



ZYGMUND-TYPE INEQUALITIES FOR AN OPERATOR PRESERVING INEQUALITIES BETWEEN POLYNOMIALS

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ABSTRACT. In this paper, we present certain new L_p inequalities for \mathcal{B}_n -operators which include some known polynomial inequalities as special cases.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let \mathcal{P}_n denote the space of all complex polynomials $P(z) = \sum_{j=0}^n a_j z^j$ of degree n . For $P \in \mathcal{P}_n$, define

$$\begin{aligned} \|P(z)\|_0 &:= \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \log |P(e^{i\theta})| d\theta \right\}, \\ \|P(z)\|_p &:= \left\{ \frac{1}{2\pi} \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{1/p}, \quad 0 < p < \infty, \\ \|P(z)\|_\infty &:= \max_{|z|=1} |P(z)|, \quad m := \min_{|z|=1} |P(z)|, \end{aligned}$$

and denote for any complex function $\psi : \mathbb{C} \rightarrow \mathbb{C}$ the composite function of P and ψ , defined by $(P \circ \psi)(z) := P(\psi(z))$ ($z \in \mathbb{C}$), as $P \circ \psi$.

If $P \in \mathcal{P}_n$, then

$$\|P'(z)\|_p \leq n \|P(z)\|_p, \quad p \geq 1 \tag{1.1}$$

and

$$\|P(Rz)\|_p \leq R^n \|P(z)\|_p, \quad R > 1, \quad p > 0. \tag{1.2}$$

Inequality (1.1) was found out by Zygmund [20] whereas inequality (1.2) is a simple consequence of a result of Hardy [8]. Arestov [2] proved that (1.1) remains true for $0 < p < 1$ as well. For $p = \infty$, the inequality (1.1) is due to Bernstein

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(for reference, see [11, 15, 18]) whereas the case $p = \infty$ of inequality (1.2) is a simple consequence of the maximum modulus principle (see [11, 12, 15]). Both the inequalities (1.1) and (1.2) can be sharpened if we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$. In fact, if $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then inequalities (1.1) and (1.2) can be respectively replaced by

$$\|P'(z)\|_p \leq n \frac{\|P(z)\|_p}{\|1+z\|_p}, \quad p \geq 0 \quad (1.3)$$

and

$$\|P(Rz)\|_p \leq \frac{\|R^n z + 1\|_p}{\|1+z\|_p} \|P(z)\|_p, \quad R > 1, \quad p > 0. \quad (1.4)$$

Inequality (1.3) is due to De-Bruijn [7](see also [3]) for $p \geq 1$. Rahman and Schmeisser [1] extended it for $0 < p < 1$, whereas the inequality (1.4) was proved by Boas and Rahman [6] for $p \geq 1$ and later it was extended for $0 < p < 1$ by Rahman and Schmeisser [14]. For $p = \infty$, the inequality (1.3) was conjectured by Erdős and later verified by Lax [9] whereas inequality (1.4) was proved by Ankeny and Rivlin [1].

As a compact generalization of inequalities (1.3) and (1.4), Aziz and Rather [5] proved that if $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1, |\beta| \leq 1, R > r \geq 1$ and $p > 0$,

$$\|P(Rz) + \phi_n(R, r, \alpha, \beta) P(rz)\|_p \leq \frac{C_p}{\|1+z\|_p} \|P(z)\|_p \quad (1.5)$$

where

$$C_p = \|(R^n + \phi_n(R, r, \alpha, \beta)r^n)z + (1 + \phi_n(R, r, \alpha, \beta))\|_p \quad (1.6)$$

and

$$\phi_n(R, r, \alpha, \beta) = \beta \left\{ \left(\frac{R+1}{r+1} \right)^n - |\alpha| \right\} - \alpha. \quad (1.7)$$

If we take $\beta = 0, \alpha = 1$ and $r = 1$ in (1.5) and divide two sides of (1.5) by $R - 1$ then make $R \rightarrow 1$, we obtain inequality (1.3). Whereas inequality (1.4) is obtained from (1.5) by taking $\alpha = \beta = 0$.

Rahman [13] (see also Rahman and Schmeisser [15, p. 538]) introduced a class \mathcal{B}_n of operators B that maps $P \in \mathcal{P}_n$ into itself. That is, the operator B carries $P \in \mathcal{P}_n$ into a polynomial

$$B[P](z) := \lambda_0 P(z) + \lambda_1 \binom{nz}{2} \frac{P'(z)}{1!} + \lambda_2 \binom{nz}{2}^2 \frac{P''(z)}{2!} \quad (1.8)$$

where λ_0, λ_1 and λ_2 are such that all the zeros of

$$u(z) := \lambda_0 + C(n, 1)\lambda_1 z + C(n, 2)\lambda_2 z^2, \quad C(n, r) = n!/r!(n-r)!,$$

lie in the half plane

$$|z| \leq |z - n/2|. \quad (1.9)$$

While extending Bernstein type inequalities to \mathcal{B}_n operators, they [13] proved that if $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then

$$|B[P \circ \sigma](z)| \leq \frac{1}{2} \{R^n |\Lambda_n| + |\lambda_0|\} \|P(z)\|_\infty \quad \text{for } |z| = 1, \quad (1.10)$$

(see [13, Inequalities (5.2) and (5.3)]) where $\sigma(z) = Rz$, $R \geq 1$ and

$$\Lambda_n := \lambda_0 + \lambda_1 \frac{n^2}{2} + \lambda_2 \frac{n^3(n-1)}{8}. \quad (1.11)$$

As an extension of inequality (1.10) to L_p -norm, recently W.M. Shah and A. Liman [19] while seeking the desired extension, have made an incomplete attempt [19, Theorem 2] by claiming to have proved that if $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for each $R \geq 1$ and $p \geq 1$,

$$\|B[P \circ \sigma](z)\|_p \leq \frac{R^n |\Lambda_n| + |\lambda_0|}{\|1+z\|_p} \|P(z)\|_p, \quad (1.12)$$

where $B \in \mathcal{B}_n$ and $\sigma(z) = Rz$ and Λ_n is defined by (1.11).

Rather and Shah [17] pointed an error in the proof of (1.12), they not only provided a correct proof but also extended it for $0 \leq p < 1$ as well. They proved:

Theorem A. *If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish for $|z| < 1$, then for $0 \leq p < \infty$ and $R > 1$,*

$$\|B[P \circ \sigma](z)\|_p \leq \frac{\|R^n \Lambda_n z + \lambda_0\|_p}{\|1+z\|_p} \|P(z)\|_p, \quad (1.13)$$

$B \in \mathcal{B}_n$, $\sigma(z) = Rz$ and Λ_n is defined by (1.11). The result is sharp as shown by $P(z) = az^n + b$, $|a| = |b| = 1$.

Recently, Rather and Suhail Gulzar [16] obtained the following result which is a generalization of Theorem A.

Theorem B. *If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish for $|z| < 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$, $0 \leq p < \infty$ and $R > 1$,*

$$\|B[P \circ \sigma](z) - \alpha B[P](z)\|_p \leq \frac{\|(R^n - \alpha)\Lambda_n z + (1 - \alpha)\lambda_0\|_p}{\|1+z\|_p} \|P(z)\|_p, \quad (1.14)$$

where $B \in \mathcal{B}_n$, $\sigma(z) = Rz$ and Λ_n is defined by (1.11). The result is best possible and equality in (1.14) holds for $P(z) = az^n + b$, $|a| = |b| = 1$.

If we take $\alpha = 0$ in Theorem B, we obtain Theorem A.

In this paper, we investigate the dependence of

$$\|B[P \circ \sigma](z) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](z)\|_p$$

on $\|P(z)\|_p$ for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq 1$, $0 \leq p < \infty$, $\sigma(z) := Rz$, $\rho(z) := rz$ and $\phi_n(R, r, \alpha, \beta)$ is given by (1.7), and establish certain generalized L_p -mean extensions of the inequality (1.10) for $0 \leq p < \infty$ and also a generalization of (1.5). In this direction, we first present the following result which is a compact generalization of the inequalities (1.3), (1.4), (1.5) and (1.10) for $0 \leq p < 1$ as well.

