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BERGMAN KERNEL ESTIMATES AND TOEPLITZ OPERATORS ON HOLOMORPHIC LINE BUNDLES

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Abstract. We characterize operator-theoretic properties (boundedness, compactness, and Schatten class membership) of Toeplitz operators with positive measure symbols on Bergman spaces of holomorphic hermitian line bundles over Kähler Cartan-Hadamard manifolds in terms of geometric or operator-theoretic properties of measures.

1. Introduction and statment of results

The purpose of this paper is to extend the standard theory dealing with boundedness, compactness, and Schatten class membership of Toeplitz operators with nonnegative measure symbols on generalized Bargmann-Fock spaces [6, 12, 14–16, 21, 22, 25, 30] to Bergman spaces of holomorphic sections of hermitian holomorphic line bundles over Kähler Cartan-Hadamard manifolds. As an application, we give a characterization of self-holomorphic maps whose composition operators bounded, compact or belongs to the Schatten ideal class which extend previous results for generalized Bargmann-Fock spaces [4, 27–29, 34].
Let (M, g) be a complex Hermitian manifold and $(L, h) \longrightarrow M$ be a holomorphic Hermitian line bundle. For $p \in]0, \infty]$, define $\mathcal{F}^p(M, L)$ the $\mathbb{C}$-vector space of holomorphic sections $s : M \longrightarrow L$ such that

$$\|s\|_2 := \Big(\int_M |s|_h^p dv_g\Big)^{\frac{1}{p}} < \infty.$$

Let P the orthogonal projection from the Hilbert space of $L^2(M, L)$ onto its closed subspace $\mathcal{F}^2(M, L)$. Let $K \in C^\infty(M \times M, L \otimes \overline{L})$ the reproducing (or Bergman)

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kernel of P, that is

$$K(z,w) = \sum_{j=1}^{d} s_j(z) \otimes \overline{s_j(w)},$$

where $\overline{L}$ is the conjugate bundle of L, (s_j) is an orthonormal basis for $\mathcal{F}^2(M,L)$ and $d = dim\mathcal{F}^2(M,L) \leq \infty$.

The first result of this paper is a pointwise estimate for the Bergman kernel of L in spirit of those obtained in [1,5,20] for $n=1$ and in [8,18,26] for $n \geq 2$.

Theorem 1.1. *Let (M,g) be a Stein Kähler manifold with bounded geometry. Let $(L,h) \longrightarrow (M,g)$ be a Hermitian holomorphic line bundle with bounded curvature such that*

$$c(L) + Ricci(g) \geq a\omega_g$$

for some positive constant a. There are constants α, $C > 0$ such that for all $z, w \in M$,

$$|K(z,w)| \leq Ce^{-\alpha d_g(z,w)}.$$

For a positive measure μ, the Toeplitz operator T_μ with symbol μ is defined formally by

$$T_\mu s(z) = \int_M K(z,w) \bullet s(w) d\mu(w),$$

where $z \to K(z,w) \bullet s(w) \in L_z$ is the holomorphic section of L defined

$$K(z,w) \bullet s(w) := \sum_{j=1}^{d} < s(w), s_j(w) >_{L_w} s_j(z).$$

Let $\tilde{\mu} : M \to \mathbb{R}^+$ be the Berezin transform of μ:

$$\tilde{\mu}(z) := \int_M |k_z(w)|^2 d\mu(w),$$

where

$$k_z(w) := \frac{K(w,z)}{\sqrt{|K(z,z)|}}.$$

Let $T : H_1 \to H_2$ be a compact operator from two Hilbert spaces H_1 and H_2 and

$$Tf = \sum_{n=0}^{\infty} \lambda_n < f, e_n > \sigma_n, \quad f \in H_1,$$

its Schmidt decomposition where (e_n) (resp. (σ_n)) is an orthonormal basis of H_1 (resp. H_2) and (λ_n) is a sequence with $\lambda_n > 0$ and $\lambda_n \to 0$ (see [30]). For $0 < p \leq \infty$, the compact operator T belongs to the Schatten-von Neumann p-class $\mathcal{S}_p(H_1, H_2)$ if and only if

$$\|T\|_{\mathcal{S}_p}^p := \sum_{n=0}^{\infty} \lambda_n^p < \infty.$$

Let (N, ω_N) be a Hermitian manifold. Let $\Phi : N \to M$ be a holomorphic map and

$$\begin{array}{rcl} C_\Phi : \mathcal{F}^2(M,L) & \longrightarrow & \mathcal{F}^2(N, \Phi^* L) \\ s & \longrightarrow & s \circ \Phi \end{array}$$

the composition operator associated to Φ. We define the transform B_Φ (related to the usual Berezin transform) associated to Φ to be a function on M as follows:

$$B_\Phi(z)^2 = \int_M |K(z,w)|^2 d\nu_\Phi(w),$$

where ν_Φ is the pull-back measure defined as follows : for all Borel set $E \subset M$

$$\nu_\Phi(E) = \int_N \mathbf{1}_{\Phi^{-1}(E)}(w) dv_{\omega_N}(w).$$

Our second result of this paper is the characterization of operator-theoretic properties (boundedness, compactness, and Schatten class membership) of Toeplitz operators with positive measure symbols on Bergman space of holomorpic sections which extend those for generalized Bargmann-Fock spaces.

Theorem 1.2. *Let (M, g) be a Kähler Cartan-Hadamard manifold with bounded geometry and uniformly subexponentially volume growth. Let $(L, h) \longrightarrow (M, g)$ be a holomorphic Hermitian line bundle with bounded curvature such that*

$$c(L) + Ricci(g) \geq a\omega_g,$$

for some positive constant a. Let μ be a positive measure on M. Let $p \in [1, +\infty]$. The following conditions are equivalent
(a) The operator $T_\mu : \mathcal{F}^p(M,L) \longrightarrow \mathcal{F}^p(M,L)$ is bounded $(1 \leq p \leq \infty)$.
(b) μ is a Carleson measure.
(c) $\tilde{\mu}$ is bounded on M.
(d) There exists $\delta > 0$ such that the function $z \to \mu(B_g(z,\delta))$ is bounded.

Theorem 1.3. *Let (M, g) be a Kähler Cartan-Hadamard manifold with bounded geometry and uniformly subexponentially volume growth. Let $(L, h) \longrightarrow (M, g)$ be a holomorphic Hermitian line bundle with bounded curvature such that*

$$c(L) + Ricci(g) \geq a\omega_g,$$

for some positive constant a. Let μ be a positive measure on M. Let $p \in [1, \infty]$. The following conditions are equivalent :
(a) The operator $T_\mu : \mathcal{F}^2(M,L) \longrightarrow \mathcal{F}^2(M,L)$ is compact.
(b) μ is a vanishing Carleson measure.
(c) $\lim\limits_{d_g(z,z_0)\to\infty} \tilde{\mu}(z) = 0$.
(d) There exists $\delta > 0$ such that $\lim\limits_{d_g(z,z_0)\to\infty} \mu(B_g(z,\delta)) = 0$

Theorem 1.4. *Let (M, g) be a Kähler Cartan-Hadamard manifold with bounded geometry and uniformly subexponentially volume growth. Let $(L, h) \longrightarrow (M, g)$ be a holomorphic Hermitian line bundle with bounded curvature such that*

$$c(L) + Ricci(g) \geq a\omega_g$$

for some positive constant a. Let μ be a positive measure on M. The following conditions are equivalent :
(a) The operator $T_\mu : \mathcal{F}^2(M,L) \longrightarrow \mathcal{F}^2(M,L)$ belongs to $\mathcal{S}_p$ $(0 < p \leq \infty)$.
(b) $\tilde{\mu} \in L^p(M, dv_g)$.
(c) There exists $\delta > 0$ such that the function $z \to \mu(B_g(z,\delta)) \in L^p(M, dv_g)$.
(d) There exists $\delta > 0$ and an r-lattice (a_j) such that $\{\mu(B_g(a_j,\delta))\} \in \ell^p(\mathbb{N})$.
Moreover, there is a positive constant C such that

$$\frac{1}{C}\|\tilde{\mu}\|_{L^p(M,dv_g)} \leq \|T_\mu\|_{\mathcal{S}_p} \leq C\|\tilde{\mu}\|_{L^p(M,dv_g)}.$$

For boundedness, compactness, and Schatten class membership of composition operators, we have the following result which extends those for Bargmann-Fock spaces.

Theorem 1.5. *Let (M,g) be a Kähler Cartan-Hadamard manifold with bounded geometry and uniformly subexponentially volume growth. Let $(L,h) \longrightarrow (M,g)$ be a holomorphic Hermitian line bundle with bounded curvature such that*

$$c(L) + Ricci(g) \geq a\omega_g$$

for some positive constant a. Let (N, ω_N) be a Hermitian manifold. Let $\Phi : M \to M$ be a holomorphic map and

$$\begin{array}{rcl} C_\Phi : \mathcal{F}^2(M,L) & \longrightarrow & \mathcal{F}^2(N, \Phi^*L) \\ s & \longrightarrow & s \circ \Phi \end{array}$$

the composition operator associated to Φ. Let $0 < p < \infty$. Then
(i) C_Φ is bounded if and only if ν_Φ is a Carleson measure for $\mathcal{F}^2(M,L)$ if and only if B_Φ is bounded.
(ii) C_Φ is compact if and only if ν_Φ is a vanishing Carleson measure for $\mathcal{F}^2(M,L)$ if and only if B_Φ vanishes at infinity.
(iii) C_Φ is in Schatten p-class if and only if $B_\Phi \in L^p(M, dv_g)$. Morever there is a positive constant C such that

$$\frac{1}{C}\|B_\Phi\|_{L^p(M,dv_g)} \leq \|C_\Phi\|_{\mathcal{S}_p} \leq C\|B_\Phi\|_{L^p(M,dv_g)}.$$

Characterizations of bounded, compact and Schatten class Toeplitz operators with positive measure symbols on generalized Bargmann-Fock space or on weighted Bergman spaces of bounded strongly pseudoconvex domains, in terms of Carleson measures and the Berezin transform, depend strongly on off-diagonal exponential decay of the Bergman kernel. In the spirit of [8], we establish a similar off-diagonal decay of the Bergman kernel associated to holomorphic Hermitian line bundles whose curvature is uniformly comparable to the metric form.

