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FULLY S-IDEMPOTENT MODULES

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ABSTRACT. Let R be a commutative ring with identity, let S be a multiplicatively closed subset of R, and let M be an R-module. A submodule N of M is said to be *idempotent* if $N = (N :_R M)^2M$. Also, M is said to be *fully idempotent* if every submodule of M is idempotent. The aim of this paper is to introduce the concept of fully S-idempotent modules as a generalization of fully idempotent modules and investigate some properties of this class of modules.

1. Introduction

Throughout this paper, R will denote a commutative ring with identity and $\mathbb{Z}$ will denote the ring of integers. Also, S will denote a multiplicatively closed subset of R.

Let M be an R-module. The module M is said to be a *multiplication module* if for every submodule N of M, there exists an ideal I of R such that $N = IM$ [5]. It is easy to see that M is a multiplication module if and only if $N = (N :_R M)M$ for each submodule N of M. A submodule N of M is said to be *idempotent* if $N = (N :_R M)^2M$. Also, M is said to be *fully idempotent* if every submodule of M is idempotent [4].

In [1], the authors introduced and investigated the concept of S-multiplication modules as a generalization of multiplication modules. An R-module M is said to be an *S-multiplication module* if for each submodule N of M, there exist $s \in S$ and an ideal I of R such that $sN \subseteq IM \subseteq N$ [1]. One can see that M is an S-multiplication module if and only if for each submodule N of M there exists

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$s \in S$ such that $sN \subseteq (N :_R M)M \subseteq N$. The current studies on S-versions of some important classes of modules/rings can be found in [9, 10].

In this paper, we introduce the concept of fully S-idempotent R-modules as a generalization of fully idempotent modules and provide some useful information concerning this new class of modules. We say that a submodule N of an R-module M is an *S-idempotent submodule* if there exists $s \in S$ such that $sN \subseteq (N :_R M)^2 M \subseteq N$ (Definition 2.1(b)). We say that an R-module M is a *fully S-idempotent module* if every submodule of M is an S-idempotent submodule (Definition 2.1(c)). Clearly every fully idempotent R-module is a fully S-idempotent R-module (Remark 2.3(b)). Example 2.4 shows that the converse is not true in general. In Theorem 2.8, we characterize the fully idempotent R-modules. Also, we characterize the fully S-idempotent R-modules, where S satisfying the maximal multiple condition (Proposition 2.9). Let M_i be an R_i-module for $i = 1, 2, \dots, n$ and let $S_1, \dots, S_n$ be multiplicatively closed subsets of $R_1, \dots, R_n$, respectively. Assume that $M = M_1 \times \cdots \times M_n$, $R = R_1 \times \cdots \times R_n$, and $S = S_1 \times \cdots \times S_n$. Then we show that the following statements are equivalent:

(a) M is a fully S-idempotent module;
(b) M_i is a fully S_i-idempotent module for each $i \in \{1, 2, \dots, n\}$.

Also, among other results, it is shown that (Theorem 2.15) if M is an S-multiplication R-module and N is a submodule of M, then the following statements are equivalent:

(a) N is an S-pure submodule of M;
(b) N is an S-multiplication R-module and N is an S-idempotent submodule of M.

Finally, we prove that if M is a fully S-idempotent R-module, then M is a fully S-pure R-module. The converse holds if M is an S-multiplication R-module (Corollary 2.16).

2. Main results

Definition 2.1. (a) We say that an element x of an R-module M is an *S-idempotent element* if there exist $s \in S$ and $a \in (Rx :_R M)$ such that $sx = ax$.

(b) We say that a submodule N of an R-module M is an *S-idempotent submodule* if there exists $s \in S$ such that $sN \subseteq (N :_R M)^2 M \subseteq N$.

(c) We say that an R-module M is a *fully S-idempotent module* if every submodule of M is an S-idempotent submodule.

Example 2.2. Let M be an R-module with $\mathrm{Ann_R}(M) \cap S \neq \emptyset$. Then clearly, M is a fully S-idempotent R-module.

The following remarks can be immediately followed from Definition 2.1.

Remark 2.3. Let M be an R-module. Then we have the following properties:

(a) The submodules zero and M are always S-idempotent submodules of M. So each simple R-module is a fully S-idempotent R-module.
(b) Every fully idempotent R-module is a fully S-idempotent R-module.