Theorem 1.1. *If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq 1$ and $0 \leq p < \infty$,*

$$\begin{aligned} & \|B[P \circ \sigma](z) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](z)\|_p \\ & \leq \frac{\|(R^n + \phi_n(R, r, \alpha, \beta)r^n)\Lambda_n z + (1 + \phi_n(R, r, \alpha, \beta))\lambda_0\|_p}{\|1+z\|_p} \|P(z)\|_p \end{aligned} \quad (1.15)$$

where $B \in \mathcal{B}_n$, $\sigma(z) := Rz$, $\rho(z) := rz$, Λ_n and $\phi_n(R, r, \alpha, \beta)$ are defined by (1.7) and (1.11) respectively. The result is best possible and equality in (1.15) holds for $P(z) = az^n + b$, $|a| = |b| \neq 0$

Remark 1.2. If we take $\lambda_1 = \lambda_2 = 0$ in (1.15), we obtain inequality (1.5).

For $\beta = 0$, inequality (1.15) reduces the following result.

Corollary 1.3. *If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for every real or complex number α with $|\alpha| \leq 1$, $R > r \geq 1$ and $0 \leq p < \infty$,*

$$\begin{aligned} & \|B[P \circ \sigma](z) - \alpha B[P \circ \rho](z)\|_p \\ & \leq \frac{\|(R^n - \alpha r^n)\Lambda_n z + (1 - \alpha)\lambda_0\|_p}{\|1 + z\|_p} \|P(z)\|_p \end{aligned} \quad (1.16)$$

where $B \in \mathcal{B}_n$, $\sigma(z) := Rz$, $\rho(z) := rz$ and Λ_n is defined by (1.11). The result is best possible and equality in (1.16) holds for $P(z) = az^n + b$, $|a| = |b| \neq 0$.

Remark 1.4. For taking $\alpha = 0$ in (1.16), we obtain Theorem A and for $r = 1$ in (1.16), we get Theorem B.

Instead of proving Theorem 1.1, we prove the following more general result which includes Theorem 1.1 as a special case.

Theorem 1.5. *If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for $\alpha, \beta, \delta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $|\delta| \leq 1$, $R > r \geq 1$ and $0 \leq p < \infty$,*

$$\begin{aligned} & \left\| B[P \circ \sigma](z) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](z) \right. \\ & \quad \left. + \delta \frac{\left(|R^n + \phi_n(R, r, \alpha, \beta) r^n| |\Lambda_n| - |1 + \phi_n(R, r, \alpha, \beta)| |\lambda_0| \right) m}{2} \right\|_p \\ & \leq \frac{\|(R^n + \phi_n(R, r, \alpha, \beta) r^n) \Lambda_n z + (1 + \phi_n(R, r, \alpha, \beta)) \lambda_0\|_p}{\|1 + z\|_p} \|P(z)\|_p \end{aligned} \quad (1.17)$$

where $B \in \mathcal{B}_n$, $\sigma(z) := Rz$, $\rho(z) := rz$, $m = \min_{|z|=1} |P(z)|$ and $\phi_n(R, r, \alpha, \beta)$, Λ_n are defined by (1.7) and (1.11), respectively. The result is best possible and equality in (1.15) holds for $P(z) = az^n + b$, $|a| = |b| \neq 0$.

Remark 1.6. For $\delta = 0$ in (1.17), we get Theorem 1.1.

The next corollary which is a generalization of (1.5) follows by taking $\lambda_1 = \lambda_2 = 0$ in (1.17).

Corollary 1.7. *If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for $\alpha, \beta, \delta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $|\delta| \leq 1$, $R > r \geq 1$ and $0 \leq p < \infty$,*

$$\begin{aligned} & \left\| P(Rz) + \phi_n(R, r, \alpha, \beta) P(rz) \right. \\ & \quad \left. + \delta \frac{\left(|R^n + \phi_n(R, r, \alpha, \beta) r^n| - |1 + \phi_n(R, r, \alpha, \beta)| \right) m}{2} \right\|_p \end{aligned}$$

$$\leq \frac{\|(R^n + \phi_n(R, r, \alpha, \beta)r^n)z + (1 + \phi_n(R, r, \alpha, \beta))\|_p}{\|1 + z\|_p} \|P(z)\|_p \quad (1.18)$$

where $m = \min_{|z|=1} |P(z)|$ and $\phi_n(R, r, \alpha, \beta)$ is defined by (1.7). The result is best possible and equality in (1.18) holds for $P(z) = az^n + b$, $|a| = |b| \neq 0$.

2. LEMMAS

For the proofs of these theorems, we need the following lemmas. The first Lemma is easy to prove.

Lemma 2.1. *If $P \in \mathcal{P}_n$ and $P(z)$ has all its zeros in $|z| \leq 1$, then for every $R \geq r \geq 1$ and $|z| = 1$,*

$$|P(Rz)| \geq \left(\frac{R+1}{r+1}\right)^n |P(rz)|.$$

The following Lemma follows from [10, Corollary 18.3, p. 65].

Lemma 2.2. *If all the zeros of polynomial $P \in \mathcal{P}_n$ lie in $|z| \leq 1$, then all the zeros of the polynomial $B[P](z)$ also lie in $|z| \leq 1$.*

Lemma 2.3. *If $F \in \mathcal{P}_n$ has all its zeros in $|z| \leq 1$ and $P(z)$ is a polynomial of degree at most n such that*

$$|P(z)| \leq |F(z)| \text{ for } |z| = 1,$$

then for every $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq 1$, and $|z| \geq 1$,

$$\begin{aligned} |B[P \circ \sigma](z) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](z)| \\ \leq |B[F \circ \sigma](z) + \phi_n(R, r, \alpha, \beta) B[F \circ \rho](z)| \end{aligned} \quad (2.1)$$

where $B \in \mathcal{B}_n$, $\sigma(z) := Rz$, $\rho(z) := rz$, Λ_n and $\phi_n(R, r, \alpha, \beta)$ are defined by (1.11) and (1.7) respectively.

Proof. Since the polynomial $F(z)$ of degree n has all its zeros in $|z| \leq 1$ and $P(z)$ is a polynomial of degree at most n such that

$$|P(z)| \leq |F(z)| \text{ for } |z| = 1, \quad (2.2)$$

therefore, if $F(z)$ has a zero of multiplicity s at $z = e^{i\theta_0}$, then $P(z)$ has a zero of multiplicity at least s at $z = e^{i\theta_0}$. If $P(z)/F(z)$ is a constant, then the inequality (2.1) is obvious. We now assume that $P(z)/F(z)$ is not a constant, so that by the maximum modulus principle, it follows that

$$|P(z)| < |F(z)| \text{ for } |z| < 1.$$

Suppose $F(z)$ has m zeros on $|z| = 1$ where $0 \leq m \leq n$, so that we can write

$$F(z) = F_1(z)F_2(z)$$

where $F_1(z)$ is a polynomial of degree m whose all zeros lie on $|z| = 1$ and $F_2(z)$ is a polynomial of degree exactly $n - m$ having all its zeros in $|z| < 1$. This implies with the help of inequality (2.2) that

$$P(z) = P_1(z)F_1(z)$$

where $P_1(z)$ is a polynomial of degree at most $n - m$. Now, from inequality (2.2), we get

$$|P_1(z)| \leq |F_2(z)| \text{ for } |z| = 1$$

where $F_2(z) \neq 0$ for $|z| = 1$. Therefore for every $\lambda \in \mathbb{C}$ with $|\lambda| > 1$, a direct application of Rouché's theorem shows that the zeros of the polynomial $P_1(z) - \lambda F_2(z)$ of degree $n - m \geq 1$ lie in $|z| < 1$. Hence the polynomial

