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HILBERT–SCHMIDTNESS OF FOURIER INTEGRAL OPERATORS IN SG CLASSES

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Abstract. In this paper, we define a particular class of Fourier integral operators with **SG**-symbol. These classes of operators turn out to be bounded on the spaces $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing functions and turn out to be Hilbert–Schmidt on $L^2(\mathbb{R}^n)$.

1. Introduction

In the early 1960s, the theory of pseudo-differential operators was born, and it later evolved into the theory of partial differential equations; see [17].

As a result, many subjects in these two theories, such as hypo-ellipticity of operators and **SG**-symbols [6, 7], are intimately related.

First and foremost, the **SG**-calculus on $\mathbb{R}^n$ may be traced back to early 1970s efforts by Cordes [8] and Parenti [21]. Schrohe [24] expanded on the idea by demonstrating that the **SG**-operators can be defined on a **SG**-manifold, a class that includes important noncompact manifolds.

In the framework of **SG**-manifolds with boundary, Erkip and Schrohe [16] studied boundary value problems as well. Melrose [19] devised the so-called scattering calculus on asymptotically Euclidean spaces, which coincides with the **SG**-calculus on $\mathbb{R}^n$. The book [9] by Cordes is a standard reference for **SG**-theory on $\mathbb{R}^n$ and on manifolds with ends. Shubin presented several classes of symbol satisfying global estimates on $\mathbb{R}^n$ to explore the features of Schrödinger operators with polynomially rising potential (see, e.g., [2–4, 26, 27]).

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Schulze [25], in particular, employed **SG**-symbols as an element in his pseudo-differential calculus on manifolds with singularities. Also, **SG**-symbols have been investigated by a number of authors in their classical form. Egorov and Schulze [15], Witt [28], and Coriasco and Panarese [12] are only a few examples.

However, this theory fails to be adequate for studying **SG** hyperbolic problems, and one is then forced to examine a wider class of operators, the so-called **SG**-Fourier integral operators. The corresponding classes of Fourier integral operators have been incorporated in the **SG**-calculus by Coriasco [10], together with their application for solving **SG**-hyperbolic problems.

The works [13, 14, 23] deal with the global L^p-boundedness ($1 < p < \infty$) of **SG**-Fourier integral operators.

Furthermore, Maniccia and Panarese [18], Nicola [20], Battisti and Coriasco [5], and Coriasco and Maniccia [11] have looked at the notion of noncommutative trace and the spectrum theory for **SG**-operators.

The aim of this paper is to study the Hilbert–Schmidtness of **SG**-Fourier integral operators. Let us now describe the plan of this article. In the second section, we introduce the relevant notations and preliminaries about Hilbert–Schmidt operators that will be used throughout the paper.

In the last section, we will go over some fundamental definitions and theorems from the theory of **SG**-Fourier integral operators, which will serve as the starting point for our main result.

We end this section by providing a motivation for the study of the topic of **SG**-Fourier integral operators.

Inspired by certain restriction problems, we consider first-order homogeneous systems of the form

$$\begin{cases} \partial_t u - iK(t)u \;=\; 0, \quad t \in [0,T], \quad T > 0, \\ \\ u(0) = u_0, \end{cases}$$

where K is a $(\nu \times \nu)$-matrix of pseudo-differential operators with symbol $k = (k_{ij})$ such that $k_{ij}(t; x, \xi) \in \mathcal{C}^\infty([0,T], \mathbf{SG}^{(1,1)})$ while u_0 is a vector valued function in L^2.

Furthermore, we will make the assumption that K is hyperbolic with diagonal principal part and constant multiplicities, k satisfies

$$\begin{cases} \partial_t k = k_0 + k_1, \\ k_0 \in \mathcal{C}^\infty([0,T], \mathbf{SG}^{(0,0)}), \\ k_1 \in \mathcal{C}^\infty([0,T], \mathbf{SG}^{(1,1)}), \end{cases}$$

where $k_1 = diag(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_\sigma)$, $\tilde{\lambda}_j = diag(\lambda_j, \ldots, \lambda_j)$ is an $l_j \times l_j$ diagonal matrix, with $\nu \geq l_j \geq 1$ and l_j is the multiplicity of $\lambda_j, j = 1, \ldots, \sigma \leq \nu$.

For each $\lambda_j \in \mathcal{C}^\infty([0,T], \mathbf{SG}^{(1,1)})$, we have

$$\lambda_{j+1}(t; x, \xi) - \lambda_j(t; x, \xi) \geq C_j \langle \xi \rangle \langle x \rangle, \quad j = 1, \ldots, \sigma - 1, \tag{1.1}$$

for suitable $C_j > 0$.

