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ON SOME FRACTIONAL INTEGRAL INEQUALITIES OF HERMITE-HADAMARD TYPE FOR r-PREINVEX FUNCTIONS

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Abstract. In this paper, we prove Hermite-Hadamard type inequalities for r-preinvex functions via fractional integrals. The results presented here would provide extensions of those given in earlier works.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on an interval I of real numbers and $a,\ b \in I$ with $a < b$. The following inequality holds

$$f\left(\tfrac{a+b}{2}\right) \leq \frac{1}{b-a}\int\limits_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

The double inequality (1.1) is known, in the literature, as the Hermite–Hadamard integral inequality for convex functions. Both inequalities hold in the reversed direction if f is concave. The inequality (1.1) has been extended and generalized for various classes of convex functions via different approaches, see [4, 7, 10]. For several recent results concerning the inequality (1.1) we refer the interested reader to [1–12, 14–16, 18], and references therein.

2. Preliminaries

Let K be a nonempty subset of $\mathbb{R}^n$ and let $\eta : K \times K \to \mathbb{R}^n$ be a function.

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Definition 2.1. ([19]) Let $u \in K$. We say K is invex at u with respect to η if, for each $v \in K$

$$u + t\eta(v,u) \in K,\ \ t \in [0,1]\,. \tag{2.1}$$

K is said to be an invex set with respect to η if K is invex at each $u \in K$.

Definition 2.2. ([13]) The function f on the invex set K is said to be preinvex with respect to η, if

$$f(u + t\eta(v,u)) \leq (1-t)f(u) + tf(v), \quad (u, v \in K,\ \ t \in [0,1]). \tag{2.2}$$

Definition 2.3. ([17]) A positive function f on the invex set K is said to be logarithmically preinvex, if

$$f(u + t\eta(v,u)) \leq f^{1-t}(u) f^{t}(v) \tag{2.3}$$

for all $u, v \in K$ and $t \in [0,1]\,.$

Definition 2.4. ([17]) The function f on the invex set K is said to be r-preinvex with respect to η, if

$$f(u + t\eta(v,u)) \leq M_r(f(u), f(v); t)$$

holds for all $u,\ v \in K$ and $t \in [0,1]$, where

$$M_r\left(x,y;t\right) = \begin{cases} \left[(1-t)x^r + ty^r\right]^{\frac{1}{r}} & ,\ r \neq 0 \\ x^{1-t}y^t & ,\ r = 0 \end{cases}$$

is the weighted power mean of order r for positive numbers x and y.

Definition 2.5. ([18]) Let $f \in L^1[a,b]$. The Riemann-Liouville fractional integrals $J^{\alpha}_{a^+} f\left(x\right)$ and $J^{\alpha}_{b^-} f\left(x\right)$ of order $\alpha > 0$ are defined, respectively, by

$$J^{\alpha}_{a^+} f\left(x\right) = \tfrac{1}{\Gamma(\alpha)} \int\limits_a^x \left(x-t\right)^{\alpha-1} f\left(t\right) dt, \quad x > a \tag{2.4}$$

and

$$J^{\alpha}_{b^-} f\left(x\right) = \tfrac{1}{\Gamma(\alpha)} \int\limits_x^b (t-x)^{\alpha-1} f\left(t\right) dt, \quad x < b, \tag{2.5}$$

where $\Gamma\left(\alpha\right) = \int\limits_0^{\infty} e^{-u}u^{\alpha-1}du$ is the Gamma function and $J^0_{a^+}f(x) = J^0_{b^-}f(x) = f(x)$.

The main purpose of this paper is to establish Hermite-Hadamard type inequalities for Riemann-Liouville fractional integral using r-preinvex functions. Then, we give some interesting results of Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals. Some special cases are also discussed.

3. Main results

Theorem 3.1. *Let $K = [a, a+\eta(b,a)]$ be an interval of real numbers with interior K°, $a,\ b \in K^\circ$, and $a < a+\eta(b,a)$. Let $f : K \to (0,\infty)$ be an r-preinvex function on the interval K°, then*

$$\left(J^{\alpha}_{(a+\eta(b,a))^-}f\right)(a) \le \frac{[\eta(b,a)]^{\alpha}}{\Gamma(\alpha)}\left\{B\left(\alpha, \frac{1}{r}+1\right)f^r(a) + \frac{r}{\alpha r+1}f^r(b)\right\}^{\frac{1}{r}} \tag{3.1}$$

holds for all $0 < r \le 1$.

