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ON SOME FRACTIONAL INTEGRAL INEQUALITIES OF HERMITE-HADAMARD TYPE FOR r-PREINVEX FUNCTIONS

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Abstract. In this paper, we prove Hermite-Hadamard type inequalities for r-preinvex functions via fractional integrals. The results presented here would provide extensions of those given in earlier works.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on an interval I of real numbers and $a,\ b \in I$ with $a < b$. The following inequality holds

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

The double inequality (1.1) is known, in the literature, as the Hermite–Hadamard integral inequality for convex functions. Both inequalities hold in the reversed direction if f is concave. The inequality (1.1) has been extended and generalized for various classes of convex functions via different approaches, see [4, 7, 10]. For several recent results concerning the inequality (1.1) we refer the interested reader to [1–12, 14–16, 18], and references therein.

2. Preliminaries

Let K be a nonempty subset of $\mathbb{R}^n$ and let $\eta : K \times K \to \mathbb{R}^n$ be a function.

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Definition 2.1. ([19]) Let $u \in K$. We say K is invex at u with respect to η if, for each $v \in K$

$$u + t\eta(v,u) \in K,\ t \in [0,1]. \tag{2.1}$$

K is said to be an invex set with respect to η if K is invex at each $u \in K$.

Definition 2.2. ([13]) The function f on the invex set K is said to be preinvex with respect to η, if

$$f(u + t\eta(v,u)) \leq (1-t)f(u) + tf(v), \quad (u, v \in K,\ t \in [0,1]). \tag{2.2}$$

Definition 2.3. ([17]) A positive function f on the invex set K is said to be logarithmically preinvex, if

$$f(u + t\eta(v,u)) \leq f^{1-t}(u) f^{t}(v) \tag{2.3}$$

for all $u, v \in K$ and $t \in [0,1]$.

Definition 2.4. ([17]) The function f on the invex set K is said to be r-preinvex with respect to η, if

$$f(u + t\eta(v,u)) \leq M_r(f(u), f(v); t)$$

holds for all $u,\ v \in K$ and $t \in [0,1]$, where

$$M_r(x,y;t) = \begin{cases} [(1-t)x^r + ty^r]^{\frac{1}{r}} & , r \neq 0 \\ x^{1-t}y^t & , r = 0 \end{cases}$$

is the weighted power mean of order r for positive numbers x and y.

Definition 2.5. ([18]) Let $f \in L^1[a,b]$. The Riemann-Liouville fractional integrals $J^{\alpha}_{a^+} f(x)$ and $J^{\alpha}_{b^-} f(x)$ of order $\alpha > 0$ are defined, respectively, by

$$J^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\,dt, \quad x > a \tag{2.4}$$

and

$$J^{\alpha}_{b^-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\,dt, \quad x < b, \tag{2.5}$$

where $\Gamma(\alpha) = \int_0^\infty e^{-u}u^{\alpha-1}du$ is the Gamma function and $J^0_{a^+}f(x) = J^0_{b^-}f(x) = f(x)$.

The main purpose of this paper is to establish Hermite-Hadamard type inequalities for Riemann-Liouville fractional integral using r-preinvex functions. Then, we give some interesting results of Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals. Some special cases are also discussed.

3. Main results

Theorem 3.1. *Let $K = [a, a+\eta(b,a)]$ be an interval of real numbers with interior K°, $a,\ b \in K^\circ$, and $a < a+\eta(b,a)$. Let $f : K \to (0,\infty)$ be an r-preinvex function on the interval K°, then*

$$\left(J^{\alpha}_{(a+\eta(b,a))^-} f\right)(a) \le \frac{[\eta(b,a)]^\alpha}{\Gamma(\alpha)} \left\{ B\left(\alpha, \frac{1}{r}+1\right) f^r(a) + \frac{r}{\alpha r+1} f^r(b) \right\}^{\frac{1}{r}} \tag{3.1}$$

holds for all $0 < r \le 1$.