$$f(z) = F_1(z)(P_1(z) - \lambda F_2(z)) = P(z) - \lambda F(z)$$

has all its zeros in $|z| \leq 1$ with at least one zero in $|z| < 1$, so that we can write

$$f(z) = (z - te^{i\delta})H(z)$$

where $t < 1$ and $H(z)$ is a polynomial of degree $n - 1$ having all its zeros in $|z| \leq 1$. Applying Lemma 2.1 to the polynomial $f(z)$ with $k = 1$, we obtain for every $R > r \geq 1$ and $0 \leq \theta < 2\pi$,

$$\begin{aligned} |f(Re^{i\theta})| &= |Re^{i\theta} - te^{i\delta}| |H(Re^{i\theta})| \\ &\geq |Re^{i\theta} - te^{i\delta}| \left(\frac{R+1}{r+1}\right)^{n-1} |H(re^{i\theta})| \\ &= \left(\frac{R+1}{r+1}\right)^{n-1} \frac{|Re^{i\theta} - te^{i\delta}|}{|re^{i\theta} - te^{i\delta}|} |(re^{i\theta} - te^{i\delta})H(re^{i\theta})| \\ &\geq \left(\frac{R+1}{r+1}\right)^{n-1} \left(\frac{R+t}{r+t}\right) |f(re^{i\theta})|. \end{aligned}$$

This implies for $R > r \geq 1$ and $0 \leq \theta < 2\pi$,

$$\left(\frac{r+t}{R+t}\right) |f(Re^{i\theta})| \geq \left(\frac{R+1}{r+1}\right)^{n-1} |f(re^{i\theta})|. \quad (2.3)$$

Since $R > r \geq 1 > t$ so that $f(Re^{i\theta}) \neq 0$ for $0 \leq \theta < 2\pi$ and $\frac{1+r}{1+R} > \frac{r+t}{R+t}$, from inequality (2.3), we obtain $R > r \geq 1$ and $0 \leq \theta < 2\pi$,

$$|f(Re^{i\theta})| > \left(\frac{R+1}{r+1}\right)^n |f(re^{i\theta})|. \quad (2.4)$$

Equivalently,

$$|f(Rz)| > \left(\frac{R+1}{r+1}\right)^n |f(rz)|$$

for $|z| = 1$ and $R > r \geq 1$. Hence for every $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$ and $R > r \geq 1$, we have

$$\begin{aligned} |f(Rz) - \alpha f(rz)| &\geq |f(Rz)| - |\alpha| |f(rz)| \\ &> \left\{ \left(\frac{R+1}{r+1}\right)^n - |\alpha| \right\} |f(rz)|, \quad |z| = 1. \end{aligned}$$

Also, inequality (2.4) can be written in the form

$$|f(re^{i\theta})| < \left(\frac{r+1}{R+1}\right)^n |f(Re^{i\theta})| \quad (2.5)$$

for every $R > r \geq 1$ and $0 \leq \theta < 2\pi$. Since $f(Re^{i\theta}) \neq 0$ and $\left(\frac{r+1}{R+1}\right)^n < 1$, from inequality (2.5), we obtain for $0 \leq \theta < 2\pi$ and $R > r \geq 1$,

$$|f(re^{i\theta})| < |f(Re^{i\theta})|.$$

Equivalently,

$$|f(rz)| < |f(Rz)| \text{ for } |z| = 1.$$

Since all the zeros of $f(Rz)$ lie in $|z| \leq (1/R) < 1$, a direct application of Rouché's theorem shows that the polynomial $f(Rz) - \alpha f(rz)$ has all its zeros in $|z| < 1$ for every $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$. Applying Rouché's theorem again, it follows from (2.4) that for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1, |\beta| \leq 1$ and $R > r \geq 1$, all the zeros of the polynomial

$$\begin{aligned} T(z) &= f(Rz) - \alpha f(rz) + \beta \left\{ \left(\frac{R+1}{r+1} \right)^n - |\alpha| \right\} f(rz) \\ &= f(Rz) + \phi_n(R, r, \alpha, \beta) f(rz) \\ &= (P(Rz) - \lambda F(Rz)) + \phi_n(R, r, \alpha, \beta) (P(rz) - \lambda F(rz)) \\ &= (P(Rz) + \phi_n(R, r, \alpha, \beta) P(rz)) - \lambda (F(Rz) + \phi_n(R, r, \alpha, \beta) F(rz)) \end{aligned}$$

lie in $|z| < 1$ for every $\lambda \in \mathbb{C}$ with $|\lambda| > 1$. Using Lemma 2.2 and the fact that B is a linear operator, we conclude that all the zeros of polynomial

$$\begin{aligned} W(z) &= B[T](z) \\ &= (B[P \circ \sigma](z) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](z)) \\ &\quad - \lambda (B[F \circ \sigma](z) + \phi_n(R, r, \alpha, \beta) B[F \circ \rho](z)) \end{aligned}$$

also lie in $|z| < 1$ for every λ with $|\lambda| > 1$. This implies

$$\begin{aligned} |B[P \circ \sigma](z) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](z)| \\ \leq |B[F \circ \sigma](z) + \phi_n(R, r, \alpha, \beta) B[F \circ \rho](z)| \end{aligned} \quad (2.6)$$

for $|z| \geq 1$ and $R > r \geq 1$. If inequality (2.6) is not true, then there exists a point $z = z_0$ with $|z_0| \geq 1$ such that

$$|B[P \circ \sigma](z_0) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](z_0)| > |B[F \circ \sigma](z_0) + \phi_n(R, r, \alpha, \beta) B[F \circ \rho](z_0)|.$$

But all the zeros of $F(Rz)$ lie in $|z| < 1$, therefore, it follows (as in case of $f(z)$) that all the zeros of $F(Rz) + \phi_n(R, r, \alpha, \beta) F(rz)$ lie in $|z| < 1$. Hence by Lemma 2.2, all the zeros of $B[F \circ \sigma](z) + \phi_n(R, r, \alpha, \beta) B[F \circ \rho](z)$ also lie in $|z| < 1$, which shows that

$$B[F \circ \sigma](z_0) + \phi_n(R, r, \alpha, \beta) B[F \circ \rho](z_0) \neq 0.$$

We take

$$\lambda = \frac{B[P \circ \sigma](z_0) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](z_0)}{B[F \circ \sigma](z_0) + \phi_n(R, r, \alpha, \beta) B[F \circ \rho](z_0)},$$

then λ is a well defined real or complex number with $|\lambda| > 1$ and with this choice of λ , we obtain $W(z_0) = 0$. This contradicts the fact that all the zeros of $W(z)$ lie in $|z| < 1$. Thus (2.6) holds and this completes the proof of Lemma 2.3. \square

Lemma 2.4. *If $P \in \mathcal{P}_n$ and $P(z)$ has all its zeros in $|z| \leq 1$, then for every $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1, |\beta| \leq 1$ and $|z| \geq 1$,*

$$\begin{aligned} |B[P \circ \sigma](z) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](z)| \\ \geq |R^n + \phi_n(R, r, \alpha, \beta) r^n| |\Lambda_n| |z|^n m \end{aligned} \quad (2.7)$$

where $m = \min_{|z|=1} |P(z)|$, $B \in \mathcal{B}_n$, $\sigma(z) = Rz$, $\rho(z) = rz$, Λ_n and $\phi_n(R, r, \alpha, \beta)$ are defined by (1.11) and (1.7), respectively.