Proof. Since f is an r-preinvex function and $r > 0$, we have

$$f(a+t\eta(b,a)) \le [tf^r(b) + (1-t)f^r(a)]^{\frac{1}{r}}$$

for all $t \in [0,1]$. Then,

$$\begin{aligned}
\frac{\Gamma(\alpha)}{(\eta(b,a))^{\alpha}}\left(J^{\alpha}_{(a+\eta(b,a))^-}f\right)(a) &= \frac{1}{(\eta(b,a))^{\alpha}}\int_a^{a+\eta(b,a)}(u-a)^{\alpha-1}f(u)\,du \\
&= \int_0^1 t^{\alpha-1}f(a+t\eta(b,a))dt \\
&\le \int_0^1 t^{\alpha-1}[tf^r(b)+(1-t)f^r(a)]^{\frac{1}{r}}\,dt \\
&= \int_0^1 \left[t^{r(\alpha-1)+1}f^r(b) + t^{r(\alpha-1)}(1-t)f^r(a)\right]^{\frac{1}{r}}dt.
\end{aligned}$$

Using Minkowski's inequality, we have

$$\begin{aligned}
&\int_0^1 \left[t^{r(\alpha-1)+1}f^r(b) + t^{r(\alpha-1)}(1-t)f^r(a)\right]^{\frac{1}{r}}dt \\
&\le \left\{\left[\int_0^1 t^{\alpha-1+\frac{1}{r}}f(b)dt\right]^r + \left[\int_0^1 t^{\alpha-1}(1-t)^{\frac{1}{r}}f(a)dt\right]^r\right\}^{\frac{1}{r}} \\
&= \left\{f^r(b)\frac{r}{\alpha r+1} + f^r(a)B\left(\alpha, \frac{1}{r}+1\right)\right\}^{\frac{1}{r}},
\end{aligned}$$

and the proof is complete.

Remark 3.2. Under the same conditions as in Theorem 3.1, with $\alpha = 1,\ r = 1$ and $\eta(b,a) = b-a$, we have

$$\frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}.$$

□

Theorem 3.3. *Let $f,\ g : K = [a, a+\eta(b,a)] \to (0,\infty)$ be r-preinvex and s-preinvex functions respectively on the interval of real numbers K°, $a,\ b \in K^{\circ}$ with $a < a+\eta(b,a)$, then*

$$\begin{aligned}
&\left(J^{\alpha}_{(a+\eta(b,a))^{-}} fg\right)(a) \\
&\leq \frac{[\eta(b,a)]^{\alpha}}{2\Gamma(\alpha)} \left\{ \left(B\left(\frac{2(\alpha-1)}{r}+1, \frac{2}{r}+1\right) f^{r}(a) + \frac{r}{2\alpha+r} f^{r}(b) \right)^{\frac{2}{r}} \right. \\
&\left. + \left(B\left(\frac{2(\alpha-1)}{s}+1, \frac{2}{s}+1\right) g^{s}(a) + \frac{s}{2\alpha+s} g^{s}(b) \right)^{\frac{2}{s}} \right\}
\end{aligned} \tag{3.2}$$

holds for $0 < r,\ s \leq 2$.

Proof. Since f is a r-preinvex function and g is a s-preinvex function, by the hypothesis, we have

$$f(a+t\eta(b,a) \leq [tf^{r}(b) + (1-t)f^{r}(a)]^{\frac{1}{r}} \tag{3.3}$$

and

$$g(a+t\eta(b,a) \leq [tg^{s}(b) + (1-t)g^{s}(a)]^{\frac{1}{s}} \tag{3.4}$$

for $t \in [0,1]$. By using the inequality (3.3) and (3.4), we get

$$\begin{aligned}
&\frac{1}{[\eta(b,a)]^{\alpha}} \int_{a}^{a+\eta(b,a)} (u-a)^{(\alpha-1)\left(\frac{1}{r}+\frac{1}{s}\right)} f(u)\, g(u) du \\
&= \int_{0}^{1} t^{(\alpha-1)\left(\frac{1}{r}+\frac{1}{s}\right)} f(a+t\eta(b,a)) g\left(a+t\eta(b,a)\right) dt \\
&\leq \int_{0}^{1} t^{(\alpha-1)\left(\frac{1}{r}+\frac{1}{s}\right)} \left[tf^{r}(b) + (1-t)f^{r}(a)\right]^{\frac{1}{r}} \left[tg^{s}(b) + (1-t)g^{s}(a)\right]^{\frac{1}{s}} dt.
\end{aligned} \tag{3.5}$$