Proof. Since f is an r-preinvex function and $r > 0$, we have

$$f(a+t\eta(b,a)) \le [tf^r(b) + (1-t)f^r(a)]^{\frac{1}{r}}$$

for all $t \in [0,1]$. Then,

$$\begin{aligned}
\frac{\Gamma(\alpha)}{(\eta(b,a))^\alpha}\left(J^{\alpha}_{(a+\eta(b,a))^-} f\right)(a) &= \frac{1}{(\eta(b,a))^\alpha} \int_a^{a+\eta(b,a)} (u-a)^{\alpha-1} f(u)\,du \\
&= \int_0^1 t^{\alpha-1} f(a+t\eta(b,a))dt \\
&\le \int_0^1 t^{\alpha-1} [tf^r(b) + (1-t)f^r(a)]^{\frac{1}{r}}\, dt \\
&= \int_0^1 \left[t^{r(\alpha-1)+1} f^r(b) + t^{r(\alpha-1)}(1-t) f^r(a)\right]^{\frac{1}{r}} dt.
\end{aligned}$$

Using Minkowski's inequality, we have

$$\begin{aligned}
&\int_0^1 \left[t^{r(\alpha-1)+1} f^r(b) + t^{r(\alpha-1)}(1-t) f^r(a)\right]^{\frac{1}{r}} dt \\
&\le \left\{ \left[\int_0^1 t^{\alpha-1+\frac{1}{r}} f(b)dt\right]^r + \left[\int_0^1 t^{\alpha-1}(1-t)^{\frac{1}{r}} f(a)dt\right]^r \right\}^{\frac{1}{r}} \\
&= \left\{ f^r(b)\frac{r}{\alpha r+1} + f^r(a) B\left(\alpha, \frac{1}{r}+1\right) \right\}^{\frac{1}{r}},
\end{aligned}$$

and the proof is complete.

Remark 3.2. Under the same conditions as in Theorem 3.1, with $\alpha = 1,\ r = 1$ and $\eta(b,a) = b-a$, we have

$$\frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}.$$

□

Theorem 3.3. *Let $f,\ g : K = [a, a+\eta(b,a)] \to (0,\infty)$ be r-preinvex and s-preinvex functions respectively on the interval of real numbers K°, $a,\ b \in K^{\circ}$ with $a < a+\eta(b,a)$, then*

$$\begin{aligned}
&\left(J^{\alpha}_{(a+\eta(b,a))^-} fg\right)(a)\\
&\leq \frac{[\eta(b,a)]^{\alpha}}{2\Gamma(\alpha)}\left\{\left(B\left(\frac{2(\alpha-1)}{r}+1,\frac{2}{r}+1\right)f^{r}(a)+\frac{r}{2\alpha+r}f^{r}(b)\right)^{\frac{2}{r}}\right. \qquad (3.2)\\
&\left.+\left(B\left(\frac{2(\alpha-1)}{s}+1,\frac{2}{s}+1\right)g^{s}(a)+\frac{s}{2\alpha+s}g^{s}(b)\right)^{\frac{2}{s}}\right\}
\end{aligned}$$

holds for $0 < r,\ s \leq 2$.

Proof. Since f is a r-preinvex function and g is a s-preinvex function, by the hypothesis, we have

$$f(a+t\eta(b,a) \leq [tf^{r}(b)+(1-t)f^{r}(a)]^{\frac{1}{r}} \qquad (3.3)$$

and

$$g(a+t\eta(b,a) \leq [tg^{s}(b)+(1-t)g^{s}(a)]^{\frac{1}{s}} \qquad (3.4)$$

for $t \in [0,1]$. By using the inequality (3.3) and (3.4), we get

$$\begin{aligned}
&\frac{1}{[\eta(b,a)]^{\alpha}}\int_{a}^{a+\eta(b,a)}(u-a)^{(\alpha-1)\left(\frac{1}{r}+\frac{1}{s}\right)}f(u)g(u)du\\
&=\int_{0}^{1}t^{(\alpha-1)\left(\frac{1}{r}+\frac{1}{s}\right)}f(a+t\eta(b,a))g(a+t\eta(b,a))\,dt \qquad (3.5)\\
&\leq \int_{0}^{1}t^{(\alpha-1)\left(\frac{1}{r}+\frac{1}{s}\right)}[tf^{r}(b)+(1-t)f^{r}(a)]^{\frac{1}{r}}[tg^{s}(b)+(1-t)g^{s}(a)]^{\frac{1}{s}}\,dt.
\end{aligned}$$