Proof. By hypothesis, all the zeros of $P(z)$ lie in $|z| \leq 1$ and

$$m|z|^n \leq |P(z)| \quad \text{for } |z| = 1.$$

We first show that the polynomial $g(z) = P(z) - \lambda m z^n$ has all its zeros in $|z| \leq 1$ for every $\lambda \in \mathbb{C}$ with $|\lambda| < 1$. This is obvious if $m = 0$, that is if $P(z)$ has a zero on $|z| = 1$. Henceforth, we assume $P(z)$ has all its zeros in $|z| < 1$, then $m > 0$ and it follows by Rouché's theorem that the polynomial $g(z)$ has all its zeros in $|z| < 1$ for every $\lambda \in \mathbb{C}$ with $|\lambda| < 1$. Proceeding similarly as in the proof of Lemma 2.3, we obtain that for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1, |\beta| \leq 1$ and $R > r \geq 1$, all the zeros of the polynomial

$$\begin{aligned} H(z) &= g(Rz) - \alpha g(rz) + \beta \left\{ \left(\frac{R+1}{r+1} \right)^n - |\alpha| \right\} g(rz) \\ &= g(Rz) + \phi_n(R, r, \alpha, \beta) g(rz) \\ &= (P(Rz) - \lambda R^n z^n m) + \phi_n(R, r, \alpha, \beta) (P(rz) - \lambda r^n z^n m) \\ &= (P(Rz) + \phi_n(R, r, \alpha, \beta) P(rz)) - \lambda (R^n + \phi_n(R, r, \alpha, \beta) r^n) m z^n \end{aligned}$$

lie in $|z| < 1$. Applying Lemma 2.1 to $H(z)$ and noting that B is a linear operator, it follows that all the zeros of polynomial

$$\begin{aligned} B[H](z) &= \{B[P \circ \sigma](z) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](z)\} \\ &\quad - \lambda (R^n + \phi_n(R, r, \alpha, \beta) r^n) m B[z^n] \end{aligned} \quad (2.8)$$

lie in $|z| < 1$. This gives

$$\begin{aligned} |B[P \circ \sigma](z) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](z)| \\ \geq |R^n + \phi_n(R, r, \alpha, \beta) r^n| |\Lambda_n| |z|^n m \quad \text{for } |z| \geq 1. \end{aligned} \quad (2.9)$$

If (2.9) is not true, then there is point w with $|w| \geq 1$ such that

$$|B[P \circ \sigma](w) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](w)| < |R^n + \phi_n(R, r, \alpha, \beta) r^n| |\Lambda_n| |w|^n m.$$

We choose

$$\lambda = \frac{B[P \circ \sigma](w) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](w)}{R^n + \phi_n(R, r, \alpha, \beta) r^n |\Lambda_n| |w|^n m},$$

then clearly $|\lambda| < 1$ and with this choice of λ , from (2.8), we get $B[H](w) = 0$ with $|w| \geq 1$. This is clearly a contradiction to the fact that all the zeros of $H(z)$ lie in $|z| < 1$. Thus for every $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1, |\beta| \leq 1$,

$$|B[P \circ \sigma](z) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](z)| \geq |R^n + \phi_n(R, r, \alpha, \beta) r^n| |\Lambda_n| |z|^n m$$

for $|z| \geq 1$ and $R > r \geq 1$. □

Lemma 2.5. *If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for every $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1, |\beta| \leq 1, R > r \geq 1$ and $|z| \geq 1$,*

$$\begin{aligned} & |B[P \circ \sigma](z) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](z)| \\ & \leq |B[P^* \circ \sigma](z) + \phi_n(R, r, \alpha, \beta) B[P^* \circ \rho](z)| \end{aligned}$$

where $P^*(z) := z^n \overline{P(1/\bar{z})}$, $B \in \mathcal{B}_n$, $\sigma(z) := Rz$, $\rho(z) := rz$, and $\phi_n(R, r, \alpha, \beta)$ is defined by (1.7).

Proof. By hypothesis the polynomial $P(z)$ of degree n does not vanish in $|z| < 1$, therefore, all the zeros of the polynomial $P^*(z) = z^n \overline{P(1/\bar{z})}$ of degree n lie in $|z| \leq 1$. Applying Lemma 2.3 with $F(z)$ replaced by $P^*(z)$, it follows that

$$\begin{aligned} & |B[P \circ \sigma](z) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](z)| \\ & \leq |B[P^* \circ \sigma](z) + \phi_n(R, r, \alpha, \beta) B[P^* \circ \rho](z)| \end{aligned}$$

for $|z| \geq 1, |\alpha| \leq 1, |\beta| \leq 1$ and $R > r \geq 1$. This proves the Lemma 2.5. \square

Lemma 2.6. *If $P \in \mathcal{P}_n$ and $P(z)$ has no zeros in $|z| < 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1, R > r \geq 1$ and $|z| \geq 1$,*

$$\begin{aligned} & |B[P \circ \sigma](z) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](z)| \\ & \leq |B[P^* \circ \sigma](z) + \phi_n(R, r, \alpha, \beta) B[P^* \circ \rho](z)| \\ & \quad - \left(|R^n + \phi_n(R, r, \alpha, \beta) r^n| |\Lambda_n| - |1 + \phi_n(R, r, \alpha, \beta)| |\lambda_0| \right) m, \quad (2.10) \end{aligned}$$

where $P^*(z) = z^n \overline{P(1/\bar{z})}$, $m = \min_{|z|=1} |P(z)|$, $B \in \mathcal{B}_n$, $\sigma(z) = Rz$, $\rho(z) = rz$, Λ_n and $\phi_n(R, r, \alpha, \beta)$ are given by (1.11) and (1.7), respectively.

Proof. By hypothesis $P(z)$ has all its zeros in $|z| \geq 1$ and

$$m \leq |P(z)| \quad \text{for } |z| = 1. \quad (2.11)$$

We show $F(z) = P(z) + \lambda m$ does not vanish in $|z| < 1$ for every $\lambda \in \mathbb{C}$ with $|\lambda| < 1$. This is obvious if $m = 0$ that is, if $P(z)$ has a zero on $|z| = 1$. So we assume all the zeros of $P(z)$ lie in $|z| > 1$, then $m > 0$ and by the maximum modulus principle, it follows from (2.11) that

$$m < |P(z)| \quad \text{for } |z| < 1. \quad (2.12)$$

Now if $F(z) = P(z) + \lambda m = 0$ for some z_0 with $|z_0| < 1$, then

$$P(z_0) + \lambda m = 0.$$

This implies

$$|P(z_0)| = |\lambda| m \leq m, \quad \text{for } |z_0| < 1$$

which is clearly contradiction to (2.12). Thus the polynomial $F(z)$ does not vanish in $|z| < 1$ for every λ with $|\lambda| < 1$. Applying Lemma 2.3 to the polynomial $F(z)$, we get

$$\begin{aligned} & |B[F \circ \sigma](z) + \phi_n(R, r, \alpha, \beta) B[F \circ \rho](z)| \\ & \leq |B[F^* \circ \sigma](z) + \phi_n(R, r, \alpha, \beta) B[F^* \circ \rho](z)| \end{aligned}$$

for $|z| = 1$ and $R > r \geq 1$. Replacing $F(z)$ by $P(z) + \lambda m$, we obtain

$$\begin{aligned} & |B[P \circ \sigma](z) + \phi_n(R, r, \alpha, \beta)B[P \circ \rho](z) + \lambda(1 + \phi_n(R, r, \alpha, \beta))\lambda_0 m| \\ & \leq |B[P^* \circ \sigma](z) + \phi_n(R, r, \alpha, \beta)B[P^* \circ \rho](z) \\ & \quad + \bar{\lambda}(R^n + \phi_n(R, r, \alpha, \beta)r^n)\Lambda_n z^n m| \end{aligned} \quad (2.13)$$