This paper consists of five sections. In the next section, we will recall some definitions and properties of Kähler manifolds, Bergman Kernel of line bundles, $\bar{\partial}$-methods and Toeplitz operators. In Section 3, we provide useful estimates for Bergman kernel and we prove Theorem 1.1. In Section 4, we will prove Theorems 1.2, 1.3. In Section 5, we will prove Theorems 1.4 and 1.5.

2. Preliminary

2.1. **Curvatures in Kählerian Geometry.** Let (M, J, g) be a complex n-manifold with a Riemannian metric g which is Hermitian i.e.,

$$g(JX, JY) = g(X, Y),\ \forall X, Y \in TM \text{ (real tangent vectors)}$$

and a complex structure $J : TM \to TM$ i.e $J^2 = -Id_{TM}$. Assume furthermore that g is Kähler i.e., the real 2-form

$$\omega_g(X, Y) = g(JX, Y)$$

is closed. In local coordinates $z^1, z^2, \cdots, z^n$ of M

$$g = \sum_{i,j=1}^{n} g_{i\bar{j}} dz^i \otimes d\bar{z^j}, \quad \omega = \frac{\sqrt{-1}}{2} \sum_{i,j=1}^{n} g_{i\bar{j}} dz^i \wedge d\bar{z^j}.$$

The coefficients of the curvature tensor R of g are given by

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial \bar{z}^l} + \sum_{p,q=1}^{n} g^{q\bar{p}} \frac{\partial g_{i\bar{p}}}{\partial z^k} \frac{\partial g_{q\bar{j}}}{\partial \bar{z}^l}.$$

The sectional curvature of a 2-plane $\sigma \subset T_xM$ is defined as

$$K(\sigma) := R(X, Y, Y, X) = R(X, Y, JY, JX),$$

where $\{X, Y\}$ is an orthonormal basis of σ.

Definition 2.1. We say that (M, g) has non-positive sectional curvature if

$$K(\sigma) \leq 0 \quad \text{for all } 2-\text{plane } \sigma \subset TM.$$

A Cartan-Hadamad manifold is a simply conneceted complete manifold with negative sectional curvature. Since each point in a Cartan-Hadamard manifold is a pole, then the square of the distance function at such point is smooth.

The Ricci curvature of g is the $(1, 1)$-form

$$Ric(g) := \frac{i}{2\pi} \sum_{i,jk,l=1}^{n} g^{k\bar{l}} R_{i\bar{j}k\bar{l}} dz^i \wedge d\bar{z}^j.$$

In local coordinates

$$Ric(g) = -\frac{i}{2\pi} \sum_{i,j=1}^{n} \frac{\partial^2 \log \det(g_{k\bar{l}})}{\partial z^i \partial \bar{z}^j} dz^j \wedge d\bar{z}^l.$$

Definition 2.2. We say that the Ricci curvature of (M, g) has lower bound $C \in \mathbb{R}$ if

$$Ric(g)(X, X) \geq C\omega_g(X, X) \quad \text{for all } X \in T^{(1,0)}M.$$

Denote by $d_g(z, w)$ the Riemannian distance from $z \in X$ to $w \in X$ and $B(z, r) = \{w \in M \ : \ d_g(w, z) < r\}$ the open geodesic ball. The manifold (M, g) is complete if (M, d_g) is a complete metric space.

Given (M, g) a Riemannian manifold, we say that a family (Ω_k) of open subsets of M is a uniformly locally finite covering of M if the following holds:

(Ω_k) is a covering of M, and there exits an integer N such that each point $x \in M$ has a neighborhood which intersects at most N of the Ω_k. One then has the following Gromov's Packing Lemma [11].

Lemma 2.3. *Let (M, g) be a smooth, compete Riemannian n-manifold with Ricci curvature bounded from below by some K real, and let $\rho > 0$ be given. There exists a sequence (x_i) of points of M such that for every $r \geq \rho$:*
(i) the family $(B_g(x_i, r))$ is a uniformly locally finite covering of M, and there is an upper bound for N in terms of n, r, ρ, and K,
(ii) for any $i \neq j$, $B_g(x_i \frac{\rho}{2}) \cap B_g(x_j, \frac{\rho}{2}) = \emptyset$.

Definition 2.4. We say that the volume of (M, g) grows uniformly subexponentially if and only if for any $\epsilon > 0$ there exists a constant $C < \infty$ such that, for all $r > 0$ and all $z \in M$

$$vol_g(B(z, r)) \leq Ce^{\epsilon r} vol_g(B(z, 1)).$$

Definition 2.5. A Hermitian manifold (M, g) is said to have bounded geometry if there exist positive numbers R and c such that for all $z \in M$ there exists a biholomorphic mapping $F_z : (U, 0) \subset \mathbb{C}^n \longrightarrow (V, z) \subset M$ such that
(i) $F_z(0) = z$,
(ii) $B_g(z, R) \subset F_z(U)$ and
(iii) $\frac{1}{c} g_e \leq F_z^* g \leq c g_e$ on $F_z^{-1}(B_g(z, R))$ where g_e is the Euclidean metric.

By (iii)

$$\forall\ w \in B_g(z, R)\ :\ \frac{1}{c} \|F_z^{-1}(w)\|_e \leq d_g(w, z) \leq c \|F_z^{-1}(w)\|_e.$$

Remark 2.6. If a Hermitian manifold (M, g) has bounded geometry, then the geodesic exponential map $\exp_z : T_z^{\mathbb{R}} M \to M$ is defined on a ball $B(0, r) \subset T_z^{\mathbb{R}} M$ for any $r < R$ and provide a diffeomorphism of this ball onto the ball $B_g(z, r) \subset M$. It follows that the manifold (M, g) is complete.

Remark 2.7. It is well known that if (M, g) has bounded geometry and $Ric(g) \geq Kg$, then (M, g) satisfies the uniform ball size condition ([7] Prop. 14) i.e., for every $r \in \mathbb{R}^+$

$$\inf_{z \in M} vol(B_g(z, r)) > 0 \quad \text{and} \quad \sup_{z \in M} vol(B_g(z, r)) < \infty.$$

Also by Volume Comparison Theorem [3], there are nonnegative constants C, α, β such that

$$vol_g(B_g(z, r)) \leq Cr^\alpha e^{\beta r}, \quad \forall\ r \geq 1,\ z \in M.$$

Bounded geometry allows one to produce an exhaustion function which behaves like the distance function and whose gradient and hessian are bounded on M [23].

Lemma 2.8. *Let (M, g) be a Hermitian manifold with bounded geometry. For every $z \in M$ there exists a smooth function $\Psi_z : M \longrightarrow \mathbb{R}$ such that*
(i) $C_1 d_g(., z) \leq \Psi_z \leq C_2(d_g(., z) + 1)$,
(ii) $|\partial \Psi_z|_g \leq C_3$, and
(iii) $-C_4 \omega_g \leq i\partial\bar{\partial}\Psi_z \leq C_5 \omega_g$.

Furthermore, the constants in $(i), (ii)$ and (iii) depend only on the constants associated with the bounded geometry of (M, g).

2.2. **Bergman Kernel of Line Bundles.** Let L be a holomorphic Hermitian line bundle over a complex manifold M, and let (U_j) be a covering of the manifold by open sets over which L is locally trivial. A section s of L is then represented by a collection of complex valued functions s_j on U_j that are related by the transition functions (g_{jk}) of the bundle

$$s_j = g_{jk} s_k \quad \text{on } U_j \cap U_k.$$

We say that s is holomorphic if each s_i is holomorphic on U_j and we write $\bar{\partial} s = 0$. The conjugate bundle of L is the Hermitian anti-holomorphic line bundle $\overline{L}$ whose transition functions are $(\overline{g}_{jk})$. A metric h on L is given by a collection of real valued functions Φ_j on U_j, related so that

$$|f_j|^2 e^{-\Phi_j} =: |s|_h^2$$

is globally well defined. We will write h for the collection (Φ_j) , and refer to h the metric on L. We say that L is positive, $L > 0$, if h can be chosen smooth with curvature

$$c(L) := i\partial\bar{\partial}\Phi_j$$

strictly positive, and that L is semipositive, $L \geq 0$, if it has a smooth metric of semipositive curvature. We say that h is a singular metric if each Φ_j is only plurisubharmonic.

Definition 2.9. A holomorphic Hermitian line bundle $(L, h) \longrightarrow (M, g)$ has bounded curvature if

$$-M\omega_g \leq c(L) \leq M\omega_g,$$

for some positive constant M.

Fix $p \in [1, +\infty]$. Consider the Lebesgue spaces

$$\begin{aligned} L^p(M, L) &:= \{s : M \longrightarrow L \ : \ \|s\|_p := \Big(\int_M |u|_h^p dv_g\Big)^{\frac{1}{p}} < \infty\}, \\ L^\infty(M, L) &:= \{s : M \longrightarrow L \ : \ \|s\|_\infty := \sup_{z \in M} |s(z)|_h < \infty\}, \end{aligned}$$

and the Bergman spaces of holomorphic sections

$$\begin{aligned} \mathcal{F}^p(M, L) &:= \{s \in L^p(M, L) \ : \ \bar{\partial} s = 0\} \\ \mathcal{F}^\infty(M, L) &:= \{s \in L^\infty(M, L) \ : \ \bar{\partial} s = 0\}. \end{aligned}$$

Let us note an important property of the space $\mathcal{F}^2(X, L)$ which follows from the Cauchy estimates for holomorphic functions. Namely, for every compact set $G \subset M$ there exists $C_G > 0$ such that

$$\sup_{z \in G} |s(x)| \leq C_G \|s\|_2 \text{ for all } s \in \mathcal{F}^2(X, L). \tag{2.1}$$

We deduce that $\mathcal{F}^2(M, L)$ is a closed subspace of $L^2(M, L)$. One can show also that $\mathcal{F}^2(M, L)$ is separable (cf. [31, p. 30]).