The usual wave operator $\partial_t^2 - \Delta_x,\ x \in \mathbb{R}^n$, is not **SG**-hyperbolic (its characteristic equation has real solutions $\pm|\xi|$, but they do not satisfy (1.1)). Indeed the operator $\partial_t^2 + (1+x^2)(1-\partial_x^2)$, $x \in \mathbb{R}$, can be taken as the standard model of second-order **SG**-hyperbolic operator (or **SG**-wave operator in one spatial dimension). In fact, the roots of the characteristic equation are $\lambda_{1,2} = \pm\langle x\rangle\langle\xi\rangle$ and obviously satisfy (1.1). More generally, we can consider operators of the form

$$L = \partial_t^2 + 2k_1(x)\partial_t\partial_x + k_2(x)\partial_x^2 + \langle \cdot, x\rangle^2, \tag{1.2}$$

where $k_1 \in \mathbf{SG}^{(0,1)}$ and $k_2 \in \mathbf{SG}^{(0,2)}$ satisfying the following condition:

$$\text{There exists } C > 0, \text{ for all } x \in \mathbb{R}, \text{ such that } C^{-1} \leq \frac{k_1^2(x) - k_2(x)}{\langle x\rangle^2} \leq C. \tag{1.3}$$

Assumption (1.3) assures that (1.1) is satisfied. Under such conditions, operators of the form (1.2) are strictly **SG**-hyperbolic (the characteristic roots have constant multiplicity one). As an example of operator with multiple characteristics in the **SG** environment, we can consider

$$L = \partial_t^2[\partial_t^2 + (1+x^2)(1-\partial_t^2)],$$

whose distinct characteristic roots $\lambda_{1,2} = \pm\langle x\rangle\langle\xi\rangle$ and $\lambda_3 = 0$ (double) again satisfy (1.1).

2. Preliminaries

We assume $n \in \mathbb{N}$ throughout the whole paper unless otherwise noted. In particular, $n \neq 0$. For all $x, \xi \in \mathbb{R}^n$, we define

$$\langle x, \xi\rangle := \sum_{j=0}^{n} x_j\xi_j \quad \textit{and} \quad \widehat{d\xi} := (2\pi)^{-n} d\xi.$$

Additionally, let us recall weight functions defined by

$$\langle\xi\rangle := \left(1 + |\xi|^2\right)^{1/2}$$

and

$$\lambda(x,\xi) := \left(1 + |x|^2 + |\xi|^2\right)^{1/2}.$$

Partial derivatives with respect to a variable $x \in \mathbb{R}^n$ scaled with the factor $-i$ are denoted by

$$D_x^\alpha := (-i)^{|\alpha|}\partial_x^\alpha := (-i)^{|\alpha|}\partial_{x_1}^{\alpha_1}\dots\partial_{x_n}^{\alpha_n},$$

where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ is a multi-index and $|\alpha| = \sum_{j=1}^n \alpha_j$ is the length of α.

Considering two Fréshet spaces E and F, the set $\mathcal{L}(E,F)$ contains of all linear and bounded operators $A : E \to F$. If $E = F$, then we also just write $\mathcal{L}(E)$.

Let $\mathcal{S}(\mathbb{R}^n)$ be the space of rapidly decreasing smooth functions (Schwartz space). We define the Fourier transform $\hat{u}$ and its inverse $\mathcal{F}^{-1}(u)$ of $u \in \mathcal{S}(\mathbb{R}^n)$ by

$$\hat{u}(\xi) = \mathcal{F}(u)(\xi) = \int_{\mathbb{R}^n} e^{-ix\xi}u(x)dx \quad \textit{and} \quad \mathcal{F}^{-1}(u)(x) = \int_{\mathbb{R}^n} e^{i\xi x}u(x)\widehat{d\xi}.$$

For all $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we set

$$d_x\varphi = (\partial_{x_1}\varphi, \dots, \partial_{x_n}\varphi),$$

and

$$\nabla_\xi \varphi = (\partial_{\xi_1}\varphi, \ldots, \partial_{\xi_n}\varphi)^t.$$

Lemma 2.1. *Let $s < \frac{-n}{2}$. Then $\langle x\rangle^s \in L^2(\mathbb{R}^n)$ for all $x \in \mathbb{R}^n$.*

Proof. An alternative approach can be found in [22, Lemma 1.3]. □

In what follows, we will look at the Hilbert–Schmidt operators, which are another significant class of bounded operators. The Hilbert–Schmidt operator class has a Hilbert space structure that is natural.