Using Cauchy's inequality for (3.5), we have

$$\begin{aligned}
&\int_{0}^{1} \left[t^{\alpha} f^{r}(b) + t^{\alpha-1}(1-t)f^{r}(a)\right]^{\frac{1}{r}} \left[t^{\alpha} g^{s}(b) + t^{\alpha-1}(1-t)g^{s}(a)\right]^{\frac{1}{s}} dt \\
&\leq \frac{1}{2} \left\{ \int_{0}^{1} \left[t^{\alpha} f^{r}(b) + t^{\alpha-1}(1-t)f^{r}(a)\right]^{\frac{2}{r}} dt + \int_{0}^{1} \left[t^{\alpha} g^{s}(b) + t^{\alpha-1}(1-t)g^{s}(a)\right]^{\frac{2}{s}} dt \right\} \\
&= \frac{1}{2} \left\{ I_1 + I_2 \right\}.
\end{aligned}$$

Using Minkowski's inequality for I_1 and I_2, we hawe

$$I_1 = \int_0^1 \left[t^{\alpha} f^r(b) + t^{\alpha-1}(1-t) f^r(a)\right]^{\frac{2}{r}} dt$$

$$\leq \left\{ \left(\int_0^1 t^{\frac{2}{r}\alpha} f^2(b) dt \right)^{\frac{r}{2}} + \left(\int_0^1 t^{\frac{2(\alpha-1)}{r}} (1-t)^{\frac{2}{r}} f^2(a) dt \right)^{\frac{r}{2}} \right\}^{\frac{2}{r}}$$

$$= \left\{ f^r(b) \frac{r}{2\alpha + r} + f^r(a) B\left(\frac{2(\alpha-1)}{r} + 1, \frac{2}{r} + 1 \right) \right\}^{\frac{2}{r}},$$

and

$$I_2 = \int_0^1 \left[t^{\alpha+1} g^s(b) + t^{\alpha}(1-t) g^s(a)\right]^{\frac{2}{s}} dt$$

$$\leq \left\{ g^s(b) \frac{s}{2\alpha + s} + g^s(a) B\left(\frac{2(\alpha-1)}{s} + 1, \frac{2}{s} + 1 \right) \right\}^{\frac{2}{s}}.$$

Combining I_1 and I_2 leads to (3.2) and the proof is complete. □

Corollary 3.4. *Under the same conditions as in Theorem 3.3, if $r = s = 2$, we have*

$$\frac{\Gamma(\alpha)}{(\eta(b,a))^{\alpha}} \left(J^{\alpha}_{(a+\eta(b,a))^-} fg \right)(a) \leq \frac{f^2(a) + f^2(b) + g^2(a) + g^2(b)}{2(\alpha+1)}.$$

Corollary 3.5. *Under the same conditions as in Theorem 3.3, if $\eta(b,a) = b - a$ and $r = s = 2$, we have*

$$\frac{\Gamma(\alpha)}{(b-a)^{\alpha}} J^{\alpha}_{b^+} fg(a) \leq \frac{f^2(a) + f^2(b) + g^2(a) + g^2(b)}{2(\alpha+1)}.$$

Corollary 3.6. *Under the same conditions as in Theorem 3.3, if $\alpha = 1$ and $r = s = 2$, we have the inequality*

$$\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u) g(u) du \leq \frac{f^2(a) + f^2(b) + g^2(a) + g^2(b)}{4}$$

in [17].

Corollary 3.7. *Under the same conditions as in Theorem 3.3, if $\alpha = 1$, $\eta(b,a) = b - a$ and $r = s = 2$, we have*

$$\frac{1}{b-a} \int_a^b f(u) g(u) du \leq \frac{f^2(a) + f^2(b) + g^2(a) + g^2(b)}{4}.$$

Corollary 3.8. *Under the same conditions as in Theorem 3.3, if $\alpha = 1$, $r = s = 2$, and $f(x) = g(x)$, we have the inequality*

$$\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f^2(u) du \leq \frac{f^2(a) + f^2(b)}{2}$$

in [17].