(c) Every fully S-idempotent R-module is an S-multiplication R-module.
(d) If $S \subseteq \mathrm{U(R)}$, then every fully S-idempotent R-module is a fully idempotent R-module, where $\mathrm{U(R)}$ is the set of units in R.
(e) If $S_1 \subseteq S_2$ are multiplicatively closed subsets of R and M is a fully S_1-idempotent R-module, then M is a fully S_2-idempotent R-module.
(f) If N is an S-idempotent submodule of M, then by a similar argument to the proof of $((b) \Rightarrow (c))$ in [4, Lemma 2.2], one can see that there is $s \in S$ such that

$$sN \subseteq \mathrm{Hom}_R(M, N)N,$$

where $\mathrm{Hom}_R(M, N)N = \sum\{\varphi(N) : \varphi \in \mathrm{Hom}_R(M, N)\}$.

The following examples show that the converse of Remark 2.3(b, c, f) is not true in general.

Example 2.4. Take the $\mathbb{Z}$-module $M = \mathbb{Z}_{p^\infty}$ for a prime number p. Then we know that all proper submodules of M are of the form $G_t = \langle 1/p^t + \mathbb{Z}\rangle$ for some $t \in \mathbb{N} \cup \{0\}$ and $(G_t :_{\mathbb{Z}} M) = 0$. Therefore, M is not a fully idempotent $\mathbb{Z}$-module. Now, take the multiplicatively closed subset $S = \{p^n : n \in \mathbb{N} \cup \{0\}\}$ of $\mathbb{Z}$. Then $p^t G_t = 0 \subseteq (G_t :_{\mathbb{Z}} M)^2 M \subseteq G_t$. Hence, G_t is an S-idempotent submodule of M for each $t \in \mathbb{N} \cup \{0\}$. So, M is a fully S-idempotent $\mathbb{Z}$-module.

Example 2.5. Take the multiplicatively closed subset $S = \mathbb{Z} \setminus 2\mathbb{Z}$ of $\mathbb{Z}$. Then $\mathbb{Z}_4$ is an S-multiplication $\mathbb{Z}$-module. Indeed $\mathbb{Z}_4$ is not a fully S-idempotent $\mathbb{Z}$-module, because $2\mathbb{Z}_4$ is not an S-idempotent submodule of $\mathbb{Z}_4$.

Example 2.6. Let p be a prime number. Take the multiplicatively closed subset $S = \mathbb{Z} \setminus p\mathbb{Z}$ of $\mathbb{Z}$. Then one can see that the submodule $N = \mathbb{Z}_p \oplus 0$ of the $\mathbb{Z}$-module $M = \mathbb{Z}_p \oplus \mathbb{Z}_p$ is not S-idempotent, but $sN \subseteq \mathrm{Hom}_{\mathbb{Z}}(M, N)N = N$ for each $s \in S$.

The saturation S^* of S is defined as $S^* = \{x \in R : x/1 \text{ is a unit of } S^{-1}R\}$. It is obvious that S^* is a multiplicatively closed subset of R containing S [8].

A submodule N of an R-module M is said to be *S-copure* if there exists $s \in S$ such that $s(N :_M I) \subseteq N + (0 :_M I)$ for every ideal I of R [7].

Proposition 2.7. Let M be an R-module. Then we have the following properties:

(a) M is a fully S-idempotent R-module if and only if M is a fully S^*-idempotent R-module.
(b) If M is a fully S-idempotent R-module, then every submodule of M is a fully S-idempotent R-module.
(c) If M is an S-multiplication R-module and N is an S-copure submodule of M, then N is S-idempotent.

Proof. (a) Let M be a fully S-idempotent R-module. Since $S \subseteq S^*$, by Remark 2.3(e), M is a fully S^*-idempotent R-module. For the converse, assume that M is a fully S^*-idempotent module and that N is a submodule of M. Then there exists $x \in S^*$ such that $xN \subseteq (N :_R M)^2 M$. As $x \in S^*$, $x/1$ is a unit of $S^{-1}R$ and so $(x/1)(a/s) = 1$ for some $a \in R$ and $s \in S$. This yields that $us = uxa$ for some $u \in S$. Thus we have $usN = uxaN \subseteq xN \subseteq (N :_R M)^2 M$. Therefore, M is a fully S-idempotent R-module.