Definition 2.2. Let H_1 and H_2 be two Hilbert spaces. A bounded linear operator $A : H_1 \to H_2$ is called a Hilbert–Schmidt operator if for some orthonormal basis $\{e_n\}_{n=0}^{\infty}$ in H_1, we have

$$\sum_{n=0}^{\infty} \|Ae_n\|_{H_2}^2 < +\infty. \tag{2.1}$$

The set of all Hilbert–Schmidt operators $A : H_1 \to H_2$ is denoted by $\mathcal{C}_2\left(H_1, H_2\right)$, or $\mathcal{C}_2\left(H\right)$ in the case when $H_1 = H_2 = H$.

Remark 2.3. The Hilbert–Schmidt norm, also known as the Frobenius norm of the operator A, is defined as the square root of the left-hand side of (2.1) and is denoted by $\|.\|_2$.

Proposition 2.4. *Let $A \in \mathcal{C}_2\left(H\right)$.*

(1) *The Hilbert–Schmidt norm $\|\cdot\|_2$ is independent of the choice of orthonormal basis;*
(2) $\|A^*\|_2 = \|A\|_2$ *;*
(3) $\|A\| \leq \|A\|_2$*, where* $\|\cdot\|$ *is the usual operator norm;*
(4) *Every operator $A \in \mathcal{C}_2(H_1, H_2)$ is a compact operator.*

Lemma 2.5. *If $T \in \mathcal{L}\left(H\right)$, then $AT, TA \in \mathcal{C}_2\left(H\right)$ and*

$$\max\left\{\|AT\|_2, \|TA\|_2\right\} \leq \|T\|\|A\|_2.$$

Proof. [26] contains the proof of the above lemma. □

Now, let $\mathbb{R}^n$ be a space with a positive measure and let $H_1 = H_2 = L^2\left(\mathbb{R}^n\right)$. In this situation, the operators $A \in \mathcal{C}_2(H_1, H_2)$ are described as follows.

Theorem 2.6. *The operators $A \in \mathcal{C}_2(L^2\left(\mathbb{R}^n\right))$ are exactly those that can be represented as*

$$Au\left(x\right) = \int_{\mathbb{R}^n} k\left(x, y\right) u\left(y\right) dy, \tag{2.2}$$

with a kernel $k \in L^2\left(\mathbb{R}^{2n}\right)$. We then also have

$$\|A\|_2 = \|k\|_{L^2(\mathbb{R}^n)}. \tag{2.3}$$

For more details about this class of operators you can see [26].

3. SG-Fourier integral operators

Let $m = (m_1, m_2) \in \mathbb{R}^2$, and let ρ_j, δ_j be real numbers with $0 \leq \delta_j < \rho_j \leq 1,\ j \in \{1, 2\}$, and denote $\rho = (\rho_1, \rho_2)$ and $\delta = (\delta_1, \delta_2)$.

Definition 3.1. We say that a function $a(x, \xi) \in C^\infty(\mathbb{R}^{2n})$ is a symbol of class $\mathbf{SG}^m_{\rho,\delta}$ if for any $\alpha, \beta \in \mathbb{N}^n$, there exists $C_{\alpha,\beta} > 0$ such that

$$\left| D^\alpha_x D^\beta_\xi a(x, \xi) \right| \leq C_{\alpha,\beta} \langle x \rangle^{m_1 - \rho_1 |\alpha| + \delta_1 |\beta|} \langle \xi \rangle^{m_2 - \rho_2 |\beta| + \delta_2 |\alpha|},$$

for all $x, \xi \in \mathbb{R}^n$.

For $a \in \mathbf{SG}^m_{\rho,\delta}$, we can put the set of weighted semi-norms defined by

$$|a|_{\alpha,\beta} := \sup_{(x,\xi) \in \mathbb{R}^n} \langle x \rangle^{-m_1 + \rho_1 |\alpha| - \delta_1 |\beta|} \langle \xi \rangle^{-m_2 + \rho_2 |\beta| - \delta_2 |\alpha|} \left| D^\alpha_x D^\beta_\xi a(x, \xi) \right|.$$

Then, in terms of the topology produced by these semi-norms, $\mathbf{SG}^m_{\rho,\delta}$ is a Fréchet space.

Remark 3.2. When $\rho = (1, 1)$ and $\delta = (0, 0)$, we write $\mathbf{SG}^m$ instead of $\mathbf{SG}^m_{\rho,\delta}$.