Theorem 3.9. *Let $f,\ g : K = [a, a+\eta(b,a)] \to (0,\infty)$ be r-preinvex and s-preinvex functions, respectively, on the interval of real numbers K°, $a,b \in K^\circ$ with $a < a+\eta(b,a)$. If $r > 1$ and $\frac{1}{r}+\frac{1}{s}=1$, then*

$$\left(J^{\alpha}_{(a+\eta(b,a))^-} fg\right)(a)$$
$$\leq \frac{[\eta(b,a)]^\alpha}{\Gamma(\alpha)}\left(\left(f^r(a)B(\alpha,2)+\frac{f^r(b)}{\alpha+1}\right)^{\frac{1}{r}}+\left(g^s(a)B(\alpha,2)+\frac{g^s(b)}{\alpha+1}\right)^{\frac{1}{s}}\right).$$

Proof. Since f is a r-preinvex function and g is a s-preinvex function, for $t\in[0,1]$, we have

$$f(a+t\eta(b,a) \leq [tf^r(b)+(1-t)f^r(a)]^{\frac{1}{r}} \tag{3.6}$$

and

$$g(a+t\eta(b,a) \leq [tg^s(b)+(1-t)g^s(a)]^{\frac{1}{s}}. \tag{3.7}$$

From (3.6) and (3.7), we get

$$\frac{1}{[\eta(b,a)]^\alpha}\int_a^{a+\eta(b,a)}(u-a)^{(\alpha-1)\left(\frac{1}{r}+\frac{1}{s}\right)} f(u)\,g(u)du$$
$$=\int_0^1 t^{(\alpha-1)\left(\frac{1}{r}+\frac{1}{s}\right)} f(a+t\eta(b,a))g\left(a+t\eta(b,a)\right)dt$$
$$\leq \int_0^1 t^{(\alpha-1)\left(\frac{1}{r}+\frac{1}{s}\right)}\left[tf^r(b)+(1-t)f^r(a)\right]^{\frac{1}{r}}\left[tg^s(b)+(1-t)g^s(a)\right]^{\frac{1}{s}}dt.$$

By virtue of Hölder's inequality, we have

$$\int_0^1\left[t^\alpha f^r(b)+t^{\alpha-1}(1-t)f^r(a)\right]^{\frac{1}{r}}\left[t^\alpha g^s(b)+t^{\alpha-1}(1-t)g^s(a)\right]^{\frac{1}{s}}dt$$
$$\leq\left\{\int_0^1\left[t^\alpha f^r(b)+t^{\alpha-1}(1-t)f^r(a)\right]dt\right\}^{\frac{1}{r}}+\left\{\int_0^1\left[t^\alpha g^s(b)+t^{\alpha-1}(1-t)g^s(a)\right]dt\right\}^{\frac{1}{s}}$$
$$=\frac{[\eta(b,a)]^\alpha}{\Gamma(\alpha)}\left\{\left(f^r(b)\frac{1}{\alpha+1}+f^r(a)B(\alpha,2)\right)^{\frac{1}{r}}+\left(g^s(b)\frac{1}{\alpha+1}+g^s(a)B(\alpha,2)\right)^{\frac{1}{s}}\right\}.$$

The proof is done. □

Corollary 3.10. *Under the same conditions as in Theorem 3.9, if $r=s=2$, we have*

$$\left(J^{\alpha}_{(a+\eta(b,a))^-} fg\right)(a)$$
$$\leq \frac{[\eta(b,a)]^\alpha}{\Gamma(\alpha)}\left(\sqrt{f^2(a)B(\alpha,2)+\frac{f^2(b)}{\alpha+1}}+\sqrt{g^2(a)B(\alpha,2)+\frac{g^2(b)}{\alpha+1}}\right).$$

Corollary 3.11. *Under the same conditions as in Theorem 3.9, if* $r = s = 2$, $\eta(b,a) = b - a$, *we have*

$$\frac{(b-a)^{\alpha}}{\Gamma(\alpha)} J_{b+}^{\alpha} fg(a) \leq \sqrt{f^2(a) B(\alpha, 2) + \frac{f^2(b)}{\alpha+1}} + \sqrt{g^2(a) B(\alpha, 2) + \frac{g^2(b)}{\alpha+1}}.$$