Now choosing the argument of λ in the right hand side of (2.13) such that

$$\begin{aligned} & |B[P^* \circ \sigma](z) + \phi_n(R, r, \alpha, \beta)B[P^* \circ \rho](z) + \bar{\lambda}(R^n + \phi_n(R, r, \alpha, \beta)r^n)\Lambda_n z^n m| \\ & = |B[P^* \circ \sigma](z) + \phi_n(R, r, \alpha, \beta)B[P^* \circ \rho](z)| \\ & \quad - |\bar{\lambda}| |R^n + \phi_n(R, r, \alpha, \beta)r^n| |\Lambda_n| |z|^n m. \end{aligned}$$

for $|z| = 1$, which is possible by Lemma 2.4, we get

$$\begin{aligned} & |B[P \circ \sigma](z) + \phi_n(R, r, \alpha, \beta)B[P \circ \rho](z)| - |\lambda| |1 + \phi_n(R, r, \alpha, \beta)| |\lambda_0| m \\ & \leq |B[P^* \circ \sigma](z) + \phi_n(R, r, \alpha, \beta)B[P^* \circ \rho](z)| \\ & \quad - |\lambda| |R^n + \phi_n(R, r, \alpha, \beta)r^n| |\Lambda_n| |z|^n m. \end{aligned}$$

Equivalently,

$$\begin{aligned} & |B[P \circ \sigma](z) + \phi_n(R, r, \alpha, \beta)B[P \circ \rho](z)| \\ & \leq |B[P^* \circ \sigma](z) + \phi_n(R, r, \alpha, \beta)B[P^* \circ \rho](z)| \\ & \quad - |\lambda| \left(|R^n + \phi_n(R, r, \alpha, \beta)r^n| |\Lambda_n| - |1 + \phi_n(R, r, \alpha, \beta)| |\lambda_0| \right) m. \end{aligned} \quad (2.14)$$

Letting $|\lambda| \rightarrow 1$ in (2.14) we obtain inequality (2.10) and this completes the proof of Lemma 2.6. \square

Next we describe a result of Arestov [2].

For $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_n) \in \mathbb{C}^{n+1}$ and $P(z) = \sum_{j=0}^n a_j z^j$, we define

$$C_\gamma P(z) = \sum_{j=0}^n \gamma_j a_j z^j.$$

The operator C_γ is said to be admissible if it preserves one of the following properties:

- (i) $P(z)$ has all its zeros in $\{z \in \mathbb{C} : |z| \leq 1\}$,
- (ii) $P(z)$ has all its zeros in $\{z \in \mathbb{C} : |z| \geq 1\}$.

The result of Arestov may now be stated as follows.

Lemma 2.7. [2, Theorem 2] *Let $\phi(x) = \psi(\log x)$ where ψ is a convex non-decreasing function on \mathbb{R} . Then for all $P \in \mathcal{P}_n$ and each admissible operator C_γ ,*

$$\int_0^{2\pi} \phi(|C_\gamma P(e^{i\theta})|) d\theta \leq \int_0^{2\pi} \phi(c(\gamma, n)|P(e^{i\theta})|) d\theta,$$

where $c(\gamma, n) = \max(|\gamma_0|, |\gamma_n|)$.

In particular Lemma 2.7 applies with $\phi : x \rightarrow x^p$ for every $p \in (0, \infty)$ and $\phi : x \rightarrow \log x$ as well. Therefore, we have for $0 \leq p < \infty$,

$$\left\{ \int_0^{2\pi} \phi(|C_\gamma P(e^{i\theta})|^p) d\theta \right\}^{1/p} \leq c(\gamma, n) \left\{ \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{1/p}. \quad (2.15)$$

From Lemma 2.7, we deduce the following result.

Lemma 2.8. *If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for each $p > 0$, $R > 1$ and η real, $0 \leq \eta < 2\pi$,*

$$\begin{aligned} & \int_0^{2\pi} |(B[P \circ \sigma](e^{i\theta}) + \phi_n(R, r, \alpha, \beta)B[P \circ \rho](e^{i\theta}))e^{i\eta} \\ & \quad + (B[P^* \circ \sigma]^*(e^{i\theta}) + \phi_n(R, r, \bar{\alpha}, \bar{\beta})B[P^* \circ \rho]^*(e^{i\theta}))|^p d\theta \\ & \leq |(R^n + \phi_n(R, r, \alpha, \beta)r^n)\Lambda_n e^{i\eta} + (1 + \phi_n(R, r, \bar{\alpha}, \bar{\beta}))\bar{\lambda}_0|^p \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \end{aligned}$$

where $B \in \mathcal{B}_n$, $\sigma(z) := Rz$, $\rho(z) := rz$, $B[P^* \circ \sigma]^*(z) := (B[P^* \circ \sigma](z))^*$, Λ_n and $\phi_n(R, r, \alpha, \beta)$ are defined by (1.11) and (1.7), respectively.

Proof. Since $P(z)$ does not vanish in $|z| < 1$ and $P^*(z) = z^n \overline{P(1/\bar{z})}$, by Lemma 2.5, we have for $R > r \geq 1$,

$$\begin{aligned} & |B[P \circ \sigma](z) + \phi_n(R, r, \alpha, \beta)B[P \circ \rho](z)| \\ & \leq |B[P^* \circ \sigma](z) + \phi_n(R, r, \alpha, \beta)B[P^* \circ \rho](z)| \end{aligned} \quad (2.16)$$

Also, since

$P^*(Rz) + \phi_n(R, r, \alpha, \beta)P^*(rz) = R^n z^n \overline{P(1/R\bar{z})} + \phi_n(R, r, \alpha, \beta)r^n z^n \overline{P(1/r\bar{z})}$, therefore,

$$\begin{aligned} & B[P^* \circ \sigma](z) + \phi_n(R, r, \alpha, \beta)B[P^* \circ \rho](z) \\ & = \lambda_0 (R^n z^n \overline{P(1/R\bar{z})} + \phi_n(R, r, \alpha, \beta)r^n z^n \overline{P(1/r\bar{z})}) + \lambda_1 \left(\frac{nz}{2} \right) \left(nR^n z^{n-1} \overline{P(1/R\bar{z})} \right. \\ & \quad \left. - R^{n-1} z^{n-2} \overline{P'(1/R\bar{z})} + \phi_n(R, r, \alpha, \beta) (nr^n z^{n-1} \overline{P(1/r\bar{z})} - r^{n-1} z^{n-2} \overline{P'(1/r\bar{z})}) \right) \\ & \quad + \frac{\lambda_2}{2!} \left(\frac{nz}{2} \right)^2 \left(n(n-1)R^n z^{n-2} \overline{P(1/R\bar{z})} - 2(n-1)R^{n-1} z^{n-3} \overline{P'(1/R\bar{z})} \right. \\ & \quad \left. + R^{n-2} z^{n-4} \overline{P''(1/R\bar{z})} + \phi_n(R, r, \alpha, \beta) (n(n-1)r^n z^{n-2} \overline{P(1/r\bar{z})} \right. \\ & \quad \left. - 2(n-1)r^{n-1} z^{n-3} \overline{P'(1/r\bar{z})} + r^{n-2} z^{n-4} \overline{P''(1/r\bar{z})}) \right), \end{aligned}$$

and hence

$$\begin{aligned} & B[P^* \circ \sigma]^*(z) + \phi(R, r, \bar{\alpha}, \bar{\beta})B[P^* \circ \rho]^*(z) \\ & = (B[P^* \circ \sigma](z) + \phi_n(R, r, \alpha, \beta)B[P^* \circ \rho](z))^* \\ & = \left(\bar{\lambda}_0 + \bar{\lambda}_1 \frac{n^2}{2} + \bar{\lambda}_2 \frac{n^3(n-1)}{8} \right) (R^n P(z/R) + \phi(R, r, \bar{\alpha}, \bar{\beta})r^n P(z/r)) \\ & \quad - \left(\bar{\lambda}_1 \frac{n}{2} + \bar{\lambda}_2 \frac{n^2(n-1)}{4} \right) \left(R^{n-1} z P'(z/R) + \phi(R, r, \bar{\alpha}, \bar{\beta})r^{n-1} z P'(z/r) \right) \\ & \quad + \bar{\lambda}_2 \frac{n^2}{8} \left(R^{n-2} z^2 P''(z/R) + \phi(R, r, \bar{\alpha}, \bar{\beta})r^{n-2} z^2 P''(z/r) \right). \end{aligned} \quad (2.17)$$