Definition 2.10. The Bergman projection is the orthogonal projection

$$P \ : \ L^2(M,L) \longrightarrow \mathcal{F}^2(M,L).$$

In view of (2.1), the Riesz Representation Theorem shows that for a fixed $z \in M$ there exists a section $K(z,.) \in L^2(M, L_z \otimes \overline{L})$ such that

$$s(z) = \int_M K(z,w) \bullet s(w) dv_g \text{ for all } s \in \mathcal{F}^2(M,L). \tag{2.2}$$

The distribution kernel K is called the Bergman Kernel of $(L,h) \longrightarrow (M,g)$. If $\mathcal{F}^2(M,L) = 0$ we have of course $K(z,z) = 0$ for all $z \in M$. If $\mathcal{F}^2(M,L) \neq 0$, consider an orthonormal basis $(s_j)_{j=1}^d$ of $\mathcal{F}^2(X,L)$ (where $1 \leq d \leq \infty$). By estimates (1.1)

$$K(z,w) = \sum_{j=1}^d s_j(z) \otimes \overline{s_j(w)} \in L_z \otimes \overline{L_w},$$

where the right hand side converges on every compact together with all its derivatives (see [31, p.62]). Thus $K(z,w) \in C^\infty(M \times M, L \otimes \overline{L})$. It follows that

$$(Ps)(z) = \int_M K(z,w) \bullet s(w) dv_g(w), \text{ for all } s \in L^2(M,L),$$

that is $K(.,.)$ is the integral kernel of the Bergman projection P. Since

$$\begin{aligned} |K(z,w)|^2 &= \sum_{j=1}^d \sum_{k=1}^d < s_j(z) \otimes \overline{s_j(w)}, s_k(z) \otimes \overline{s_k(w)} >_{L_z \otimes \overline{L_w}} \\ &= \sum_j \sum_k < s_j(z), s_k(z) >_{L_z} \overline{< s_j(w), s_k(w) >_{L_w}}, \end{aligned}$$

then $K(z,w)$ is symmetric

$$|K(z,w)| = |K(w,z)|.$$

The function $|K(z,z)|$ is called the Bergman function of $\mathcal{F}^2(M,L)$. It satisfies

$$|K(z,z)| = \int_M |K(z,w)|^2 dv_g(w).$$

2.3. **$\bar{\partial}$-Methods.** We recall Demailly's Theorem [9], which generalizes Hörmander's L^2 estimates [13] (Theorem 2.2.1, p. 104) for forms with values in a line bundle.

Theorem 2.11. *Let (X,ω) be a complete Kähler manifold, (L,h) a holomorphic Hermitian line bundle over X, and let ϕ be a locally integrable function over X. If the curvature $c(L)$ is such that*

$$c(L) + Ric(\omega) + i\partial\bar{\partial}\phi \geq \gamma\omega,$$

for some positive and continuous function γ on X, then for all $v \in L^2_{(0,1)}(X, L, loc)$, $\bar{\partial}$-closed and such that

$$\int_X \gamma^{-1} |v|^2 dv_\omega < \infty.$$

There exists $u \in L^2(X, L)$ *such that*

$$\bar{\partial} u = v \quad \text{and} \quad \int_X |u|_h^2 dv_\omega \leq \int_X \gamma^{-1} |v|^2_{\omega,h} dv_\omega.$$

Also, we recall J. McNeal-D. Varolin's Theorem [19] (Theorem 2.2.1, p. 104), which generalizes Berndtsson-Delin's improved L^2-estimate of $\bar{\partial}$-equation having minimal L^2-norm [2, 8] for forms with values in a line bundle.

Theorem 2.12. *Let* (M, g) *be a Stein Kähler manifold, and* $(L, h) \longrightarrow (M, g)$ *a holomorphic Hermitian line bundle with Hermitian metric* h*. Suppose there exists a smooth function* $\eta : M \to \mathbb{R}$ *and a positive i.e., strictly positive Hermitian* $(1, 1)$*-form* Θ *on* M *such that*

$$c(L) + Ric(g) + i\partial\bar{\partial}\eta - i\partial\eta \wedge \bar{\partial}\eta \geq \Theta.$$

Let v *be an* L*-valued* $(0, 1)$*-form such that* $v = \bar{\partial} u$ *for some* L*-valued section* u *satisfying*

$$\int_M |u|_h^2 dv_g < \infty.$$

Then the solution u_0 *of* $\bar{\partial} u = v$ *having minimal* L^2*-norm i.e.,*

$$\int_M < u_0, \sigma > dv_g = 0 \text{ for all } \sigma \in \mathcal{F}^2(M, L)$$

satisfies the estimate

$$\int_M |u_0|_h^2 e^\eta dv_g \leq \int_M |v|^2_{\Theta,h} e^\eta dv_g.$$

3. Estimates for the Bergman Kernel

3.1. Weighted Bergman Inequalities.

Proposition 3.1. *Let* (M, g) *be a complete noncompact Kähler manifold with bounded geometry and lower Ricci curvature bound. Let* $(L, h) \longrightarrow (M, g)$ *be a Hermitian holomorphic line bundle with bounded curvature. Fix* $p \in]0, \infty[$*. Then for each* $r > 0$ *there exists a constant* C_r *such that if* $s \in \mathcal{F}^2(M, L)$*, then*

$$|s(z)|^p \leq C_r^p \int_{B_g(z,r)} |s|^p dv_g. \tag{3.1}$$

In particular, $\mathcal{F}^p(M, L) \subset \mathcal{F}^\infty(M, L)$ *and*

$$|\nabla |s(z)|^p|_g(z) \leq C_r^p \int_{B_g(z,r)} |s|^p dv_g. \tag{3.2}$$

Proof. Since (M, g) has bounded geometry, there exists positive numbers R and c such that for all $z \in M$ there exists a biholomorphic mapping $\Psi_z : (U, 0) \subset \mathbb{C}^n \longrightarrow (V, z) \subset M$ such that
(i) $\Psi_z(0) = z$,
(ii) $B_g(z, R) \subset \Psi_z(U)$ and

(iii) $\frac{1}{c}g_e \leq \Psi_z^* g \leq c g_e$ on $\Psi_z^{-1}(B_g(z,R))$, where g_e is the Euclidean metric.
Consider the $(1,1)$-form defined on $B_e(0,\delta(R)) \subset\subset \Psi_z^{-1}(B_g(z,R)) \subset \mathbb{C}^n$ by

$$\Theta := \Psi_z^* c(L).$$

Since $-K\omega_g \leq c(L) \leq K\omega_g$, by [25, Lemma 4.1] , there exists a function $\phi \in C^2(B_e(0,\delta))$ such that

$$i\partial\bar{\partial}\phi = \Theta \quad \text{and} \quad \sup_{B_e(0,\delta)} (|\phi| + |d\phi|_{g_e}) \leq M.$$

On $B_g(z,\eta) \subset\subset \Psi_z(B_e(0,\delta(R)))$, consider the C^2-function

$$\psi := \phi \circ \Psi_z^{-1}$$

. By (iii) we have

$$i\partial\bar{\partial}\psi = c(L) \quad \text{and} \quad \sup_{B_g(z,\eta)} (|\psi| + |\nabla\psi|_g) \leq M^{'},$$

where $M^{'}$ and η depend only on R and c.
Let e be a frame of L around $z \in B_g(z,\eta)$ and $\Phi(w) = -\log|e(w)|^2$. Then $i\partial\bar{\partial}\psi = i\partial\bar{\partial}\Phi$ on $B_g(z,\eta)$. Hence the function

$$\rho(w) = \Phi(w) - \Phi(z) + \psi(z) - \psi(w)$$

is pluriharmonic. Then $\rho = \Re(F)$ for some holomorphic function F with $\Im(F)(z) = 0$ and

$$\sup_{B_g(z,\eta)} |\Phi - \Phi(z) - \Re(F)| = \sup_{B_g(z,\eta)} |\psi - \psi(z)| \leq C, \tag{3.3}$$

$$\sup_{B_g(z,\eta)} |\nabla(\Phi - \Phi(z) - \Re(F))|_g = \sup_{B_g(z,\eta)} |\nabla\psi|_g \leq C. \tag{3.4}$$

We can suppose $0 < r \leq \eta$. According to [17], for all $z \in M$ and all holomorphic function f on $B_g(z,\eta)$ and all $\zeta \in B_g(z,\eta/2)$

$$|f(\zeta)|^p \leq \frac{C}{\mathrm{Vol}(B_g(\zeta,\eta/2))} \int_{B_g(\zeta,\eta)} |f(w)|^p dv_g,$$

where C depend only in K, n, η. Since g has bounded geometry $\mathrm{Vol}(B_g(z,\eta/2)) \succeq 1$ uniformly in z. Hence

$$|f(\zeta)|^p \leq C \int_{B_g(\zeta,\eta)} |f(w)|^p dv_g.$$

Let $s \in \mathcal{F}^p(M,L)$ and $s = fe$ on $B_g(z,\eta)$. By (2.3) we have have

$$\begin{aligned} |s|_h^p &= |fe^{-\frac{F}{2}}|^p e^{-\frac{p}{2}\Phi(z)} e^{-\frac{p}{2}(\Phi - \Phi(z) - \Re(F))} \\ &\leq C^p |fe^{-\frac{F}{2}}|^p e^{-\frac{p}{2}\Phi(z)}. \end{aligned}$$

By mean value inequality

$$\begin{aligned} |f(z)e^{-\frac{F(z)}{2}}|^p e^{-\frac{p}{2}\Phi(z)} &\leq c_r^p \int_{B_g(z,r)} |fe^{-\frac{F}{2}}|^p e^{-\frac{p}{2}\Phi(z)} dv_g \\ &\leq C_r^p \int_{B_g(z,r)} |fe^{-\Phi(w)}|^p dv_g. \end{aligned}$$

Hence

$$|s(z)|_h^p \leq C_r^p \int_{B_g(z,r)} |s|^p dv_g.$$

By (2.3) and (2.4)

$$\begin{aligned}
|\nabla |s|_h^p|_g &\leq e^{-\frac{p}{2}\Phi(z)} e^{-\frac{p}{2}(\Phi-\Phi(z)-\Re(F)))} |\nabla |fe^{-\frac{F}{2}}|^p| \\
&+ \frac{p}{2}|fe^{-\frac{F}{2}}|^p e^{-\frac{p}{2}\Phi(z)} e^{-\frac{p}{2}(\Phi-\Phi(z)-\Re(F))} |\nabla(\Phi-\Phi(z)-\Re(F))|_g \\
&\leq e^{-\frac{p}{2}\Phi(z)} e^{-\frac{p}{2}(\Phi-\Phi(z)-\Re(F))} |\nabla |fe^{-\frac{F}{2}}|^p| \\
&+ \frac{p}{2}|s|_h^p e^{-\frac{p}{2}(\Phi-\Phi(z)-\Re(F))} |\nabla(\Phi-\Phi(z)-\Re(F))|_g \\
&\leq C^p \big(e^{-\frac{p}{2}\Phi(z)} |\nabla |fe^{-\frac{F}{2}}|^p| + \frac{p}{2}|s|_h^p\big).
\end{aligned}$$