Moreover, we define

$$\mathbf{SG}^\infty = \bigcup_{m \in \mathbb{R}^2} \mathbf{SG}^m, \quad \mathbf{SG}^{-\infty} = \bigcap_{m \in \mathbb{R}^2} \mathbf{SG}^m = \mathcal{S}(\mathbb{R}^{2n}).$$

Proposition 3.3. *Let $m, m' \in \mathbb{R}$.*

(i) *If $a \in \mathbf{SG}^m$ and $b \in \mathbf{SG}^{m'}$, then $ab \in \mathbf{SG}^{m+m'}$.*
(ii) *If $m \leq m'$, then $\mathbf{SG}^m \subseteq \mathbf{SG}^{m'}$.*
(iii) *If $a \in \mathbf{SG}^m$, then $\partial^\alpha_x \partial^\beta_\xi a \in \mathbf{SG}^{m''}$ such that*

$$m'' = m - |\alpha|\, e_1 - |\beta|\, e_2,$$

where $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

Proof. The proof is based on Leibniz's formula. □

Definition 3.4 (Phase functions)**.** We will call a phase function any smooth real valued function satisfying the following conditions:

(H_1) For all $(\alpha, \beta) \in \mathbb{N}^{2n}$, there exists $C_{\alpha,\beta} > 0$ such that

$$\left| \partial^\alpha_x \partial^\beta_\xi \varphi(x, \xi) \right| \leq C_{\alpha,\beta} \lambda^{2 - |\alpha| - |\beta|}(x, \xi).$$

(H_2) There exists ϱ_0 such that

$$\inf_{x,\xi \in \mathbb{R}^n} \left| \det \frac{\partial^2 \varphi}{\partial x \partial \xi}(x, \xi) \right| \geq \varrho_0.$$

For suitable constants $C_1, C_2 > 0$ and denote by Φ the set of all phase functions.

In addition, we define for all $\varepsilon > 0$, the set of all regular phase functions, denoted by Φ_ε, as follows:

$$\Phi_\varepsilon = \left\{ \varphi \in \Phi : \quad \text{for all } x, \xi \in \mathbb{R}^n \ \left| det \left(\frac{\partial^2 \varphi}{\varphi_{x_i} \varphi_{\xi_j}} \right) \right| \geq \varepsilon \right\}.$$

Example 3.5. Consider the function given by

$$\varphi\left(x,\xi\right)=\sum_{|\alpha|+|\beta|=2}K_{\alpha,\beta}x^{\alpha}\xi^{\beta},\qquad \text{for all } \left(x,\xi\right)\in\mathbb{R}^{2n},$$

where $K_{\alpha,\beta}\in\mathbb{R}$ for all $\alpha,\beta\in\mathbb{N}^n$.

Then $\varphi\left(x,\xi\right)$ verifies (H_1) and (H_2).

Let $a\in\mathbf{SG}^m$ and let $\varphi\in\Phi$. Then

$$A_{a,\varphi}u(x)=\int_{\mathbb{R}^n}e^{i\varphi(x,\xi)}a(x,\xi)\hat{u}(\xi)\widehat{d\xi}\quad \text{for all } u\in\mathcal{S}(\mathbb{R}^n),\tag{3.1}$$

defines the associated Fourier integral operator in **SG** classes (**SG**-Fourier integral operator). In particular, if $\varphi=\langle\cdot,\cdot\rangle$, then $A_{a,\varphi}:=Op\left(a\right)$ is called a **SG**-pseudo-differential operator.

To give a meaning to the right-hand side of (3.1) , we use the oscillatory integral method.

So we consider $g\in\mathcal{S}\left(\mathbb{R}^{2n}\right)$ with $g\left(0_{\mathbb{R}^{2n}}\right)=1$. If $a\in\mathbf{SG}^m$, then we define

$$a_r\left(x,\xi\right)=g\left(x/r,\xi/r\right)a\left(x,\xi\right),\qquad r>0.$$

Theorem 3.6. *If $\varphi\in\Phi$ and $a\in\mathbf{SG}^m$, then the following statements hold:*

(1) *For all $u\in\mathcal{S}\left(\mathbb{R}^n\right)$, $\lim_{r\to\infty}A_{a_r,\varphi}u\left(x\right)$ exists for every $x\in\mathbb{R}^n$ and is independent of the choice of the function g. We set then*

$$A_{a,\varphi}u:=\lim_{r\to\infty}A_{a_r,\varphi}u.$$

(2) *$A_{a,\varphi}\in\mathcal{L}\left(\mathcal{S}\left(\mathbb{R}^n\right)\right)$ and $A_{a,\varphi}\in\mathcal{L}\left(\mathcal{S}'\left(\mathbb{R}^n\right)\right)$.*

Proof. See [1, theorem 2.6] or [3]. □

Now, we have the following result concerning the Hilbert–Schmidtness of **SG**-Fourier integral operators.