Using Cauchy's inequality for (3.5), we have

$$\begin{aligned}
&\int_{0}^{1}[t^{\alpha}f^{r}(b)+t^{\alpha-1}(1-t)f^{r}(a)]^{\frac{1}{r}}[t^{\alpha}g^{s}(b)+t^{\alpha-1}(1-t)g^{s}(a)]^{\frac{1}{s}}\,dt\\
&\leq \frac{1}{2}\left\{\int_{0}^{1}[t^{\alpha}f^{r}(b)+t^{\alpha-1}(1-t)f^{r}(a)]^{\frac{2}{r}}\,dt+\int_{0}^{1}[t^{\alpha}g^{s}(b)+t^{\alpha-1}(1-t)g^{s}(a)]^{\frac{2}{s}}\,dt\right\}\\
&=\frac{1}{2}\{I_1+I_2\}.
\end{aligned}$$

Using Minkowski's inequality for I_1 and I_2, we hawe

$$I_1 = \int_0^1 \left[t^{\alpha} f^r(b) + t^{\alpha-1}(1-t) f^r(a)\right]^{\frac{2}{r}} dt$$

$$\leq \left\{ \left(\int_0^1 t^{\frac{2}{r}\alpha} f^2(b) dt \right)^{\frac{r}{2}} + \left(\int_0^1 t^{\frac{2(\alpha-1)}{r}} (1-t)^{\frac{2}{r}} f^2(a) dt \right)^{\frac{r}{2}} \right\}^{\frac{2}{r}}$$

$$= \left\{ f^r(b) \frac{r}{2\alpha + r} + f^r(a) B\left(\frac{2(\alpha-1)}{r} + 1, \frac{2}{r} + 1 \right) \right\}^{\frac{2}{r}},$$

and

$$I_2 = \int_0^1 \left[t^{\alpha+1} g^s(b) + t^{\alpha}(1-t) g^s(a)\right]^{\frac{2}{s}} dt$$

$$\leq \left\{ g^s(b) \frac{s}{2\alpha + s} + g^s(a) B\left(\frac{2(\alpha-1)}{s} + 1, \frac{2}{s} + 1 \right) \right\}^{\frac{2}{s}}.$$

Combining I_1 and I_2 leads to (3.2) and the proof is complete. □

Corollary 3.4. *Under the same conditions as in Theorem 3.3, if $r = s = 2$, we have*

$$\frac{\Gamma(\alpha)}{(\eta(b,a))^{\alpha}} \left(J^{\alpha}_{(a+\eta(b,a))^-} fg \right)(a) \leq \frac{f^2(a) + f^2(b) + g^2(a) + g^2(b)}{2(\alpha+1)}.$$

Corollary 3.5. *Under the same conditions as in Theorem 3.3, if $\eta(b,a) = b - a$ and $r = s = 2$, we have*

$$\frac{\Gamma(\alpha)}{(b-a)^{\alpha}} J^{\alpha}_{b^+} fg(a) \leq \frac{f^2(a) + f^2(b) + g^2(a) + g^2(b)}{2(\alpha+1)}.$$

Corollary 3.6. *Under the same conditions as in Theorem 3.3, if $\alpha = 1$ and $r = s = 2$, we have the inequality*

$$\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u) g(u) du \leq \frac{f^2(a) + f^2(b) + g^2(a) + g^2(b)}{4}$$

in [17].

Corollary 3.7. *Under the same conditions as in Theorem 3.3, if $\alpha = 1$, $\eta(b,a) = b - a$ and $r = s = 2$, we have*

$$\frac{1}{b-a} \int_a^b f(u) g(u) du \leq \frac{f^2(a) + f^2(b) + g^2(a) + g^2(b)}{4}.$$

Corollary 3.8. *Under the same conditions as in Theorem 3.3, if $\alpha = 1$, $r = s = 2$, and $f(x) = g(x)$, we have the inequality*

$$\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f^2(u) du \leq \frac{f^2(a) + f^2(b)}{2}$$

in [17].

Theorem 3.9. *Let $f,\ g : K = [a, a + \eta(b, a)] \to (0, \infty)$ be r-preinvex and s-preinvex functions, respectively, on the interval of real numbers K°, $a, b \in K^\circ$ with $a < a + \eta(b, a)$. If $r > 1$ and $\frac{1}{r} + \frac{1}{s} = 1$, then*

$$\left(J^{\alpha}_{(a+\eta(b,a))^-} fg\right)(a)$$
$$\leq \frac{[\eta(b,a)]^\alpha}{\Gamma(\alpha)} \left(\left(f^r(a) B(\alpha, 2) + \frac{f^r(b)}{\alpha+1} \right)^{\frac{1}{r}} + \left(g^s(a) B(\alpha, 2) + \frac{g^s(b)}{\alpha+1} \right)^{\frac{1}{s}} \right).$$