By mean value inequality (Cauchy formula for partial derivatives), there exists $c_r > 0$ such that

$$\begin{aligned}
|\nabla |fe^{-\frac{F}{2}}|^p|(z) e^{-\frac{p}{2}\Phi(z)} &\leq c_r^p \int_{B_g(z,r)} |fe^{-\frac{F}{2}}|^p e^{-\frac{p}{2}\Phi(z)} dv_g \\
&\leq C_r^p \int_{B_g(z,r)} |s|^p dv_g.
\end{aligned}$$

From this, we get (2.2). □

3.2. **Slow Growth of Bergman Sections.**

Lemma 3.2. *Let (M, g) be a Kähler manifold with bounded geometry and lower Ricci curvature bound. Let $(L, h) \longrightarrow (M, g)$ be a Hermitian holomorphic line bundle with bounded curvature. Then there exists $\delta > 0$ with the following properties: if $z \in M$, $s \in \mathcal{F}^p(M, L)$, $\|s\|_p \leq 1$, then*

$$|s(z)|_h \geq a \Longrightarrow |s(w)|_h \geq \frac{a}{2}, \ \forall\ w \in B_g(z, \delta).$$

Proof. Let $R > \delta > 0$. By (3.2) of Proposition 3.1 and mean value theorem for all $w \in B_g(z, R/2)$

$$\begin{aligned}
||s(w)|_h^p - |s(z)|_h^p| &\leq C_r^p d_g(w,z) \Big(\int_{B_g(z,R)} |s(\zeta)|^p dv_g\Big) \\
&\leq \delta C_R^p \|s\|_p^p.
\end{aligned}$$

Hence if δ is small enough

$$\forall w \in B_g(z, \delta) \ : \ |s(w)|_h^p \geq a^p - \delta C_R^p \geq \frac{a^p}{2^p}.$$

□

3.3. **One-Point Interpolation with Uniform L^p Estimates.**

Proposition 3.3. *Let (M, g) be a Kähler Cartan-Hadamard manifold with bounded geometry. Let $(L, h) \longrightarrow (M, g)$ be a Hermitian holomorphic line bundle with bounded curvature such that*

$$c(L) + Ricci(g) \geq a\omega_g,$$

for some positive constant a. Let $p \in [1, +\infty]$. If $p \neq 2$ or $p \neq \infty$, suppose further that

$$\sup_{z\in M} \int_M e^{-\beta d_g(w,z)} dv_g < \infty,$$

for all $\beta > 0$. Then there exists $C > 0$ such that for each $z \in M$ and $\lambda \in L_z$ there exists $s \in \mathcal{F}^p(M, L)$ such that

$$s(z) = \lambda \text{ and } \|s\|_p \leq C|\lambda|_{L_z}.$$

Proof. Let $z \in M$ and fix a smooth function χ with compact support on $B_g(z, Rc^{-1}/2)$ (R and c are constants in Definition 2.3) such that
(i) $0 \leq \chi \leq 1$,
(ii) $\chi\, |_{B_g(z,Rc^{-1}/4)}= 1$,
(iii) $|\bar{\partial}\chi|_g \preceq 1$.
Let s_0 be a holomorphic section of L around $B_g(z, \eta)$ such that $s_0(z) \neq 0$. Since (L, h) is g-regular, for all $w \in B_g(z, \eta)$

$$\Phi(w) \simeq \Phi(z) + \Re(F(w)).$$

Let $\lambda \in L_z = \{z\} \times \mathbb{C}$. Consider the local section

$$s_z(w) = \lambda(w)e^{\Phi(z)+\Re(F(w)}s_0(w)$$

and the $(0, 1)$-form with values on L

$$v(w) = \bar{\partial}(\chi.s_z)(w) = \bar{\partial}\gamma(w).s_z(w)$$

Let $\Phi_z \in C^\infty(M)$ as in Lemma 2.8 and choose $\epsilon > 0$ small enough such that

$$c(L) + Ricci(g) - \epsilon\partial\bar{\partial}\Phi_z \geq g \text{ on } M.$$

By (iii) in Definition 2.5 of bounded geometry

$$c^{-2n}dv_e \leq \Psi_z^* dv_g \leq c^{2n}dv_e \text{ on } \Psi_z^{-1}(B_g(z, \eta)).$$

Hence

$$\text{Vol}_g(B(z, \eta)) \asymp 1 \text{ uniformly in } z \in M.$$

Since M is Cartan-Hadamard $d_g^2(., z)$ is smooth. By comparison theorem for the Hessian [10] the function $w \in M \to \phi_z(w) := \log d_g^2(z, z)$ is plurisousharmonic on M.

$$\begin{aligned}
\int_M |v|^2 e^{\epsilon\Phi_z} e^{-2n\phi_z} dv_g &\preceq \int_{B_g(z,\eta/2)\setminus B_g(z,\eta/4)} \frac{|v|^2 e^{2\epsilon\Phi_z}}{d_g(., z)^{2n}} dv_g \\
&\preceq |\lambda|^2 \text{Vol}_g(B_g(z, \eta/2) \setminus B_g(z, \eta/4)) \\
&\preceq |\lambda|^2 e^{-\Phi(z)} = |\lambda|^2_{L_z}
\end{aligned}$$

Since a Kähler Cartan-Hadamard manifold is Stein [32], by Lemma 2.8, there exists u such that $\bar\partial u = v$ and

$$\int_M \frac{|u(w)|^2 e^{2\epsilon\Phi_z(w)}}{d_g(w,z)^{2n}} dv_g \preceq |\lambda|^2_{L_z}.$$

Since $w \longrightarrow d_g^{-2n}(w,z)$ is not summable near z, we have $u(z) = 0$. Let

$$s(w) = \chi(w)s_z(w) - u(w).$$

Then $s(z) = \lambda$ and $\bar\partial s = 0$. Since $(2n)!e^t \geq t^{2n}$ if $t \geq 0$ and $\Phi_z \asymp d_g(.,z)$,

$$\int_M |u|^2 dv_g \preceq \int_M \frac{|v|^2 e^{2\epsilon\Phi_z}}{d_g(.,z)^{2n}} dv_g \preceq |\lambda|^2_{L_z}.$$

Thus

$$\int_M |s|^2 dv_g \leq 2\int_M |\chi s_z|^2 dv_g + 2\int_M |u|^2 dv_g \leq C|\lambda|^2_{L_z}.$$

Also

$$\begin{aligned}\int_M |s(w)|^2 e^{\epsilon\Psi_z(w)} dv_g(w) \;&\leq\; \int_M |\chi(w)|^2 e^{\epsilon\Phi_z(w)} |e(w)|^2 dv_g \\ &\quad + \int_M \frac{|u(w)|^2 e^{\epsilon\Phi_z(w)}}{d_g(w,z)^{2n}} d_g^{2n}(w,z) e^{-\epsilon\Phi_z(w)} dv_g(w).\end{aligned}$$

Since $\Phi(w) \simeq \Phi(z) + \Re(F(w))$ and $\Psi_z(w) \asymp d_g(w,z) \asymp 1$ uniformly on the support of γ and $d_g^{2n}(w,z)e^{-\epsilon\Phi_z(w)} \asymp 1$ uniformly in $z \in M$, there exists $C > 0$ independent of z such that

$$\int_M |\chi(w)|^2 e^{\epsilon\Phi_z(w)} |s_z(w)|^2 dv_g \leq C|\lambda|^2_{L_{z]}}.$$

and

$$\int_M \frac{|u(w)|^2 e^{\epsilon\Phi_z(w)}}{d_g(w,z)^{2n}} d_g^{2n}(w,z) e^{-\epsilon\Phi_z(w)} dv_g(w) \leq C|\lambda|^2_{L_{z]}}.$$

Hence

$$\int_M |s(w)|^2 e^{\epsilon\Phi_z(w)} dv_g(w) \leq C|\lambda|^2_{L_z}.$$

Since $\|\partial\bar\partial\Phi_z\|_\infty$ is uniformly bounded in $z \in M$, the line bundle $(L, he^{\epsilon\Phi_z})$ has bounded curvature. By (3.1) of Proposition 3.1

$$\begin{aligned}|s(w)|^2 \;&\preceq\; |s(w)|^2 e^{\epsilon\Phi_z} \\ &\preceq\; \int_{B_g(w,\eta)} |s(\zeta)|^2 e^{\epsilon\Phi_z(\zeta)} dv_g(\zeta) \\ &\preceq\; \int_M |s(\zeta)|^2 e^{\epsilon\Phi_z(\zeta)} dv_g(\zeta) \\ &\leq\; C|\lambda|^2_{L_z}.\end{aligned}$$

Hence $\|s\|_\infty \leq C|\lambda|_{L_z}$. Also

$$\begin{aligned} |s(w)|^2 e^{\epsilon\Phi_z(w)} &\leq C_R \int_{B_g(w,R)} |s(\zeta)|^2 e^{\epsilon\Phi_z(\zeta)} dv_g(\zeta) \\ &\leq C_R \int_M |s(\zeta)|^2 e^{\epsilon\Phi_z(\zeta)} dv_g(\zeta) \\ &\leq C_R |\lambda|^2_{L_z}. \end{aligned}$$

Thus

$$\begin{aligned} \int_M |s|^p dv_g &= \int_M \left(|s|^2 e^{\epsilon\Phi_z}\right)^{\frac{p}{2}} e^{-\frac{p}{2}\epsilon\Phi_z} dv_g \\ &\leq C|\lambda|^p \int_M e^{-\frac{p}{2}\epsilon\Phi_z} dv_g \\ &\leq C|\lambda|^p_{L_z} \int_M e^{-\frac{pC_1}{2}\epsilon d_g(w.z)} dv_g(w) \\ &\leq C^p |\lambda|^p_{L_z}. \end{aligned}$$

Finally, there exists $C > 0$ independent of $z \in M$ and $p \in [1, +\infty]$ such that

$$\|s\|_p \leq C|\lambda|_{L_z}.$$

□

3.4. **Diagonal Bounds for the Bergman Kernel.** As a consequence of (3.1) Proposition 3.1 and Proposition 3.3, we obtain the following proposition.