Proposition 3.7. *Let us recall that $A_{a,\varphi}$ on $\mathbb{R}^n$ is an integral operators of the form*

$$A_{a,\varphi}u\left(x\right)=\int_{\mathbb{R}^n}e^{i\varphi(x,\xi)}a\left(x,\xi\right)\hat{u}\left(\xi\right)\widehat{d\xi},$$

where $a\in\mathbf{SG}^m$ and $\varphi\in\Phi$.

For any $m\in\mathbb{R}^2$ such that $\max(m_1,m_2)<-\frac{n}{2}$, $A_{a,\varphi}$ can be extended as a Hilbert–Schmidt operator on $L^2\left(\mathbb{R}^n\right)$.

Proof. Let us observe that the **SG**-Fourier integral operator $A_{a,\varphi}$can be written as

$$A_{a,\varphi}u=I_{a,\varphi}\left(\mathcal{F}u\right),\quad \text{for all } u\in\mathcal{S}(\mathbb{R}^n),\tag{3.2}$$

with

$$I_{a,\varphi}u(x)=\int_{\mathbb{R}^n}e^{i\varphi(x,\xi)}a\left(x,\xi\right)u\left(\xi\right)d\xi.$$

It results so from (3.2) and using lemma 2.5 that

$$\begin{aligned} \|A_{a,\varphi}\|_2 &= \|I_{a,\varphi}\mathcal{F}\|_2\,, \\ &\leq \|I_{a,\varphi}\|_2\,\|\mathcal{F}_h\|_{\mathcal{L}(L^2(\mathbb{R}^n))}\,. \end{aligned}$$

It is enough to prove $I_{a,\varphi} \in \mathcal{C}_2\left(L^2(\mathbb{R}^n)\right)$. First, let us observe that $I_{a,\varphi}$ has a integral representation just as (2.2) with kernel $k_{a,\varphi}(x,\xi)$. In fact, a straightforward computation shows us that

$$I_{a,\varphi}u(x) := \int_{\mathbb{R}^n} k_{a,\varphi}(x,\xi)\,u(\xi)\,d\xi,$$

where

$$k_{a,\varphi}(x,\xi) := e^{i\varphi(x,\xi)}a(x,\xi)\,.$$

Now let us show that, for $k_{a,\varphi} \in L^2\left(\mathbb{R}^{2n}\right)$,

$$\begin{aligned} |k_{a,\varphi}(x,\xi)| &= \left|e^{i\varphi(x,\xi)}a(x,\xi)\right| \\ &= |a(x,\xi)| \\ &\leq C_{0,0}\langle x\rangle^{m_1}\langle\xi\rangle^{m_2}. \end{aligned}$$

Then

$$\|k_{a,\varphi}\|^2_{L^2(\mathbb{R}^{2n})} \leq C^2_{0,0}\,\|\langle x\rangle^{m_1}\|^2_{L^2(\mathbb{R}^n)}\,\|\langle\xi\rangle^{m_2}\|^2_{L^2(\mathbb{R}^n)}\,.$$

We deduce from lemma 2.1 that

$$k_{a,\varphi} \in L^2\left(\mathbb{R}^{2n}\right),$$

for all $m \in \mathbb{R}^2$ such that $\max(m_1,m_2) < \frac{-n}{2}$, and from (2.3), we have

$$\|I_{a,\varphi}\|_2 = \|k_{a,\varphi}\|_{L^2(\mathbb{R}^{2n})} < +\infty,$$

which proves that $A_{a,\varphi}$ is a Hilbert–Schmidt operator. □

Example 3.8. We consider the Hermite operator

$$L = (-\Delta + x^2)^{-\frac{3}{2}},\ \ x \in \mathbb{R}.$$

The function $a(x,\xi) = (x^2+\xi^2)^{-\frac{3}{2}}$ is a pseudo-differential symbol of operator L. Since $a \in \mathbf{SG}^{(-3/2,-3/2)}(\mathbb{R}^2)$ and $-3/2 < -1$ then L is a Hilbert–Schmidt operator on $L^2(\mathbb{R})$.

4. Conclusion and some open problems

4.1. **Conclusion.** We considered a class of Fourier integral operators defined by **SG**-symbols and smooth phase functions. We proved, under some assumptions on the symbols, that these operators are Hilbert–Schmidt on L^2.

4.2. **Some open problems.** The following conditions can be investigated:

1. The Hilbert–Schmidtness of **SG**-Fourier integral operators on L^2 with a class of nonsmooth phase functions.
2. H^s-compactness of a class of **SG**-Fourier integral operators.
3. The boundedness of **SG**-Fourier integral operators on Holder spaces.

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