(b) Let N be a submodule of M and let K be a submodule of N. Then there exists $s \in S$ such that $sK \subseteq (K :_R M)^2 M \subseteq K$. This implies that

$$s^2 K \subseteq s(K :_R M)^2 M \subseteq s(K :_R M)K \subseteq (K :_R M)(K :_R M)^2 M \subseteq (K :_R M)^3 M.$$

Thus

$$s^2 K \subseteq (K :_R M)^3 M \subseteq (K :_R N)^2 (N :_R M)M \subseteq (K :_R N)^2 N.$$

Therefore, N is fully S-idempotent.

(c) Let M be an S-multiplication R-module and let N be an S-copure submodule of M. Then there exists $s \in S$ such that

$$s(N :_M (N :_R M)) \subseteq N + (0 :_M (N :_R M)).$$

This in turn implies that $sM \subseteq N + (0 :_M (N :_R M))$. It follows that

$$s(N :_R M)M \subseteq (N :_R M)N.$$

As M is an S-multiplication module, there is an element $t \in S$ such that $tN \subseteq (N :_R M)M$. Hence, we have

$$st^2 N \subseteq st(N :_R M)M \subseteq (N :_R M)tN \subseteq (N :_R M)^2 M,$$

as needed. □

In the following theorem, we characterize the fully idempotent R-modules.

Theorem 2.8. *Let M be an R-module. Then the following statements are equivalent:*

(a) *M is a fully idempotent R-module;*
(b) *M is a fully $(R \setminus \mathfrak{p})$-idempotent R-module for each prime ideal $\mathfrak{p}$ of R;*
(c) *M is a fully $(R \setminus \mathfrak{m})$-idempotent R-module for each maximal ideal $\mathfrak{m}$ of R;*
(d) *M is a fully $(R \setminus \mathfrak{m})$-idempotent R-module for each maximal ideal $\mathfrak{m}$ of R with $M_{\mathfrak{m}} \neq 0$.*

Proof. $(a) \Rightarrow (b)$. This follows from Remark 2.3(b).

$(b) \Rightarrow (c)$ and $(c) \Rightarrow (d)$. These are clear.

$(d) \Rightarrow (a)$. Let N be a submodule of M. Take a maximal ideal $\mathfrak{m}$ of R with $M_{\mathfrak{m}} \neq 0$. As M is a fully $(R \setminus \mathfrak{m})$-idempotent module, there exists $s \notin \mathfrak{m}$ such that $sN \subseteq (N :_R M)^2 M \subseteq N$. This implies that

$$N_{\mathfrak{m}} = (sN)_{\mathfrak{m}} \subseteq ((N :_R M)^2 M)_{\mathfrak{m}} \subseteq N_{\mathfrak{m}}.$$

If $M_{\mathfrak{m}} = 0$, then clearly $N_{\mathfrak{m}} = ((N :_R M)^2 M)_{\mathfrak{m}}$. Thus we conclude that $N_{\mathfrak{m}} = ((N :_R M)^2 M)_{\mathfrak{m}}$ for each maximal ideal $\mathfrak{m}$ of R. It follows that $N = (N :_R M)^2 M$, as needed. □

A multiplicatively closed subset S of R is said to satisfy the *maximal multiple condition* if there exists $s \in S$ such that $t \mid s$ for each $t \in S$.

In the following theorem, we characterize the fully S-idempotent R-modules, where S is a multiplicatively closed subset of R satisfying the maximal multiple condition.

Proposition 2.9. Let S be a multiplicatively closed subset of R satisfying the maximal multiple condition (e.g., S is finite or $S \subseteq \mathrm{U(R)}$) and let M be an R-module. Then the following statements are equivalent:

(a) M is a fully S-idempotent module;
(b) Every cyclic submodule of M is S-idempotent;
(c) Every element of M is S-idempotent;
(d) For all submodules N and K of M, we have $s(N \cap K) \subseteq (N :_R M)(K :_R M)M$ for some $s \in S$.

Proof. $(a) \Rightarrow (b)$ and $(b) \Rightarrow (c)$ are clear.

