be the transitive point, where $u_1 = v_1^1$, $u_2 = v_2^1$, and so on. So each u_i is one of the v_j^m's coming after each other in the obvious order. Observe that u_i has the same number of 0's and 1's and any word $v \in \mathcal{L}(\Sigma_{|\mathcal{F}|})$ appears as the initial segment of infinitely many u_i's. We will show that $\mathfrak{I}$ is exact along t.

Another observation is that for any word b such as u_i whose 0's and 1's are equal, and any cylinder $[a]$, $|f_b([a])| \leq |[a]|$.

Set $[a_i] := f_{u_1 \cdots u_i}([w])$ and note that $\{|a_i|\}_{i \in \mathbb{N}}$ is a nonincreasing sequence. Moreover, if $|a_{i+1}| < |a_i|$ for some $|w|$ instances of i's along t, then call the last instance ℓ and note that then $f_{u_1 \cdots u_\ell}([w]) = X$ and so in this case, this IFS is exact along t. Otherwise, without loss of generality, assume that for all $i \in \mathbb{N}$, $|a_i| = |w|$. We will show that this latter case does not happen and so we are done.

First let $b = b_1 \cdots b_{|b|}$ be any word and let $|f_b([a])| = |[a]|$, where $a = a_1 \cdots a_{|a|}$. Let $m(a,\, b) = \min\{|f_{b_1 \cdots b_i}([a])| :\ 1 \le i \le |b|\}$, and set

$$\alpha = \alpha(a,\, b) := \max\{i :\ |f_{b_1 \cdots b_i}([a])| = m(a,\, b),\ 1 \le i \le |b|\}.$$

In other words, $\alpha(a,\, b)$ is the last instance, where $f_{b_1 \cdots b_i}([a])$ has the shortest length. Let $f_{b_1 \cdots b_\alpha}([a]) = [a'] = [a'_1 \cdots a'_{|a'|}]$ for some a', $|a'| < |a|$. In fact a' is the terminal segment of a. Since $|f_{b_1 \cdots b_{\alpha+i}}([a])| > |a'|$ for $1 \le i \le |b| - \alpha$, by the definition of f_j's, $b_{\alpha+1} = a'_1$, and in particular $f_b([a]) = [{a'_1}^{\beta(a,\, b)} a']$, where

$$\beta(a,\, b) = |a| - |a'|.$$

Now assume $|a_i| = |w|$, and set $\alpha_i = \alpha(a_i,\, u_{i+1})$ and $\beta_i = \beta(a_i,\, u_{i+1})$. If $\beta_{i+1} \le \beta_i$, then $[a_i] = [a_{i+1}]$. So if there is $M \in \mathbb{N}$ such that for $i \ge M$, $\beta_{i+1} \le \beta_i$; or equivalently, for $i \ge M$, $[a_i] = [a_M]$, then along t, we arrive at u_ℓ whose initial segment is a_M^* and then $f_{u_1 \cdots u_\ell}([w]) = X$. This violates our assumption that $|a_i| = |w|$.

So the only other possibility is that $|a_i| = |w|$ and for any $M \in \mathbb{N}$, there is an $i > M$, where $0 \le \beta_i < \beta_{i+1} \le |a_i|$, which is clearly not possible. $\square$

The following is an immediate result from the above example.

Proposition 5.3. *There is a surjective IFS, which is exact along a transitive orbit and yet none of its constituent maps are transitive.*

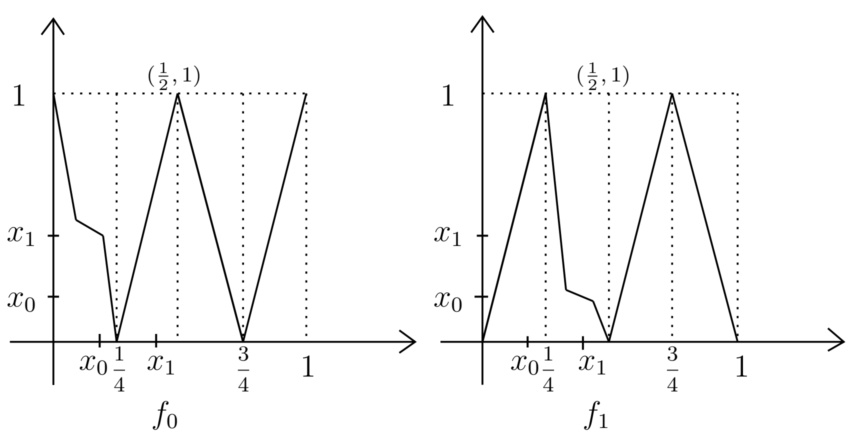

FIGURE 3. $([0, 1], f_i)$ is exact, but $\mathfrak{I} = ([0, 1], \{f_0, f_1\}, \{(01)^\infty, (10)^\infty\})$ is not even point transitive.

Next we give an example whose any map in the IFS is exact as a conventional dynamical system, though the IFS itself is not exact; somehow presenting opposite properties comparing to the previous example.

Example 5.4. Let $\mathfrak{I} = (X = [0,\, 1],\ \{f_0,\ f_1\},\ \Sigma = \{(01)^\infty,\ (10)^\infty\})$. To define f_i, choose two different points $x_0,\ x_1 \in (0,\ 1)$ and small open interval I_i around x_i such that $x_j \notin I_i$ if $j \ne i$. We aim to have

$$I_0 \xrightarrow{f_0} I_1 \xrightarrow{f_1} I_0$$

and f_0 (resp. f_1) being contracting on points of I_0 (resp. I_1) with an infimum rate $c_0 \in (\frac{1}{2}, 1)$ and elsewhere expansive with infimum rate $e_0 > 2$. An example of f_0 and f_1 can be those presented in Figure 3.

This construction guarantees that f_i being exact; however, a sufficiently small neighborhood around x_0 shrinks to a point along $\sigma = (01)^\infty$. Thus $\mathfrak{I}$ cannot be exact.

6. Conclusion and further work

Full shifts show up in many situations in dynamical systems, say, in classical IFS and various attractors. Then, the natural question is what happens if one replaces that full shift with a general subshift. Here, we considered that question for the very basic topological properties of the dynamics of an IFS, and we observed some diverse problems. However, we have left open other issues, such as ergodicity and stochastic problems, that are of interest when one is dealing with more than one map acting on a phase space. Also, dynamics along an orbit in this paper is, in fact, a non-autonomous dynamical system, and its dynamics with respect to the IFS, where that orbit belongs must be of interest. We hope that we or others can address some of these issues.

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