Also, for $|z| = 1$

$$\begin{aligned} & |B[P^* \circ \sigma](z) + \phi_n(R, r, \alpha, \beta) B[P^* \circ \rho](z)| \\ &= |B[P^* \circ \sigma]^*(z) + \phi(R, r, \bar{\alpha}, \bar{\beta}) B[P^* \circ \rho]^*(z)|. \end{aligned}$$

Using this in (2.16), we get for $|z| = 1$ and $R > r \geq 1$,

$$\begin{aligned} & |B[P \circ \sigma](z) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](z)| \\ & \leq |B[P^* \circ \sigma]^*(z) + \phi(R, r, \bar{\alpha}, \bar{\beta}) B[P^* \circ \rho]^*(z)|. \end{aligned}$$

Since all the zeros of $P^*(z)$ lie in $|z| \leq 1$, as before, all the zeros of $P^*(Rz) + \phi_n(R, r, \alpha, \beta)P^*(rz)$ lie in $|z| < 1$ for all real or complex numbers α, β with $|\alpha| \leq 1, |\beta| \leq 1$ and $R > r \geq 1$. Hence by Lemma 2.2, all the zeros of $B[P^* \circ \sigma](z) + \phi_n(R, r, \alpha, \beta)B[P^* \circ \rho](z)$ lie in $|z| < 1$, therefore, all the zeros of $B[P^* \circ \sigma]^*(z) + \phi_n(R, r, \bar{\alpha}, \bar{\beta})B[P^* \circ \rho]^*(z)$ lie in $|z| > 1$. Hence by the maximum modulus principle,

$$\begin{aligned} & |B[P \circ \sigma](z) + \phi_n(R, r, \alpha, \beta) B[P^* \circ \rho](z)| \\ & < |B[P^* \circ \sigma]^*(z) + \phi(R, r, \bar{\alpha}, \bar{\beta}) B[P^* \circ \rho]^*(z)| \end{aligned} \quad (2.18)$$

for $|z| < 1$. A direct application of Rouché's theorem shows that

$$\begin{aligned} C_\gamma P(z) &= (B[P \circ \sigma](z) + \phi_n(R, r, \alpha, \beta)B[P \circ \rho](z))e^{i\eta} \\ &+ (B[P^* \circ \sigma]^*(z) + \phi_n(R, r, \bar{\alpha}, \bar{\beta})B[P^* \circ \rho]^*(z)) \\ &= \{(R^n + \phi_n(R, r, \alpha, \beta)r^n)\Lambda_n e^{i\eta} + (1 + \phi_n(R, r, \bar{\alpha}, \bar{\beta}))\bar{\lambda}_0\} a_n z^n \\ &+ \cdots + \{(R^n + \phi_n(R, r, \bar{\alpha}, \bar{\beta})r^n)\bar{\Lambda}_n + e^{i\eta}(1 + \phi_n(R, r, \alpha, \beta))\lambda_0\} a_0 \end{aligned}$$

does not vanish in $|z| < 1$. Therefore, C_γ is an admissible operator. Applying (2.15) of Lemma 2.7, the desired result follows immediately for each $p > 0$. \square

We also need the following lemma [4].

Lemma 2.9. *If A, B, C are non-negative real numbers such that $B + C \leq A$, then for each real number γ ,*

$$|(A - C)e^{i\gamma} + (B + C)| \leq |Ae^{i\gamma} + B|.$$

3. PROOF OF THE THEOREMS

Proof of Theorem 1.5. By hypothesis $P(z)$ does not vanish in $|z| < 1$, therefore by Lemma 2.6, we have

$$\begin{aligned} & |B[P \circ \sigma](z) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](z)| \\ & \leq |B[P^* \circ \sigma](z) + \phi_n(R, r, \alpha, \beta) B[P^* \circ \rho](z)| \\ & \quad - \left(|R^n + \phi_n(R, r, \alpha, \beta)r^n| |\Lambda_n| - |1 + \phi_n(R, r, \alpha, \beta)| |\lambda_0| \right) m, \end{aligned} \quad (3.1)$$

for $|z| = 1, |\alpha| \leq 1$ and $R > r \geq 1$ where $P^*(z) = z^n \overline{P(1/\bar{z})}$. Since $B[P^* \circ \sigma]^*(z) + \phi_n(R, r, \bar{\alpha}, \bar{\beta}) B[P^* \circ \rho]^*(z)$ is the conjugate of $B[P^* \circ \sigma](z) + \phi_n(R, r, \alpha, \beta) B[P^* \circ \rho](z)$ and

$$\begin{aligned} & |B[P^* \circ \sigma]^*(z) + \phi_n(R, r, \bar{\alpha}, \bar{\beta}) B[P^* \circ \rho]^*(z)| \\ &= |B[P^* \circ \sigma](z) + \phi_n(R, r, \alpha, \beta) B[P^* \circ \rho](z)| \end{aligned}$$

Thus (3.1) can be written as

$$\begin{aligned}
 & |B[P \circ \sigma](z) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](z)| \\
 & \quad + \frac{\left(|R^n + \phi_n(R, r, \alpha, \beta) r^n| |\Lambda_n| - |1 + \phi_n(R, r, \alpha, \beta)| |\lambda_0|\right) m}{2} \\
 & \leq |B[P^* \circ \sigma]^*(z) + \phi_n(R, r, \bar{\alpha}, \bar{\beta}) B[P^* \circ \rho]^*(z)| \\
 & \quad - \frac{\left(|R^n + \phi_n(R, r, \alpha, \beta) r^n| |\Lambda_n| - |1 + \phi_n(R, r, \alpha, \beta)| |\lambda_0|\right) m}{2} \tag{3.2}
 \end{aligned}$$

for $|z| = 1$. Taking

$$\begin{aligned}
 A &= |B[P^* \circ \sigma]^*(z) + \phi_n(R, r, \bar{\alpha}, \bar{\beta}) B[P^* \circ \rho]^*(z)| \\
 B &= |B[P \circ \sigma](z) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](z)|,
 \end{aligned}$$

and

$$C = \frac{\left(|R^n + \phi_n(R, r, \alpha, \beta) r^n| |\Lambda_n| - |1 + \phi_n(R, r, \alpha, \beta)| |\lambda_0|\right) m}{2}$$

in Lemma 2.9 and noting by (3.2) that

$$B + C \leq A - C \leq A,$$

we get for every real γ ,

$$\begin{aligned}
 & \left| \left\{ |B[P^* \circ \sigma]^*(e^{i\theta}) + \phi_n(R, r, \bar{\alpha}, \bar{\beta}) B[P^* \circ \rho]^*(e^{i\theta})| \right. \right. \\
 & \quad \left. \left. - \frac{\left(|R^n + \phi_n(R, r, \alpha, \beta) r^n| |\Lambda_n| - |1 + \phi_n(R, r, \alpha, \beta)| |\lambda_0|\right) m}{2} \right\} e^{i\gamma} \right. \\
 & \quad \left. + \left\{ |B[P \circ \sigma](e^{i\theta}) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](e^{i\theta})| \right. \right. \\
 & \quad \left. \left. + \frac{\left(|R^n + \phi_n(R, r, \alpha, \beta) r^n| |\Lambda_n| - |1 + \phi_n(R, r, \alpha, \beta)| |\lambda_0|\right) m}{2} \right\} \right| \\
 & \leq \left| |B[P^* \circ \sigma]^*(e^{i\theta}) + \phi_n(R, r, \bar{\alpha}, \bar{\beta}) B[P^* \circ \rho]^*(e^{i\theta})| e^{i\gamma} \right. \\
 & \quad \left. + |B[P \circ \sigma](e^{i\theta}) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](e^{i\theta})| \right|.
 \end{aligned}$$