Proposition 3.4. *Let (M, g) be a Kähler manifold with bounded geometry and lower Ricci curvature bound. Let $(L, h) \longrightarrow (M, g)$ be a Hermitian holomorphic line bundle with bounded curvature. There is a constant $C > 0$ such that for all $z \in M$: $|K(z, z)| \preceq C$. Therefore $|K(z, w)| \leq C$, for all $z, w \in M$.*

Proof. Let (s_j) be a orthonormal basis of $\mathcal{F}^2(M, L)$. By definition of the Bergman Kernel

$$K(z, w) = \sum_j s_j(z) \otimes \overline{s_j(w)}$$

By (3.1) Proposition 3.1 the evaluation

$$\begin{aligned} ev_z \quad : \quad \mathcal{F}^2(M, L) &\longrightarrow L_z \\ s &\longrightarrow s(z) \end{aligned}$$

is continuous and

$$|K(z, z)| \preceq 1$$

uniformly in $z \in M$. Therefore

$$\begin{aligned} |K(z, w)| &\leq \sum_j |s_j(z)||s_j(w)| \\ &\leq \sqrt{|K(z, z)|}\sqrt{|K(w, w)|} \preceq 1. \end{aligned}$$

□

The following result gives bounds for the Bergman kernel in a small but uniform neighborhood of the diagonal

Proposition 3.5. *Let (M, g) be a Kähler manifold with bounded geometry and lower Ricci curvature Bound. Let $(L, h) \longrightarrow (M, g)$ be a Hermitian holomorphic line bundle with bounded curvature. There are constants $\delta, C_1, C_2 > 0$ such that for all $z \in M$ and $w \in B_g(z, \delta)$*

$$C_1|K(z, z)| \leq |K(z, w)| \leq C_2|K(z, z)|.$$

Proof. Let $z \in M$. Fix a frame e in a neighborhood U of the point z and consider an orthonormal basis $(s_j)_{j=1}^d$ of $\mathcal{F}^2(X, L)$ (where $1 \leq d \leq \infty$). In U each s_i is represented by a holomorphic function f_i such that $s_i(x) = f_i(x)e(x)$. Let

$$s_z(w) := |e(z)| \sum_{i=1}^{d} \overline{f_i(z)} s_i(w).$$

Then

$$\begin{aligned} |s_z(w)| &= \Big| \Big(\sum_{i=1}^{d} \overline{f_i(z)} s_i(w) \Big) \otimes \overline{e(z)} \Big| \\ &= \Big| \sum_{i=1}^{d} s_i(w) \otimes \overline{s_i(z)} \Big| \\ &= |K(w, z)| \end{aligned}$$

and

$$\begin{aligned} \int_M |s_z|^2 dv_g(w) &= \int_M |K(w, z)|^2 dv_g(w) \\ &= |K(z, z)| \preceq 1 \end{aligned}$$

Hence, by Lemma 3.2, there exists $C, \delta > 0$ independent of z such that

$$|K(w, z)| = |s_z(w)| \geq C|s_z(z)| = C|K(z, z)|,$$

for all $w \in B_g(z, \delta)$. □

3.5. **Off-Diagonal Decay of the Bergman Kernel.** A key tool we use is the following off-diagonal upper bound exponential decay for the Bergman kernel of L.

Theorem 3.6. *Let (M, g) be a Stein Kähler manifold with bounded geometry. Let $(L, h) \longrightarrow (M, g)$ be a Hermitian holomorphic line bundle with bounded curvature such that*

$$c(L) + Ricci(g) \geq a\omega_g,$$

for some positive constant a. There are constants α, $C > 0$ such that for all $z, w \in M$,

$$|K(z, w)| \leq Ce^{-\alpha d_g(z,w)}.$$

Proof. Let $z, w \in M$ such that $d_g(z,w) \geq \delta$ where $\delta > 0$ as in Proposition 3.4. Fix a smooth function $\chi \in C_0^\infty(B_g(w,\delta/2))$ such that
(i) $0 \leq \chi \leq 1$,
(ii) $\chi = 1$ in $B_g(w,\delta/4)$,
(iii) $|\bar\partial\chi|_g \preceq \chi$.
Let $s_z \in \mathcal{F}^2(M,L)$ defined by

$$s_z(w) := |e(z)| \sum_{i=1}^{d} \overline{f_i(z)} s_i(w),$$

where $(s_i)_{1\leq i\leq d}$ is an orthonormal basis of $\mathcal{F}^2(M,L)$ and e is a a local vframe of L arround z. Then $|s_z(w)| = |K(w,z)|$ and $\|s_z\|_2 = |K(z,z)| \preceq 1$. Also

$$s_z(w) \otimes \overline{\frac{e(z)}{|e(z)|}} = K(w,z).$$

By (3.1) Proposition 3.1

$$|s_z(w)|^2 \preceq \int_{B(w,\delta/2)} \chi(\zeta)|s_z(\zeta)|^2 dv_g \preceq \|s_z\|^2_{L^2(\chi dv_g)}$$

We have $\|s_z\|_{L^2(\chi dV_g)} = \sup_\sigma | < \sigma, s_z >_{L^2(\chi dv_g)} |$ where $\sigma \in \mathcal{F}^2(B_g(z,\delta),L)$ such that $\|\sigma\|_{L^2(\chi dv_g)} = 1$. we have

$$\begin{aligned}
\Big| < \sigma, s_z >_{L^2(\chi dv_g)} \Big|_{\mathbb{C}} &= \Big| \int_M < \chi(w)\sigma(w), s_z(w) > dv_g(w) \Big|_{\mathbb{C}} \\
&= \Big| \sum_{i=1}^{d} \int_M < \chi(w)\sigma(w), |e(z)|\overline{f_i(z)} s_i(w) > dv_g(w) \Big|_{\mathbb{C}} \\
&= \Big| \sum_{i=1}^{d} \int_M < \chi(w)\sigma(w), s_i(w) > f_i(z)|e(z)| dv_g(w) \Big|_{\mathbb{C}} \\
&= \Big| \sum_{i=1}^{d} \int_M < \chi(w)\sigma(w), s_i(w) > f_i(z)e(z) dv_g(w) \Big|_{L_z} \\
&= \Big| \sum_{i=1}^{d} \int_M < \chi(w)\sigma(w), s_i(w) > s_i(z) dv_g(w) \Big|_{L_z} \\
&= \Big| \int_M K(z,w) \bullet \chi(w)\sigma(w) dv_g(w) \Big|_{L_z} \\
&= |P(\chi\sigma)(z)|_{L_z}.
\end{aligned}$$

Since $c(L) + Ricci(g) \geq ag$, by Theorem 2.11, there exists a solution u of $\bar\partial u = \bar\partial\chi.\sigma$ such that

$$\int_M |u|^2 dv_g \preceq \int_M |\bar\partial\chi|_g^2 |\sigma|^2 dv_g < \infty.$$

Let $u_\sigma = \chi\sigma - P(\chi\sigma)$ be the solution having minimal L^2-norm of

$$\bar\partial u = \bar\partial\chi.\sigma$$

Since $\chi(z) = 0$

$$\Big| < \sigma, s_z >_{L^2(\chi dv_g)} \Big|_{\mathbb{C}} = |P(\chi\sigma)(z)|_{L_z} = |u_\sigma(z)|_{L_z}.$$

Since $B(z,\delta/2) \cap B(w,\delta/2) = \emptyset$, the section u_σ is holomorphic in $B_g(z,\delta/2)$. Let $\epsilon \in]0, 2/\delta]$, By (3.1) Proposition 3.1

$$|u_\sigma(z)|^2_{L_z} \preceq \int_{B_g(z,\delta/2)} |u_\sigma(\zeta)|^2_{L_\zeta} dv_g \preceq \int_{B_g(z,\delta/2)} e^{-\epsilon d(\zeta,z)} |u_\sigma(\zeta)|^2_{L_\zeta} dv_g. \tag{3.5}$$

Let $\eta := -\epsilon\Phi_z$ where Φ_z is as in Lemma 2.8 and $\Theta = \epsilon\omega_g$. Choose ϵ small enough such that

$$c(L) + Ricci(g) - i\epsilon\partial\bar{\partial}\Phi_z - i\epsilon^2\partial\Phi_z \wedge \bar{\partial}\Phi_z - \epsilon\omega_g \geq 0.$$

By Theorem 2.12

$$\int_M e^{-\epsilon\Phi_z} |u_\sigma|^2 dv_g \preceq \int_M e^{-\epsilon\Phi_z} |\bar{\partial}\chi|^2_g |\sigma|^2 dv_g.$$

Since $C_1 d_g(.,z) \leq \Phi_z \leq C_2(d_g(.,z)+1)$, we obtain

$$|u_\sigma(z)|^2_{L_z} \preceq \int_M e^{-\epsilon C_1 d_g(\zeta,z)} \chi(\zeta) |\sigma(\zeta)|^2 dv_g.$$

Since $\zeta \in B_g(w,\delta)$, we have

$$\begin{aligned} d_g(\zeta,z) &\geq d_g(z,w) - d_g(w,\zeta) \\ &\succeq d_g(z,w) - \delta \succeq d_g(z,w). \end{aligned}$$

Finally

$$|K(z,w)| \preceq \sup_\sigma |u_\sigma(z)|_{L_z} \preceq e^{-\alpha d_g(z,w)}.$$

$\square$

3.6. **Boundedness of the Bergman Projection on $\mathcal{F}^p(M,L)$.** The following proposition is a consequence of Theorem 3.6.

Proposition 3.7. *Let (M,g) be a Kähler Cartan-Hadamard manifold with bounded geometry such that*

$$\sup_{z\in M} \int_M e^{-\beta d_g(w,z)} dv_g < \infty,$$

for all $\beta > 0$. Let $(L,h) \longrightarrow (M,g)$ be a Hermitian holomorphic line bundle with bounded curvature such that

$$c(L) + Ricci(g) \geq a\omega_g,$$

for some positive constant a. Let $p \in [1,+\infty]$. Then the Bergman projection is bounded as a map from $L^p(M,L)$ to $\mathcal{F}^p(M,L)$.