$(c) \Rightarrow (a)$. Let N be a submodule of M and let $x \in N$. Then by the hypothesis, there exist $s_x \in S$ and $a \in (Rx :_R M)$ such that $s_x x = ax$. Hence $as_x x = a^2 x$, and so $s_x^2 x = s_x ax = as_x x = a^2 x$. Thus $s_x^2 Rx \subseteq (Rx :_R M)^2 M$. Now as S satisfying the maximal multiple condition, there exists $s \in S$ such that $sRx \subseteq (Rx :_R M)^2 M \subseteq (N :_R M)^2 M$. Therefore, $sN \subseteq (N :_R M)^2 M$, as required.

$(a) \Rightarrow (d)$. Let N and K be two submodules of M. Then for some $s \in S$, we have

$$s(N \cap K) \subseteq (N \cap K :_R M)^2 M \subseteq (N :_R M)(K :_R M)M.$$

$(d) \Rightarrow (a)$. For a submodule N of M, we have

$$sN = s(N \cap N) \subseteq (N :_R M)(N :_R M)M = (N :_R M)^2 M$$

for some $s \in S$. □

Let R_i be a commutative ring with identity, let M_i be an R_i-module for each $i = 1, 2, \ldots, n$, and let $n \in \mathbb{N}$. Assume that $M = M_1 \times M_2 \times \cdots \times M_n$ and that $R = R_1 \times R_2 \times \cdots \times R_n$. Then M is clearly an R-module with componentwise addition and scalar multiplication. Also, if S_i is a multiplicatively closed subset of R_i for each $i = 1, 2, \ldots, n$, then $S = S_1 \times S_2 \times \cdots \times S_n$ is a multiplicatively closed subset of R. Furthermore, each submodule N of M is of the form $N = N_1 \times N_2 \times \cdots \times N_n$, where N_i is a submodule of M_i.

Theorem 2.10. *Let M_i be an R_i-module for $i = 1, 2, \ldots, n$ and let $S_1, \ldots, S_n$ be multiplicatively closed subsets of $R_1, \ldots, R_n$, respectively. Assume that $M = M_1 \times \cdots \times M_n$, $R = R_1 \times \cdots \times R_n$ and $S = S_1 \times \cdots \times S_n$. Then M is a fully S-idempotent module if and only if M_i is a fully S_i-idempotent module for each $i \in \{1, 2, \ldots, n\}$.*

Proof. We use mathematical induction. If $n = 1$, then the claim is trivial. Now suppose that $n = 2$. For *only if* part, without loss of generality, we will show that M_1 is a fully S_1-idempotent R_1-module. Take a submodule N_1 of M_1. Then $N_1 \times \{0\}$ is a submodule of M. Since M is a fully S-idempotent R-module, there exists $s = (s_1, s_2) \in S_1 \times S_2$ such that $(s_1, s_2)(N_1 \times \{0\}) \subseteq (N_1 \times \{0\} :_R M)^2 M$. By focusing on the first coordinate, we have $s_1 N_1 \subseteq (N_1 :_{R_1} M_1)^2 M_1$. So M_1 is a fully S_1-idempotent R_1-module. Now assume that M_1 is a fully S_1-idempotent module and that M_2 is a fully S_2-idempotent module. Take a submodule N of M. Then N must be in the form of $N_1 \times N_2$, where $N_1 \subseteq M_1, N_2 \subseteq M_2$. Since M_1 is a fully S_1-idempotent R_1-module, there exists $s_1 \in S_1$ such that

$s_1N_1 \subseteq (N_1 :_{R_1} M_1)^2M_1$. Similarly, there exists an element $s_2 \in S_2$ such that $s_2N_2 \subseteq (N_2 :_{R_2} M_2)^2M_2$. Now, put $s = (s_1, s_2) \in S$. Then we get

$$(s_1, s_2)N \subseteq s_1N_1 \times s_2N_2 \subseteq (N_1 :_{R_1} M_1)^2M_1 \times (N_2 :_{R_2} M_2)^2M_2 \subseteq (N :_R M)^2M.$$

Hence, M is a fully S-idempotent R-module.