Corollary 3.12. *Under the same conditions as in Theorem 3.9, if* $r = s = 2$, $\eta(b,a) = b - a$ *and* $\alpha = 1$, *we have*

$$\frac{1}{b-a} \int_a^b f(u) g(u) du \leq \sqrt{\frac{f^2(a) + f^2(b)}{2}} \sqrt{\frac{g^2(a) + g^2(b)}{2}}.$$

Corollary 3.13. *Under the same conditions as in Theorem 3.9, if* $r = s = 2$ *and* $\alpha = 1$, *we have*

$$\frac{1}{[\eta(b,a)]} \int_a^{a+\eta(b,a)} f(u) g(u) du \leq \sqrt{\frac{f^2(a) + f^2(b)}{2}} \sqrt{\frac{g^2(a) + g^2(b)}{2}}.$$

References

1. A. Barani, A.G. Ghazanfari and S.S. Dragomir,*Hermite-Hadamard inequality for functions whose derivatives absolute values are preinvex*, J. Inequal. Appl. **2012** (2012), no. 247, 9 pages. doi: 10.1186/1029-242X-2012-247.
2. S.S. Dragomir and C.E.M. Pearce, *Selected Topics on Hermite–Hadamard Inequalities and Applications*, Victoria University, 2000.
3. S.S. Dragomir,J. Pecaric and L.E. Persson, *Some inequalities of Hadamard type*, Soochow J. Math. **21** (1995) 335–341.
4. I. İşcan, *Hermite–Hadamard's inequalities for preinvex function via fractional integrals and related functional inequalities*, Amer. J. Math. Anal. **1** (2013), no. 3, 33–38.
5. M.A. Latif and S.S. Dragomir, *Some Hermite-Hadamard type inequalities for functions whose partial derivatives in absloute value are preinvex on the co-oordinates*, Facta Univ. Ser. Math. Inform. **28** (2013), no. 3, 257–270.
6. M.A. Latif, S.S. Dragomir and E. Momoniat, *Some weighted integral inequalities for differentiable preinvex and prequasiinvex functions*, RGMIA **2014** (2014) 18 pages.
7. M.A. Noor, *Some new classes of nonconvex functions*. Nonlinear Funct. Anal. Appl. **11** (2006), no. 1, 165–171.
8. M.A. Noor and K.I. Noor, *Generalized preinvex functions and their properties*, J. Appl. Math. Stochastic Anal. **2006** (2006) Article ID. 12736, 13 pages. doi:10.1155/JAMSA/2006/12736.
9. M.A. Noor, K.I. Noor, M.A. Ashraf, M.U. Awan and B. Bashir, *Hermite–Hadamard inequalities for $h_{\varphi}-$convex functions* Nonlinear Anal. Forum **18** (2013) 65–76.
10. M.A. Noor, K.I. Noor and M.U. Awan, *Hermite–Hadamard inequalities for relative semi-convex functions and applications*, Filomat **28** (2014), no. 2, 221–230.
11. M.A. Noor, K.I. Noor and M.U. Awan, *Generalized convexity and integral inequalities*, Appl. Math. Inf. Sci. **9** (2015), no. 1, 233–243.
12. M.A. Noor, K.I. Noor, M.U. Awan and J. Li, *On Hermite–Hadamard type Inequalities for h-preinvex functions*, Filomat, to appear.
13. M.A. Noor, K.I. Noor, M.U. Awan and S. Khan, *Hermite–Hadamard inequalities for differantiable h_{φ}-preinvex functions*, Arab. J. Math. **4** (2015) 63–76.
14. M.Z. Sarikaya E. Set and M.E. Özdemir, *On some new inequalities of Hadamard type involving h-convex functions*, Acta Math. Univ. Comenianian, **2** (2010) 265–272.

15. M.Z. Sarikaya, N. Alp and H. Bozkurt, *On Hermite-Hadamard Type Integral Inequalities for preinvex and log-preinvex functions*, Contemp. Anal. Appl. Math. **1** (2013), no. 2, 237–252.
16. S. Varosanec, *On h-convexity*, J. Math. Anal. Appl. **326** (2007) 303–311.
17. W.-Dong Jiang, D.-Wei Nıu and F. Qi, *Some Fractional Inequalties of Hermite-Hadamard type for r-φ-Preinvex Functions*, Tamkang J. Math. **45** (2014), no. 1, 31–38.
18. S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach, Yverdon, Switzerland, 1993.
19. T. Weir and B. Mond, *Pre-invex functions in multiple objective optimization*, J. Math. Anal. Appl. **136** (1988), 29–38.

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