Proof. If $p = \infty$, we have

$$\begin{aligned}
\|Ps\|_\infty &= \Big\| \int_M K(z,w).s(w)dv_g(w)\Big\|_\infty \\
&\le \|s\|_\infty \sup_{z\in M}\int_M |K(z,w)|dv_g(w) \\
&\preceq \|s\|_\infty \sup_{z\in M}\int_M e^{-\alpha d_g(z,w)}dv_g(w) \\
&\preceq \|s\|_\infty.
\end{aligned}$$

P is bounded from $L^\infty(M,L)$ to $\mathcal{F}^\infty(M,L)$. If $p\in[1,\infty[$,

$$\begin{aligned}
\int_M |Ps(z)|^p dv_g(w) &= \int_M \Big|\int_M K(z,w).s(w)dv_g(w)\Big|^p dv_g(z) \\
&\le \int_M \Big|\int_M |s(w)||K(z,w)|dv_g(w)\Big|^p dv_g(z) \\
&\le \int_M \Big(\Big(\int_M |K(z,w)|dv_g(w)\Big)^{p-1} \\
&\quad \times \int_M |s(w)|^p|K(z,w)|dv_g(w)\Big)dv_g(z)(\text{ Jensen inequality}) \\
&\preceq \int_M \Big(\int_M e^{-\alpha d_g(w,z)}dv_g(w)\Big)^{p-1} \\
&\quad \times \int_M |s(w)|^p|K(z,w)|dv_g(w)\Big)dv_g(z).
\end{aligned}$$

Thus

$$\begin{aligned}
\int_M |Ps(z)|^p dv_g(w) &\preceq \int_M\int_M |s(w)|^p e^{-\alpha d_g(w,z)}dv_g(w)dv_g(z) \\
&\preceq \int_M |s(w)|^p dv_g(w)
\end{aligned}$$

and then P is bounded from $L^p(M,L)$ to $\mathcal{F}^p(M,L)$. □

4. Boundedness and Compactness for Toeplitz Operators

Let (M,g) be a Kähler manifold. Consider the following conditions:
(1) (M,g) is a Cartan-Hadamard manifold.
(2) (M,g) has bounded geometry.
(3) $(L,h)\longrightarrow(M,g)$ is a Hermitian holomorphic line bundle with bounded curvature such that

$$c(L)+Ricci(g)\ge a\omega_g,$$

for some positive constant a.
(4) For all $\beta>0$

$$\sup_{z\in M}\int_M e^{-\beta d_g(w,z)}dv_g(w)<\infty.$$

Remark 4.1. Let (M, g) have bounded geometry and $Ricci(g) \geq Kg$. Since

$$\int_M e^{-\beta d_g(w,z)} dv_g(w) \asymp \int_0^\infty e^{-\beta r} vol(B_g(z,r)) dr,$$

if the volume of (M, g) grows uniformly subexponentially, then it satisfies the condition (4). In particular, this is true if the volume of (M, g) grows uniformly polynomially.

4.1. Carleson Measures for $\mathcal{F}^p(M, L)$.

Definition 4.2. A positive measure μ on M is Carleson for $\mathcal{F}^p(M, L)$, $1 \leq p < \infty$, if the exists $C_{\mu,p} > 0$ such that

$$\forall\ s \in \mathcal{F}^p(M, L)\ :\ \int_M |s|^p d\mu \leq C_{\mu,p} \int_M |s|^p dv_g.$$

If $p = \infty$, the measure μ on M is Carleson for $\mathcal{F}^\infty(M, L)$ if there exist $C, r > 0$ such that $\mu(B(z, r)) \leq C$.

The following is a geometric characterization of Carleson measures established earlier for classical Bargmann-Fock space by Ortega Cerda [22] and for generalized Bargmann-Fock space by Schuster-Varolin [23].

Theorem 4.3. *Let (M, g) be a Kähler manifold which satisfies (1),(2) and (3). Let μ be a positive measure on M. Let $p \in [1, \infty[$. If $p \neq 2$ or $p \neq \infty$, suppose further*

$$\sup_{z \in M} \int_M e^{-\beta d_g(w,z)} dv_g(w) < \infty,$$

for all $\beta > 0$. The following are equivalent:
(a) The measure μ is Carleson, for $\mathcal{F}^p(M, L)$.
(b) There exists $r_0 > 0$ such that $\mu(B_g(z, r)) \leq C_{r_0}$, for any $z \in M$.
(c) For each $r > 0$ there exists $C_r > 0$ such that $\mu(B_g(z, r)) \leq C_r$ for any $z \in M$.

Proof. $(c) \implies (b)$ is trivial. For $(b) \implies (c)$, fix $r > r_0$ and an r_0-lattice (a_k) in M. There exists an integer N such that each point $z \in M$ has a neighborhood which intersects at most N of the $B_k(a_k, r_0)$'s. Hence

$$\mu(B(z, r)) \leq \sum_{k=1}^{N} \mu(B_g(a_k, r_0)) \leq N C_{r_0}$$

$(b) \implies (a)$. Let $s \in \mathcal{F}^p(M, L)$. By (3.1) of Proposition 3.1

$$\begin{aligned} \int_{B_g(a_k, r_0/2)} |s|^p d\mu &\leq \mu(B(a_k, r_0/2)) \sup_{w \in B_g(a_k, r_0/2)} |s(w)|^p \\ &\preceq \sup_{w \in B_g(a_k, r_0/2)} |s(w)|^2 \\ &\preceq \int_{B_g(a_k, r_0)} |s(w)|^p dv_g. \end{aligned}$$

Hence

$$\begin{aligned}\int_M |s|^p d\mu &\preceq \sum_{k=1}^{\infty} \int_{B_g(a_k,r_0)} |s|^p d\mu \\ &\preceq \sum_{k=1}^{\infty} \int_{B_g(a_k,r_0)} |s|^2 dv_g \\ &\preceq \int_M |s|^p dv_g\end{aligned}$$

$(a) \Longrightarrow (b)$. Let $z \in M$. By Proposition 3.3, there is a section $s_z \in \mathcal{F}^p(M,L)$ such that

$$|s_z(z)| = 1 \quad \text{and} \quad \int_M |s_z|^p dv_g \leq C,$$

for some $C > 0$ independent of z. Also, by Lemma 3.2, there exists $0 < \delta < R$ such that

$$\forall\ w \in B_g(z,\delta)\ :\ |s_z(w)| \geq \frac{1}{2}.$$

Hence

$$\begin{aligned}\mu(B_g(z,\delta)) &\preceq \int_{B_g(z,\delta)} |s_z|^p d\mu \\ &\preceq \int_M |s_z|^p d\mu \\ &\preceq \int_M |s_z|^p dv_g \\ &\preceq 1. \qquad \text{(by Carleson condition)}\end{aligned}$$

□

4.2. **Vanishing Carleson Measures for $\mathcal{F}^2(M,L)$.** Recall that a bounded linear operator $T : \mathcal{F}^2(M,L) \longrightarrow L^2(M,L,d\mu)$ is a compact operator if for all sequence $(s_j) \subset \mathcal{F}^2(M,L)$ converging weakly to zero section i.e.,

$$\forall\ \sigma \in \mathcal{F}^2(M,L)\ :\ \lim_{j\to\infty} \int_M < \sigma(w), s_j(w) > dv_g = 0.$$

We have

$$\lim_{j\to\infty} \int_M |Ts_j|^p d\mu = 0.$$

The following lemma is a consequence of Proposition 3.1, Montel's Theorem and Alaouglu's Theorem.

Lemma 4.4. *Let (s_j) be a sequence in $\mathcal{F}^2(M,L)$. The following are equivalent.*
(a) (s_j) converges weakly zero.
(b) There exists $C > 0$ such that

$$\sup_j \int_M |s_j|^2 dv_g \leq C$$

and for all compact $F \subset M$

$$\lim_{j\to\infty} \sup_{z\in F} |s_j(z) - s(z)| = 0.$$

Definition 4.5. A positive measure μ on M is a vanishing Carleson if the inclusion $\imath_\mu : \mathcal{F}^2(M, L) \longrightarrow L^2(M, L, \mu)$ is a compact operator.

Theorem 4.6. *Let (M, g) be a Kähler manifold which satisfies (1),(2) and (3). Let μ be a positive measure on M. Then the following are equivalent.*
(a) The measure μ is a vanishing Carleson for $\mathcal{F}^2(M, L)$.
(b) For every $\epsilon > 0$, there exists $r > 0$ such that $\mu(B_g(z, R)) \leq \epsilon$ for any $z \in M \setminus B_g(z_0, r)$, where $z_0 \in M$ fixed.

Proof. $(b) \Longrightarrow (a)$. Let $s \in \mathcal{F}^2(M, L)$. By Proposition (2.1)???

$$|s(z)| \preceq \int_M \mathbf{1}_{B_g(z,1)} |s|^2 dv_g.$$

Hence

$$\begin{aligned} \int_M |s(z)|^2 d\mu &\preceq \int_M \int_M \mathbf{1}_{B_g(z,1)} |s(w)|^2 dv_g(w) d\mu(z) \\ &= \int_M |s(w)|^2 \mu(B(z, 1)) dv_g(w). \end{aligned}$$

Let $(s_j) \subset \mathcal{F}^2(M; L)$ be a sequence converging weakly to zero. By Lemma 3.4 (s_j) is bounded by C on $\mathcal{F}^2(M, L)$ and converges to zero locally uniformly in M. Let $\epsilon > 0$ and $r > 0$ such that $\mu(B_g(z, 1)) < \epsilon$ for $z \in M \setminus B_g(z_0, r)$. For j large enough

$$\begin{aligned} \int_M |s_j|^2 d\mu &\preceq \int_{B_g(z_0,r)} |s_j(z)|^2 \mu(B_g(z, 1)) dv_g(z) \\ &\quad + \epsilon \int_{M\setminus B_g(z_0,r)} |s_j(z)|^2 \mu(B_g(z, 1)) dv_g(z) \\ &\preceq \int_{B_g(z_0,r)} |s_j(z)|^2 \mu(B_g(z, 1)) dv_g(z) + C\epsilon \\ &\preceq 2C\epsilon. \end{aligned}$$