Next, assume that the claim is true for $n < k$, and we will show that it is also true for $n = k$. Put $M = (M_1 \times \cdots \times M_{n-1}) \times M_n$, $R = (R_1 \times R_2 \times \cdots \times R_{n-1}) \times R_n$, and $S = (S_1 \times \cdots \times S_{n-1}) \times S_n$. By the case when $n = 2$, M is a fully S-idempotent module if and only if $M_1 \times \cdots \times M_{n-1}$ is a fully $(S_1 \times \cdots \times S_{n-1})$-idempotent $(R_1 \times R_2 \times \cdots \times R_{n-1})$-module and M_n is a fully S_n-idempotent R_n-module. Now the rest follows from the induction hypothesis. □

Let M be an R-module. The *idealization* or *trivial extension* $R \propto M = R \oplus M$ of M is a commutative ring with componentwise addition and multiplication $(a, m)(b, \acute{m}) = (ab, a\acute{m} + bm)$ for each $a, b \in R$, $m, \acute{m} \in M$ [2]. If I is an ideal of R and N is a submodule of M, then $I \propto N$ is an ideal of $R \propto M$ if and only if $IM \subseteq N$. In that case, $I \propto N$ is called a *homogeneous ideal* of $R \propto M$. Also, if $S \subseteq R$ is a multiplicatively closed subset, then $S \propto N$ is a multiplicatively closed subset of $R \propto M$ [2, Theorem 3.8].

Let I be an ideal of R. If I is a fully S-idempotent R-module, then we say that I is a *fully S-idempotent ideal* of R.

Theorem 2.11. *Let N be a submodule of an R-module M. Then the following statements are equivalent:*

(a) *N is a fully S-idempotent R-module;*
(b) *$0 \propto N$ is a fully $(S \propto 0)$-idempotent ideal of $R \propto M$;*
(c) *$0 \propto N$ is a fully $(S \propto M)$-idempotent ideal of $R \propto M$.*

Proof. $(a) \Rightarrow (b)$. Suppose that N is a fully S-idempotent R-module. Take an ideal J of $R \propto M$ contained in $0 \propto N$. Then $J = 0 \propto \acute{N}$ for some submodule $\acute{N}$ of M with $\acute{N} \subseteq N$. Since N is a fully S-idempotent module, there exists $s \in S$ with $s\acute{N} \subseteq (\acute{N} :_R N)^2N \subseteq \acute{N}$. First, note that $(J :_{R\propto M} 0 \propto N) = (\acute{N} :_R N) \propto M$. So this gives $(J :_{R\propto M} 0 \propto N)^2 = ((\acute{N} :_R N) \propto M)^2 = (\acute{N} :_R N)^2 \propto (\acute{N} :_R N)M$. Then we have $(J :_{R\propto M} 0 \propto N)^2(0 \propto N) = 0 \propto (\acute{N} :_R N)^2N$. This implies that

$$\begin{aligned}(s, 0)J =& 0 \propto s\acute{N} \subseteq 0 \propto (\acute{N} :_R N)^2N \\ =& (J :_{R\propto M} 0 \propto N)^2(0 \propto N) \subseteq J.\end{aligned}$$

It follows that $0 \propto N$ is a fully $(S \propto 0)$-idempotent ideal of $R \propto M$.

$(b) \Rightarrow (c)$. This follows from the fact that $S \propto 0 \subseteq S \propto M$ and Remark 2.3(e).

$(c) \Rightarrow (a)$. Suppose that $0 \propto N$ is a fully $(S \propto M)$-idempotent ideal of $R \propto M$. Let $\acute{N}$ be a submodule of N. Then $0 \propto \acute{N} \subseteq 0 \propto N$ and $0 \propto \acute{N}$ is an ideal of $R \propto M$. Since $0 \propto N$ is a fully $(S \propto M)$-idempotent ideal of $R \propto M$, there exists $(s, m) \in S \propto M$ such that

$$(s, m)(0 \propto \acute{N}) \subseteq ((0 \propto \acute{N}) :_{R\propto M} (0 \propto N))^2(0 \propto N) \subseteq 0 \propto \acute{N}.$$

One can easily check that

$$(0 \propto \acute{N}) :_{R\propto M} (0 \propto N) = (\acute{N} :_R N) \propto M,$$

$$((\acute{N} :_R N) \propto M)^2(0 \propto N) = 0 \propto (\acute{N} :_R N)^2 N.$$

Thus

$$\begin{aligned}(s, m)(0 \propto \acute{N}) &= 0 \propto s\acute{N} \subseteq ((\acute{N} :_R N) \propto M)^2(0 \propto N)\\ &= 0 \propto (\acute{N} :_R N)^2 N \subseteq 0 \propto \acute{N},\end{aligned}$$

and so $s\acute{N} \subseteq (\acute{N} :_R N)^2 N \subseteq \acute{N}$. Hence, N is a fully S-idempotent R-module. □

Proposition 2.12. Let M and $\acute{M}$ be R-modules. Assume that $f : M \to \acute{M}$ is an R-epimorphism. If M is a fully S-idempotent module, then $\acute{M}$ is a fully S-idempotent module.