This implies for each $p > 0$,

$$\int_0^{2\pi} \left| \left\{ |B[P^* \circ \sigma]^*(e^{i\theta}) + \phi_n(R, r, \bar{\alpha}, \bar{\beta}) B[P^* \circ \rho]^*(e^{i\theta})| \right. \right. \\
 \left. \left. - \frac{\left(|R^n + \phi_n(R, r, \alpha, \beta) r^n| |\Lambda_n| - |1 + \phi_n(R, r, \alpha, \beta)| |\lambda_0|\right) m}{2} \right\} e^{i\gamma} \right|$$

$$\begin{aligned}
& + \left\{ \left| B[P \circ \sigma](e^{i\theta}) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](e^{i\theta}) \right| \right. \\
& + \left. \frac{\left(|R^n + \phi_n(R, r, \alpha, \beta) r^n| |\Lambda_n| - |1 + \phi_n(R, r, \alpha, \beta)| |\lambda_0| \right) m}{2} \right\}^p d\theta \\
& \leq \int_0^{2\pi} \left| \left| B[P^* \circ \sigma]^*(e^{i\theta}) + \phi_n(R, r, \bar{\alpha}, \bar{\beta}) B[P^* \circ \rho]^*(e^{i\theta}) \right| e^{i\gamma} \right. \\
& \quad \left. + \left| B[P \circ \sigma](e^{i\theta}) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](e^{i\theta}) \right| \right|^p d\theta. \quad (3.3)
\end{aligned}$$

Integrating both sides of (3.3) with respect to γ from 0 to 2π , we get with the help of Lemma 2.8 for each $p > 0$,

$$\begin{aligned}
& \int_0^{2\pi} \int_0^{2\pi} \left\{ \left| B[P^* \circ \sigma]^*(e^{i\theta}) + \phi_n(R, r, \bar{\alpha}, \bar{\beta}) B[P^* \circ \rho]^*(e^{i\theta}) \right| \right. \\
& \quad \left. - \frac{\left(|R^n + \phi_n(R, r, \alpha, \beta) r^n| |\Lambda_n| - |1 + \phi_n(R, r, \alpha, \beta)| |\lambda_0| \right) m}{2} \right\} e^{i\gamma} \\
& \quad + \left\{ \left| B[P \circ \sigma](e^{i\theta}) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](e^{i\theta}) \right| \right. \\
& \quad \left. + \frac{\left(|R^n + \phi_n(R, r, \alpha, \beta) r^n| |\Lambda_n| - |1 + \phi_n(R, r, \alpha, \beta)| |\lambda_0| \right) m}{2} \right\}^p d\theta d\gamma \\
& \leq \int_0^{2\pi} \int_0^{2\pi} \left| \left| B[P^* \circ \sigma]^*(e^{i\theta}) + \phi_n(R, r, \bar{\alpha}, \bar{\beta}) B[P^* \circ \rho]^*(e^{i\theta}) \right| e^{i\gamma} \right. \\
& \quad \left. + \left| B[P \circ \sigma](e^{i\theta}) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](e^{i\theta}) \right| \right|^p d\theta d\gamma \\
& \leq \int_0^{2\pi} \left\{ \int_0^{2\pi} \left| \left| B[P^* \circ \sigma]^*(e^{i\theta}) + \phi_n(R, r, \bar{\alpha}, \bar{\beta}) B[P^* \circ \rho]^*(e^{i\theta}) \right| e^{i\gamma} \right. \right. \\
& \quad \left. \left. + \left| B[P \circ \sigma](e^{i\theta}) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](e^{i\theta}) \right| \right|^p d\gamma \right\} d\theta \\
& \leq \int_0^{2\pi} \left\{ \int_0^{2\pi} \left(\left| B[P^* \circ \sigma]^*(e^{i\theta}) + \phi_n(R, r, \bar{\alpha}, \bar{\beta}) B[P^* \circ \rho]^*(e^{i\theta}) \right| e^{i\gamma} \right. \right. \\
& \quad \left. \left. + \left(\left| B[P \circ \sigma](e^{i\theta}) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](e^{i\theta}) \right| \right)^p d\gamma \right) d\theta \right\} \\
& \leq \int_0^{2\pi} \left\{ \int_0^{2\pi} \left(\left| B[P^* \circ \sigma]^*(e^{i\theta}) + \phi_n(R, r, \bar{\alpha}, \bar{\beta}) B[P^* \circ \rho]^*(e^{i\theta}) \right| e^{i\gamma} \right. \right. \\
& \quad \left. \left. + \left(\left| B[P \circ \sigma](e^{i\theta}) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](e^{i\theta}) \right| \right)^p d\gamma \right) d\theta \right\} \\
& \leq \int_0^{2\pi} \left\{ \int_0^{2\pi} \left(\left| B[P^* \circ \sigma]^*(e^{i\theta}) + \phi_n(R, r, \bar{\alpha}, \bar{\beta}) B[P^* \circ \rho]^*(e^{i\theta}) \right| e^{i\gamma} \right. \right. \\
& \quad \left. \left. + \left(\left| B[P \circ \sigma](e^{i\theta}) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](e^{i\theta}) \right| \right)^p d\gamma \right) d\theta \right\}
\end{aligned}$$

$$\begin{aligned}
 & + \left(B[P \circ \sigma](e^{i\theta}) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](e^{i\theta}) \right) \Big| d\theta \Big\} d\gamma \\
 & \leq \int_0^{2\pi} \left| (R^n + \phi_n(R, r, \alpha, \beta) r^n) \Lambda_n e^{i\gamma} + (1 + \phi_n(R, r, \bar{\alpha}, \bar{\beta})) \bar{\lambda}_0 \right|^p d\gamma \\
 & \quad \times \int_0^{2\pi} \left| P(e^{i\theta}) \right|^p d\theta \tag{3.4}
 \end{aligned}$$

Now it can be easily verified that for every real number γ and $s \geq 1$,

$$|s + e^{i\alpha}| \geq |1 + e^{i\alpha}|.$$

This implies for each $p > 0$,

$$\int_0^{2\pi} |s + e^{i\gamma}|^p d\gamma \geq \int_0^{2\pi} |1 + e^{i\gamma}|^p d\gamma. \tag{3.5}$$

If

$$\begin{aligned}
 & |B[P \circ \sigma](e^{i\theta}) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](e^{i\theta})| \\
 & + \frac{\left(|R^n + \phi_n(R, r, \alpha, \beta) r^n| |\Lambda_n| - |1 + \phi_n(R, r, \alpha, \beta)| |\lambda_0| \right) m}{2} \neq 0,
 \end{aligned}$$

we take

$$\begin{aligned}
 & |B[P^* \circ \sigma]^*(e^{i\theta}) + \phi_n(R, r, \bar{\alpha}, \bar{\beta}) B[P^* \circ \rho]^*(e^{i\theta})| \\
 & + \frac{\left(|R^n + \phi_n(R, r, \alpha, \beta) r^n| |\Lambda_n| - |1 + \phi_n(R, r, \alpha, \beta)| |\lambda_0| \right) m}{2} \\
 s = & \frac{|B[P^* \circ \sigma]^*(e^{i\theta}) + \phi_n(R, r, \bar{\alpha}, \bar{\beta}) B[P^* \circ \rho]^*(e^{i\theta})| + \frac{\left(|R^n + \phi_n(R, r, \alpha, \beta) r^n| |\Lambda_n| - |1 + \phi_n(R, r, \alpha, \beta)| |\lambda_0| \right) m}{2}}{|B[P \circ \sigma](e^{i\theta}) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](e^{i\theta})| + \frac{\left(|R^n + \phi_n(R, r, \alpha, \beta) r^n| |\Lambda_n| - |1 + \phi_n(R, r, \alpha, \beta)| |\lambda_0| \right) m}{2}},
 \end{aligned}$$