Thus μ is a vanishing Carleson measure.
$(a) \Longrightarrow (b)$. Let $(z_j) \subset M$ such that $d_g(z_j, z_0) \longrightarrow \infty$. For each j, let $s_j \in \mathcal{F}^2(M, L)$ such that

$$|s_j(w)| = |K(w, z_j)| \text{ and } \|s_j\|_2 \asymp 1.$$

Then $s_j \longrightarrow 0$ locally uniformly in M. Since μ is vanishing Carleson

$$\lim_{j\to\infty} \int_M |s_j|^2 d\mu = 0.$$

By Proposition 3.5, there exist positive constants C_1, C_2 and δ such that

$$|K(z, w)| \geq C_1 |K(z, z)| \geq C_2,$$

for all $w \in B_g(z,\delta)$. Then

$$\begin{aligned}\int_M |s_j|^2 d\mu &\geq \int_{B_g(z_j,\delta)} |s_{z_j}|^2 d\mu \\ &= \int_{B_g(z_j,\delta)} |K(z,z_j)|^2 d\mu \\ &\succeq \mu(B_g(z_j,\delta))|K(z_j,z_j)|^2 \\ &\succeq \mu(B_g(z_j,\delta))\end{aligned}$$

since $|K(z_j,z_j)| \asymp 1$ uniformly in j. Hence

$$\lim_{j\to\infty} \mu(B_g(z_j,\delta)) = 0.$$

Since $B_g(z_j,1)$ is covered by N balls $B_g(a_{k_1},\delta),\cdots,B_g(a_{k_N},\delta)$ (δ-lattice), it follows that

$$\lim_{j\to\infty} \mu(B_g(z_j,1)) = 0.$$

$\square$

4.3. **Berezin Transforms of Carleson Measures.** Let μ be a positive meaure on M. The Berezin transform of μ is the function $\tilde{\mu}: M \to \mathbb{R}^+$ defined by

$$\tilde{\mu}(z) := \int_M |k_z(w)|^2 d\mu(w),$$

where

$$k_z(w) := \frac{K(w,z)}{\sqrt{|K(z,z)|}}.$$

Theorem 4.7. *Let (M,g) be a Kähler manifold satisfying the conditions (1),(2) and (3). Let μ be a positive measure on M. Let $p \in [1,\infty]$. If $p \neq 2$ or $p \neq \infty$, suppose further*

$$\sup_{z\in M} \int_M e^{-\beta d_g(w,z)} dv_g(w) < \infty,$$

for all $\beta > 0$. The following are equivalent.
(a) μ is Carleson for $\mathcal{F}^p(M,L)$.
(b) $\tilde{\mu}$ is bounded on M.

Proof. $(a) \Longrightarrow (b)$. For $z \in M$, let $s_z \in \mathcal{F}^2(M,L)$ such that $|s_z(w)| = |K(z,w)|$. By off-diagonal estimate $|s_z(w)| \leq Ce^{-\alpha d_g(z,w)} \preceq 1$. Let (a_i) be a lattice of M.

Since μ is Carleson by Theorem 4.1 $\mu(B_g(a_j,r)) \le C$. We have

$$\begin{aligned}
\tilde{\mu}(z) &= \frac{1}{\sqrt{|K(z,z)|}} \int_M |s_z|^2 d\mu(w) \\
&\le \sum_j \int_{B_g(a_j,r)} |s_z|^2 d\mu(w) \ (\text{ since } |K(z,z)| \asymp 1) \\
&\le \sum_j \Big(\int_{B_g(a_j,r)} |s_z|^p d\mu(w)\Big)^{\frac{1}{p}} \Big(\int_{B_g(a_j,r)} |s_z|^q d\mu(w)\Big)^{\frac{1}{q}} \\
&\le \sum_j \Big(\int_{B_g(a_j,r)} |s_z|^p d\mu(w)\Big)^{\frac{1}{p}} \mu(B_g(a_j,r))^{\frac{1}{q}} \sup_{B_g(a_j,r)} |s_z(w)| \\
&\preceq \Big(\int_M |s_z|^p d\mu(w)\Big)^{\frac{1}{p}} \\
&\preceq \Big(\int_M |s_z|^p dv_g(w)\Big)^{\frac{1}{p}} \quad (\mu \text{ is Carleson for } \mathcal{F}^p(M,L)) \\
&\preceq \Big(\int_M |s_z|^p dv_g(w)\Big)^{\frac{1}{p}} \\
&\preceq 1.
\end{aligned}$$

Hence if μ is a Carleson, then $\tilde{\mu}$ is uniformly bounded.
$(b) \Longrightarrow (a)$. Suppose that $\tilde{\mu}$ is bounded on M. Then there exists $C > 0$ such that for all $\delta > 0$ and $z \in M$

$$\int_{B_g(z,\delta)} |k_z(w)|^2 d\mu(w) \le \tilde{\mu}(z) \le C.$$

By diagonal estimates for the Bergman Kernel, there exists $C_1, \delta > 0$ independent of z such that for all $w \in B_g(z,\delta)$

$$|K(z,w)| \ge C_1 |K(z,z)|.$$

Since $|K(z,z)| \asymp 1$

$$|k_z(w)|^2 \succeq 1, \ \forall \ w \in B_g(z,\delta).$$

Hence

$$\mu(B_g(z,\delta)) \preceq 1 \quad \text{uniformly for } z \in M$$

and by Theorem 4.3 μ is Carleson for $\mathcal{F}^p(M,L)$. □

4.4. **Berezin Transforms of Vanishing Carleson Measures.**

Theorem 4.8. *Let (M,g) be a Kähler manifold satisfying the conditions (1),(2) and (3). Let μ be a positive measure on M. The following are equivalent.*
(a) μ is vanishing Carleson for $\mathcal{F}^2(M,L)$.
(b) $\lim_{d_g(z,z_0)\to\infty} \tilde{\mu}(z) = 0$.

Proof. $(a) \Longrightarrow (b)$. Let $(z_n) \in M$ such that $\lim_{n\to\infty} d_g(z_n, z_0) = \infty$. For $n \in \mathbb{N}$ let $s_n \in \mathcal{F}^2(M, L)$ such that $|s_n(w)| = |K(w, z_n)|$. Put

$$\tilde{s}_n(w) = \frac{s_n(w)}{\sqrt{|K(z_n, z_n)|}}.$$

Then $\tilde{s}_n \in \mathcal{F}^2(M, L)$. Since $|K(z_n, z_n)| \asymp 1$ uniformly in n and

$$|\tilde{s}_n(w)| \leq Ce^{-\alpha d_g(w,z_n)},$$

then $\lim\limits_{n\to\infty} \tilde{s}_n(w) = 0$ and

$$\int_M |\tilde{s}_n|^2 dv_g(w) = 1.$$

So $\tilde{s}_n \to 0$ uniformly on compacts of M. By Lemma 4.4 $\tilde{s}_n \to 0$ weakly on $\mathcal{F}^2(M, L)$. Since μ is vanishing Carleson

$$\lim_{n\to\infty} \tilde{\mu}(z_n) = \lim_{n\to\infty} \int_M |\tilde{s}_n(w)|^2 d\mu(w) = 0.$$

$(b) \Longrightarrow (a)$. Following the proof of $(b) \Longrightarrow (a)$ in Theorem 4.3 we have

$$\mu(B(z, r)) \preceq \tilde{\mu}(z).$$

Hence

$$\lim_{d_g(z,z_0)\to\infty} \mu(B(z, r)) \preceq \lim_{d_g(z,z_0)\to\infty} \tilde{\mu}(z) = 0.$$

By Theorem 4.6 μ is vanishing Carleson. $\square$

4.5. **Proof of Theorem 1.2.** $(b) \Longleftrightarrow (c)$ follows from Theorem 4.7.
$(b) \Longleftrightarrow (d)$ follows from Theorem 4.3.
$(b) \Longleftrightarrow (a)$. Suppose that μ is a Carleson measure. Fix $p \in]1, \infty[$. Let $s \in \mathcal{F}^p(M, L)$. Then

$$\int_M \Big| \int_M < s(w), K(w, z > d\mu(w) \Big|^p dv_g(z)$$

$$\begin{aligned}
&\leq \int_M \Big(\int_M |s||K(w, z)| d\mu(w) \Big)^p dv_g(z) \\
&\leq \int_M \Big(\int_M |s||K(w, z)|^{\frac{1}{p}} |K(w, z)|^{\frac{1}{q}} d\mu(w) \Big)^p dv_g(z) \\
&\leq \int_M \Big(\int_M |s(w)|^p |K(w, z)| d\mu(w) \Big) \Big(\int_M |K(z, w)| d\mu(w) \Big)^{p-1} dv_g(z).
\end{aligned}$$

Let $s_z \in \mathcal{F}^2(M,L)$ such that $|s_z(w)| = |K(w,z)|$. Then

$$\begin{aligned}
\int_M |K(w,z)| d\mu(w) &= \int_M |s_z(w)| d\mu(w) \\
&\preceq \int_M |s_z(w)| dv_g(w) \text{ (μ is Carleson for $\mathcal{F}^1(M,L)$)} \\
&= \int_M |K(w,z)| dv_g(w) \\
&\leq C \int_M e^{-\alpha d_g(w,z)} dv_g(w) \preceq 1
\end{aligned}$$

and

$$\int_M \Big(\int_M |s(w)|^2 |K(w,z)| d\mu(w) \Big) dv_g(z)$$

$$\begin{aligned}
&\leq \int_M |s(w)|^p \Big(\int_M |K(w,z)| dv_g(z) \Big) d\mu(w) \\
&\preceq \int_M |s|^p d\mu(w) \text{ (by off-diagonal estimate)} \\
&\preceq \int_M |s|^p dv_g \text{ (μ is Carleson for $\mathcal{F}^p(M,L)$).}
\end{aligned}$$

Hence

$$\int_M |T_\mu s(w)|^p dv_g(z) \leq C_\mu \int_M |s|^p dv_g.$$

If $f \in \mathcal{F}^1(M,L)$, then

$$\begin{aligned}
&\int_M \Big| \int_M < s(w), K(w,z > d\mu(w) \Big| dv_g(z) \\
&\leq \int_M \Big(\int_M |s||K(w,z)| d\mu(w) \Big) dv_g(z) \\
&\leq \int_M \Big(\int_M |s||K(w,z)| d\mu(w) \Big) dv_g(z) \\
&\leq \int_M |s(w)| \Big(\int_M |K(z,w)| dv_g(z) \Big) d\mu(w) \\
&\leq \int_M |s(w)| \Big(\int_M e^{-\alpha d_g(z,w)} dv_g(z) \Big) d\mu(w) \\
&\leq \int_M |s(w)| d\mu(w) \\
&\leq \int_M |s(w)| dv_g(w) \text{ (μ is Carleson for $\mathcal{F}^1(M,L)$).}
\end{aligned}$$

Hence

$$\int_M |T_\mu s(w)| dv_g(z) \leq C_\mu \int_M |s| dv_g.$$

If $f \in \mathcal{F}^\infty(M,L)$, then

$$\begin{aligned}
\sup_{z\in M}\left|\int_M < s(w), K(w,z > d\mu\right| &\leq \|s\|_\infty \sup_{z\in M}\int_M |K(z,w)|d\mu(w)\\
&= \|s\|_\infty \sup_{z\in M}\int_M |s_z(w)|d\mu(w)\\
&\preceq \|s\|_\infty \sup_{z\in M}\int_M |s_z(w)|dv_g(w)\\
&\preceq \|s\|_\infty \sup_{z\in M}\int_M |K(z,w)|dv_g(w)\\
&\preceq \|s\|_\infty \sup_{z\in M}\int_M e^{-\alpha d_g(z,w)}dv_g(w)\\
&\preceq \|s\|_\infty
\end{aligned}$$

Hence

$$\sup_{z\in M}|T_\mu s(z)| \leq C_\mu \sup_{z\in M}|s(z)|.$$

We conclude that $T_\mu : \mathcal{F}^p(M,L) \to \mathcal{F}^p(M,L)$ is well defined and bounded if μ is Carleson.