Proof. Let $\acute{N}$ be a submodule of $\acute{M}$. Then $N := f^{-1}(\acute{N})$ is a submodule of M. As M is a fully S-idempotent module, there exists $s \in S$ such that $sN \subseteq (N :_R M)^2 M \subseteq N$. Hence, $f(sN) \subseteq f((N :_R M)^2 M) \subseteq f(N)$. This yields that

$$s\acute{N} = sf(N) \subseteq (N :_R M)^2 f(M) = (N :_R M)^2 \acute{M} \subseteq \acute{N}.$$

Since f is an epimorphism, one can easily see that $(N :_R M) = (\acute{N} :_R \acute{M})$. Thus $s\acute{N} \subseteq (\acute{N} :_R \acute{M})^2 \acute{M} \subseteq \acute{N}$. Hence, $\acute{M}$ is a fully S-idempotent module. □

Corollary 2.13. Let M be a fully S-idempotent R-module and let N be a submodule of M. Then M/N is a fully S-idempotent R-module.

Theorem 2.14. *Let M be an R-module and let S and T be multiplicatively closed subsets of R. Put $\tilde{S} = \{s/1 \in T^{-1}R : s \in S\}$, a multiplicatively closed subset of $T^{-1}R$. Then we have the following properties:*

(a) *If M is a fully S-idempotent R-module, then $T^{-1}M$ is a fully $\tilde{S}$-idempotent $T^{-1}R$-module.*
(b) *If M is a fully S-idempotent R-module and $S \subseteq T^*$, then $T^{-1}M$ is a fully idempotent $T^{-1}R$-module.*
(c) *If M is a fully S-idempotent R-module, then $S^{-1}M$ is a fully idempotent $S^{-1}R$-module.*
(d) *If M is a finitely generated R-module, S satisfies the maximal multiple condition, and $S^{-1}M$ is a fully idempotent $S^{-1}R$-module, then M is a fully S-idempotent module.*

Proof. (a) Let N be a $T^{-1}R$-submodule of $T^{-1}M$. Then $N = T^{-1}\acute{N}$ for some submodule $\acute{N}$ of M. Since M is a fully S-idempotent module, there exists $s \in S$ with $s\acute{N} \subseteq (\acute{N} :_R M)^2 M \subseteq \acute{N}$. Then

$$(s/1)N = T^{-1}(s\acute{N}) \subseteq (T^{-1}(\acute{N} :_R M)^2)(T^{-1}M) \subseteq T^{-1}\acute{N} = N.$$

So $T^{-1}M$ is a fully $\tilde{S}$-idempotent $T^{-1}R$-module.

(b) If $S \subseteq T^*$, then $\tilde{S} \subseteq U(T^{-1}R)$. Hence, $T^{-1}M$ is a fully idempotent $T^{-1}R$-module by Remark 2.3(d) and part (a).

(c) This follows from part (b).

(d) Let $S^{-1}M$ be a fully idempotent $S^{-1}R$-module. Take a submodule N of M. Since $S^{-1}M$ is a fully idempotent $S^{-1}R$-module, we have

$$S^{-1}N = (S^{-1}N :_{S^{-1}R} S^{-1}M)^2(S^{-1}M).$$

As M is a finitely generated R-module, $(S^{-1}N :_{S^{-1}R} S^{-1}M) = S^{-1}(N :_R M)$. Thus $S^{-1}N = S^{-1}((N :_R M)^2M)$. Choose $s \in S$ with $t \mid s$ for each $t \in S$. Note that for each $m \in N$, we have $m/1 \in S^{-1}N = S^{-1}((N :_R M)^2M)$ and so there exists $t \in S$ such that $tm \in (N :_R M)^2M$, and hence $sm \in (N :_R M)^2M$. Therefore, we obtain

$$s^2N \subseteq s(N :_R M)^2M \subseteq (N :_R M)^2M \subseteq N.$$

Hence, M is a fully S-idempotent module. □

Let M be an R-module. A submodule N of M is said to be *pure* if $IN = N \cap IM$ for every ideal I of R [3]. Also, M is said to be *fully pure* if every submodule of M is pure [4]. A submodule N of M is said to be *S-pure* if there exists $s \in S$ such that $s(N \cap IM) \subseteq IN$ for every ideal I of R [6]. Moreover, M is said to be *fully S-pure* if every submodule of M is S-pure [6].