Proof. Since f is a r-preinvex function and g is a s-preinvex function, for $t \in [0, 1]$, we have

$$f(a + t\eta(b, a) \leq [tf^r(b) + (1 - t)f^r(a)]^{\frac{1}{r}} \tag{3.6}$$

and

$$g(a + t\eta(b, a) \leq [tg^s(b) + (1 - t)g^s(a)]^{\frac{1}{s}}. \tag{3.7}$$

From (3.6) and (3.7), we get

$$\frac{1}{[\eta(b,a)]^\alpha} \int_a^{a+\eta(b,a)} (u-a)^{(\alpha-1)\left(\frac{1}{r}+\frac{1}{s}\right)} f(u)\, g(u) du$$
$$= \int_0^1 t^{(\alpha-1)\left(\frac{1}{r}+\frac{1}{s}\right)} f(a + t\eta(b, a)) g(a + t\eta(b, a))\, dt$$
$$\leq \int_0^1 t^{(\alpha-1)\left(\frac{1}{r}+\frac{1}{s}\right)} [tf^r(b) + (1-t)f^r(a)]^{\frac{1}{r}} [tg^s(b) + (1-t)g^s(a)]^{\frac{1}{s}}\, dt.$$

By virtue of Hölder's inequality, we have

$$\int_0^1 \left[t^\alpha f^r(b) + t^{\alpha-1}(1-t) f^r(a)\right]^{\frac{1}{r}} \left[t^\alpha g^s(b) + t^{\alpha-1}(1-t) g^s(a)\right]^{\frac{1}{s}} dt$$
$$\leq \left\{ \int_0^1 \left[t^\alpha f^r(b) + t^{\alpha-1}(1-t) f^r(a)\right] dt \right\}^{\frac{1}{r}} + \left\{ \int_0^1 \left[t^\alpha g^s(b) + t^{\alpha-1}(1-t) g^s(a)\right] dt \right\}^{\frac{1}{s}}$$
$$= \frac{[\eta(b,a)]^\alpha}{\Gamma(\alpha)} \left\{ \left(f^r(b) \frac{1}{\alpha+1} + f^r(a) B(\alpha, 2) \right)^{\frac{1}{r}} + \left(g^s(b) \frac{1}{\alpha+1} + g^s(a) B(\alpha, 2) \right)^{\frac{1}{s}} \right\}.$$

The proof is done. □

Corollary 3.10. *Under the same conditions as in Theorem 3.9, if $r = s = 2$, we have*

$$\left(J^{\alpha}_{(a+\eta(b,a))^-} fg\right)(a)$$
$$\leq \frac{[\eta(b,a)]^\alpha}{\Gamma(\alpha)} \left(\sqrt{f^2(a) B(\alpha, 2) + \frac{f^2(b)}{\alpha+1}} + \sqrt{g^2(a) B(\alpha, 2) + \frac{g^2(b)}{\alpha+1}} \right).$$

Corollary 3.11. *Under the same conditions as in Theorem 3.9, if* $r = s = 2$, $\eta(b,a) = b - a$, *we have*

$$\frac{(b-a)^{\alpha}}{\Gamma(\alpha)} J_{b+}^{\alpha} fg(a) \leq \sqrt{f^2(a) B(\alpha, 2) + \frac{f^2(b)}{\alpha+1}} + \sqrt{g^2(a) B(\alpha, 2) + \frac{g^2(b)}{\alpha+1}}.$$

Corollary 3.12. *Under the same conditions as in Theorem 3.9, if* $r = s = 2$, $\eta(b,a) = b - a$ *and* $\alpha = 1$, *we have*

$$\frac{1}{b-a} \int_a^b f(u) g(u) du \leq \sqrt{\frac{f^2(a) + f^2(b)}{2}} \sqrt{\frac{g^2(a) + g^2(b)}{2}}.$$

Corollary 3.13. *Under the same conditions as in Theorem 3.9, if* $r = s = 2$ *and* $\alpha = 1$, *we have*

$$\frac{1}{[\eta(b,a)]} \int_a^{a+\eta(b,a)} f(u) g(u) du \leq \sqrt{\frac{f^2(a) + f^2(b)}{2}} \sqrt{\frac{g^2(a) + g^2(b)}{2}}.$$

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