then by (3.2), $s \geq 1$ and we get with the help of (3.5),

$$\begin{aligned}
 & \int_0^{2\pi} \left\{ |B[P^* \circ \sigma]^*(e^{i\theta}) + \phi_n(R, r, \bar{\alpha}, \bar{\beta}) B[P^* \circ \rho]^*(e^{i\theta})| \right. \\
 & \quad \left. - \frac{\left(|R^n + \phi_n(R, r, \alpha, \beta) r^n| |\Lambda_n| - |1 + \phi_n(R, r, \alpha, \beta)| |\lambda_0| \right) m}{2} \right\} e^{i\gamma} \\
 & + \left\{ |B[P \circ \sigma](e^{i\theta}) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](e^{i\theta})| \right. \\
 & \quad \left. + \frac{\left(|R^n + \phi_n(R, r, \alpha, \beta) r^n| |\Lambda_n| - |1 + \phi_n(R, r, \alpha, \beta)| |\lambda_0| \right) m}{2} \right\} \Big| d\theta \Big\}^p d\gamma \\
 & = \left| |B[P \circ \sigma](e^{i\theta}) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](e^{i\theta})| \right.
 \end{aligned}$$

$$\begin{aligned}
& + \left| \frac{\left(|R^n + \phi_n(R, r, \alpha, \beta) r^n| |\Lambda_n| - |1 + \phi_n(R, r, \alpha, \beta)| |\lambda_0| \right) m}{2} \right|^p \\
& \times \int_0^{2\pi} \left| e^{i\gamma} + \frac{\left| \frac{B[P^* \circ \sigma]^*(e^{i\theta}) + \phi_n(R, r, \bar{\alpha}, \bar{\beta}) B[P^* \circ \rho]^*(e^{i\theta})}{\left(|R^n + \phi_n(R, r, \alpha, \beta) r^n| |\Lambda_n| - |1 + \phi_n(R, r, \alpha, \beta)| |\lambda_0| \right) m} \right|}{\left| \frac{B[P \circ \sigma](e^{i\theta}) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](e^{i\theta})}{\left(|R^n + \phi_n(R, r, \alpha, \beta) r^n| |\Lambda_n| - |1 + \phi_n(R, r, \alpha, \beta)| |\lambda_0| \right) m} \right|} \right|^p d\gamma \\
& = \left| B[P \circ \sigma](e^{i\theta}) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](e^{i\theta}) \right| \\
& + \left| \frac{\left(|R^n + \phi_n(R, r, \alpha, \beta) r^n| |\Lambda_n| - |1 + \phi_n(R, r, \alpha, \beta)| |\lambda_0| \right) m}{2} \right|^p \\
& \times \int_0^{2\pi} \left| e^{i\gamma} + \frac{\left| \frac{B[P^* \circ \sigma]^*(e^{i\theta}) + \phi_n(R, r, \bar{\alpha}, \bar{\beta}) B[P^* \circ \rho]^*(e^{i\theta})}{\left(|R^n + \phi_n(R, r, \alpha, \beta) r^n| |\Lambda_n| - |1 + \phi_n(R, r, \alpha, \beta)| |\lambda_0| \right) m} \right|}{\left| \frac{B[P \circ \sigma](e^{i\theta}) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](e^{i\theta})}{\left(|R^n + \phi_n(R, r, \alpha, \beta) r^n| |\Lambda_n| - |1 + \phi_n(R, r, \alpha, \beta)| |\lambda_0| \right) m} \right|} \right|^p d\gamma \\
& \geq \left| B[P \circ \sigma](e^{i\theta}) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](e^{i\theta}) \right| \\
& + \left| \frac{\left(|R^n + \phi_n(R, r, \alpha, \beta) r^n| |\Lambda_n| - |1 + \phi_n(R, r, \alpha, \beta)| |\lambda_0| \right) m}{2} \right|^p \int_0^{2\pi} |1 + e^{i\gamma}|^p d\gamma.
\end{aligned} \tag{3.6}$$

For

$$\begin{aligned}
& \left| B[P \circ \sigma](e^{i\theta}) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](e^{i\theta}) \right| \\
& + \frac{\left(|R^n + \phi_n(R, r, \alpha, \beta) r^n| |\Lambda_n| - |1 + \phi_n(R, r, \alpha, \beta)| |\lambda_0| \right) m}{2} \neq 0,
\end{aligned}$$

then (3.6) is trivially true. Using this in (3.4), we conclude that for every $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1, |\beta| \leq 1$ $R > r \geq 1$ and $p > 0$,

$$\begin{aligned}
& \int_0^{2\pi} \left| B[P \circ \sigma](e^{i\theta}) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](e^{i\theta}) \right| \\
& + \left| \frac{\left(|R^n + \phi_n(R, r, \alpha, \beta) r^n| |\Lambda_n| - |1 + \phi_n(R, r, \alpha, \beta)| |\lambda_0| \right) m}{2} \right|^p d\theta \int_0^{2\pi} |1 + e^{i\gamma}|^p d\gamma
\end{aligned}$$

$$\leq \int_0^{2\pi} \left| (R^n + \phi_n(R, r, \alpha, \beta)r^n)\Lambda_n e^{i\gamma} + (1 + \phi_n(R, r, \bar{\alpha}, \bar{\beta}))\bar{\lambda}_0 \right|^p d\gamma \int_0^{2\pi} |P(e^{i\theta})|^p d\theta.$$

This gives for every δ, α, β with $|\delta| \leq 1, |\alpha| \leq 1, |\beta| \leq 1, R > r \geq 1$ and γ real

$$\begin{aligned} & \int_0^{2\pi} \left| B[P \circ \sigma](e^{i\theta}) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](e^{i\theta}) \right. \\ & \left. + \delta \frac{\left(|R^n + \phi_n(R, r, \alpha, \beta)r^n| |\Lambda_n| - |1 + \phi_n(R, r, \alpha, \beta)| |\lambda_0| \right) m}{2} \right|^p d\theta \int_0^{2\pi} |1 + e^{i\gamma}|^p d\gamma \\ & \leq \int_0^{2\pi} \left| (R^n + \phi_n(R, r, \alpha, \beta)r^n)\Lambda_n e^{i\gamma} + (1 + \phi_n(R, r, \bar{\alpha}, \bar{\beta}))\bar{\lambda}_0 \right|^p d\gamma \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \end{aligned} \quad (3.7)$$

Since

$$\begin{aligned} & \int_0^{2\pi} \left| (R^n + \phi_n(R, r, \alpha, \beta)r^n)\Lambda_n e^{i\gamma} + (1 + \phi_n(R, r, \bar{\alpha}, \bar{\beta}))\bar{\lambda}_0 \right|^p d\gamma \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \\ & = \int_0^{2\pi} \left| |(R^n + \phi_n(R, r, \alpha, \beta)r^n)\Lambda_n| e^{i\gamma} + |(1 + \phi_n(R, r, \bar{\alpha}, \bar{\beta}))\bar{\lambda}_0| \right|^p d\gamma \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \\ & = \int_0^{2\pi} \left| |(R^n + \phi_n(R, r, \alpha, \beta)r^n)\Lambda_n| e^{i\gamma} + |(1 + \phi_n(R, r, \alpha, \beta))\lambda_0| \right|^p d\gamma \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \\ & = \int_0^{2\pi} \left| (R^n + \phi_n(R, r, \alpha, \beta)r^n)\Lambda_n e^{i\gamma} + (1 + \phi_n(R, r, \alpha, \beta))\lambda_0 \right|^p d\gamma \int_0^{2\pi} |P(e^{i\theta})|^p d\theta, \end{aligned} \quad (3.8)$$

the desired result follows immediately by combining (3.7) and (3.8). This completes the proof of Theorem 1.5 for $p > 0$. To establish this result for $p = 0$, we simply let $p \rightarrow 0+$. \square

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