Theorem 2.15. *Let M be an S-multiplication R-module and let N be a submodule of M. Then the following statements are equivalent:*

(a) *N is an S-pure submodule of M;*
(b) *N is an S-multiplication R-module and N is an S-idempotent submodule of M;*
(c) *N is an S-multiplication R-module and there exists $s \in S$ such that $sK \subseteq (N :_R M)K$, for all submodules K of N;*
(d) *N is an S-multiplication R-module and there exists $s \in S$ such that $s(K :_R N)N \subseteq (K :_R M)(N :_R M)M$, for all submodules K of M.*

Proof. $(a) \Rightarrow (b)$. Let K be a submodule of N. As M is an S-multiplication module, there exists $s \in S$ such that $sK \subseteq (K :_R M)M$. Now since N is S-pure, there is an element $t \in S$ such that $(K :_R N)N \supseteq t(N \cap (K :_R N)M)$. Hence,

$$\begin{aligned}(K :_R N)N &\supseteq t(N \cap (K :_R N)M) \supseteq t(N \cap (K :_R M)M)\\ &\supseteq t(N \cap sK) = tsK.\end{aligned}$$

This implies that N is an S-multiplication R-module. Since M is an S-multiplication module, there exists $u \in S$ such that $uN \subseteq (N :_R M)M$. Now as N is S-pure, there is an element $v \in S$ such that $(N :_R M)uN \supseteq v(N \cap u(N :_R M)M)$. Therefore,

$$\begin{aligned}(N :_R M)^2M &= (N :_R M)(N :_R M)M \supseteq (N :_R M)uN\\ &\supseteq v(N \cap u(N :_R M)M) = vu(N :_R M)M \supseteq vu^2N.\end{aligned}$$

So, N is an S-idempotent submodule.

$(b) \Rightarrow (c)$. Let K be a submodule of N. Since N is an S-multiplication R-module, there exists $s \in S$ such that $sK \subseteq (K :_R N)N$. As N is S-idempotent, there is $t \in S$ such that $tN \subseteq (N :_R M)^2M$. Therefore,

$$\begin{aligned}tsK &\subseteq t(K :_R N)N = (K :_R N)tN\\ &\subseteq (K :_R N)(N :_R M)^2M = (N :_R M)(K :_R N)(N :_R M)M\\ &\subseteq (N :_R M)(K :_R N)N \subseteq (N :_R M)K.\end{aligned}$$

$(c) \Rightarrow (a)$. Let I be an ideal of R. Since $N \cap IM \subseteq N$, by part (c), there is $s \in S$ such that $s(N \cap IM) \subseteq (N :_R M)(N \cap IM)$. Hence,

$$s(N \cap IM) \subseteq (N \cap IM)(N :_R M) \subseteq IM(N :_R M) = IN.$$

Thus N is an S-pure submodule of M.

$(b) \Rightarrow (d)$. Let K be a submodule of M. Since N is S-idempotent, there is $s \in S$ such that $sN \subseteq (N :_R M)^2 M$. So

$$s(K :_R N)N \subseteq (K :_R N)(N :_R M)^2 M \subseteq (K :_R M)(N :_R M)M.$$

$(d) \Rightarrow (b)$. Take $K = N$. □

Corollary 2.16. Let M be an R-module. Then we have the following results:

(a) If M is a fully S-idempotent R-module, then M is a fully S-pure R-module.

(b) If M is an S-multiplication fully S-pure R-module, then M is a fully S-idempotent R-module.

Proof. (a) By Proposition 2.7(b), every submodule of M is a fully S-idempotent R-module. Hence, by Remark 2.3(c), every submodule of M is an S-multiplication R-module. Now the result follows from Theorem 2.15 $(b) \Rightarrow (a)$.

(b) This follows from Theorem 2.15 $(a) \Rightarrow (b)$. □

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