Conversely, suppose $T_\mu : \mathcal{F}^p(M,L) \to \mathcal{F}^p(M,L)$ is bounded. Let $s_z \in \mathcal{F}^2(M,L)$ such that $|s_z(w)| = |K(w,z)|$. By reproducing property of the Bergman kernel

$$s_z(w) = \int_M < s_z(t), K(t,w) > dv_g(t).$$

By diagonal bounds for the Bergman kernel, there exists $C, \delta > 0$ such that $|s_z(w)| \geq C$ for all $w \in B_g(z,\delta)$. We have

$$\begin{aligned}
\mu(B_g(z,\delta)) &\preceq \int_{B_g(z,\delta)} |s_z(w)|^2 d\mu(w)\\
&\preceq \int_M |s_z(w)|^2 d\mu(w)\\
&= \int_M < s_z(w), \int_M < s_z(t), K(t,w) > dv_g(t) > d\mu(w)\\
&= \int_M \Big(\int_M < s_z(w), < s_z(t), K(t,w) >> d\mu(w)\Big) dv_g(t)\\
&= \int_M \Big(\int_M < s_z(t), < s_z(w), K(w,t) >> d\mu(w)\Big) dv_g(t)\\
&= \int_M < s_z(t), \int_M < s_z(w), K(w,t) > d\mu(w) > dv_g(t)\\
&= \int_M < s_z(t), T_\mu s_z(t) > dv_g(t)\\
&\leq \|T_\mu s_z\|_p \|s_z\|_q \leq \|T_\mu\| \|s_z\|_p \|s_z\|_q \leq C.
\end{aligned}$$

Therefore by Theorem 4.3 μ is Carleson for $\mathcal{F}^p(M,L)$.

4.6. **Proof of Theorem 1.3.** $(b) \Longleftrightarrow (c)$ follows from Theorem 4.8.
$(b) \Longleftrightarrow (d)$ follows from Theorem 4.6.
$(b) \Longleftrightarrow (a)$. Suppose that μ is vanishing Carleson. Let $s \in \mathcal{F}^2(M, L)$. Let s_z the holomorphic section such that $|s_z(w)| = |K(w, z)|$. Then

$$\int_M |T_\mu(z)|^2 dv_g(z) = \int_M \Big| \int_M < s(w), K(w,z) > d\mu(w) \Big|^2 dv_g(z)$$

$$\begin{aligned}
&\leq \int_M \Big(\int_M |s(w)|^2 |K(w,z)| d\mu(w) \Big) \Big(\int_M |K(w,z)| d\mu(w) \Big) dv_g(z) \\
&= \int_M \Big(\int_M |s(w)|^2 |K(w,z)| d\mu(w) \Big) \Big(\int_M |s_z(w)| d\mu(w) \Big) dv_g(z) \\
&\preceq \int_M \Big(\int_M |s(w)|^2 |K(w,z)| d\mu(w) \Big) \Big(\int_M |s_z(w)| dv_g(w) \Big) dv_g(z) \\
&\preceq \int_M \int_M |s(w)|^2 |K(w,z)| d\mu(w) dv_g(z) \Big(\sup_{z \in M} \int_M |s_z(w)| dv_g(w) \Big) \\
&\preceq \int_M \int_M |s(w)|^2 |K(z,w)| dv_g(z) d\mu(w) \\
&\preceq \int_M |s(w)|^2 d\mu(w).
\end{aligned}$$

Hence $\|T_\mu\| \leq C \|\imath_\mu\|$ and this follows that T_μ is compact.
Conversely, suppose that $T_\mu : \mathcal{F}^2(M, L) \to \mathcal{F}^2(M, L)$ is compact. Let $(z_j) \in M$ such that $d_g(z_j, z_0) \to 0$ and $s_{z_j} \in \mathcal{F}^2(M, L)$ such that $|s_{z_j}(w)| = |K(w, z_n)|$. By off-diagonal estimate, the sequence (s_{z_j}) is bounded on $\mathcal{F}^2(M, L)$ and converges locally uniformly to zero section. Hence (s_{z_j}) converges weakly to the zero. Since T_μ is compact and

$$\Big| \int_M < T_\mu s_{z_j}, s_{z_j} > dv_g \Big| \leq \|T_\mu s_{z_j}\|_2 \|s_{z_j}\|_2,$$

we have

$$\lim_{j \to \infty} \int_M < T_\mu s_{z_j}, s_{z_j} > dv_g = 0.$$

From

$$\Big| \int_M < T_\mu s_{z_j}, s_{z_j} > dv_g \Big| = \int_M |s_{z_j}|^2 dv_g$$

and the diagonal estimates $|s_{z_j}(w)| \succeq 1$ on $B_g(z_j, \delta)$, we get

$$\lim_{j \to \infty} \mu(B_g(z_j, \delta)) \preceq \lim_{j \to \infty} \Big| \int_M < T_\mu s_{z_j}, s_{z_j} > dv_g \Big| = 0.$$

By Theorem 4.8 μ is vanishing Carleson for $\mathcal{F}^2(M, L)$.

5. Schatten Class Membership of Toeplitz Operators

Suppose that T is a compact operator between Hilbert spaces H_1 and H_2. Then T has a Schmidt decomposition, so that there are orthonormal bases (e_n)

and (σ_n) of H_1 and H_2, respectively, and a sequence (λ_n) with $\lambda_n > 0$ and $\lambda_n \to 0$ such that for all $f \in H_1$

$$Tf = \sum_{n=0}^{\infty} \lambda_n < f, e_n > \sigma_n.$$

For $0 < p \leq \infty$, such a compact operator T belongs to the Schatten-von Neumann p-class $\mathcal{S}_p = \mathcal{S}_p(H_1, H_2)$ if and only if

$$\|T\|_{\mathcal{S}_p}^p := \sum_{n=0}^{\infty} \lambda_n^p < \infty.$$

If $p \geq 1$, then $\mathcal{S}_p$ is a Banach space. If $0 < p < 1$, then $\mathcal{S}_p$ is a Frechet space. For all $T, S \in \mathcal{S}_p(H_1, H_1)$,

$$\|T + S\|_{\mathcal{S}_p}^p \leq 2(\|T\|_{\mathcal{S}_p}^p + \|S\|_{\mathcal{S}_p}^p). \tag{5.1}$$

By Proposition 6.3.3 in [33], if T is a positive operator on a Hilbert space H and $0 < p < 1$, then

$$< T^p e_m, e_m > \leq < Te_m, e_m >^p,$$

where (e_m) is an orthonormal set of H. It gives that

$$\|T\|_{\mathcal{S}_p}^p \leq \sum_{m,k}^{\infty} | < Te_m, e_k > |^p.$$

We will introduce the complex interpolation of Schatten p-class.

Lemma 5.1. *If $1 \leq p \leq \infty$, then*

$$[\mathcal{S}_{p_0}, \mathcal{S}_{p_1}]_\theta = \mathcal{S}_p$$

with equal norm for all $1 \leq p_0 < p_1 \leq \infty$ and all $\theta \in]0, 1[$, where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

We will let (a_j) denote an r-lattice of M and $\tilde{\mu}$ the Berezin transform of the positive measure on M. For $z \in M$ let $s_z \in \mathcal{F}^2(M, L)$ such that

$$s_z(w) \otimes \overline{\frac{e(z)}{|e(z)|}} = K(w, z),$$

where e is a frame of L around z.

Lemma 5.2. *If T is a positive operator on $\mathcal{F}^2(M, L)$, then*

$$tr(T) \asymp \int_M \tilde{T}(z) dv_g(z),$$

where

$$\tilde{T}(z) = \int_M < Ts_z(w), s_z(w) > dv_g(w)$$

is the Berezin transform of T. In particular, T is trace-class if and only if the integral above converges.

Proof. Since T is positive, then $T = R^2$ for some $R \geq 0$. Let (e_j) is an orthonormal basis of $\mathcal{F}^2(M, L)$. Then

$$\begin{aligned}
\operatorname{tr}(T) &= \sum_{j=1}^{\infty} < Te_j, e_j > \asymp \sum_{j=1}^{\infty} \|Re_j\|^2 \\
&= \sum_{j=1}^{\infty} \int_M |Re_j(z)|^2 dv_g(z) \\
&= \int_M \sum_{j=1}^{\infty} |Re_j(z)|^2 dv_g(z).
\end{aligned}$$

Hence

$$\begin{aligned}
\operatorname{tr}(T) &= \int_M \sum_{j=1}^{\infty} \Big| \int_M < Re_j(w), K(w,z) > dv_g(w) \Big|^2 dv_g(z) \\
&= \int_M \sum_{j=1}^{\infty} \Big| \int_M < Re_j(w), s_z(w) \otimes \overline{\frac{e(z)}{|e(z)|}} > dv_g(w) \Big|^2 dv_g(z) \\
&= \int_M \sum_{j=1}^{\infty} \Big| \int_M < Re_j(w), s_z(w) > \overline{\frac{e(z)}{|e(z)|}} dv_g(w) \Big|^2 dv_g(z) \\
&= \int_M \sum_{j=1}^{\infty} \Big| \int_M < Re_j(w), s_z(w) > dv_g(w) \Big|^2 dv_g(z) \\
&= \int_M \sum_{j=1}^{\infty} \Big| \int_M < e_j(w), Rs_z(w) > dv_g(w) \Big|^2 dv_g(z) \\
&\asymp \int_M \|Rs_z\|^2 dv_g(z) \asymp \int_M < Ts_z, s_z > dv_g(z) = \int_M \tilde{T}(z) dv_g(z).
\end{aligned}$$

□

Corollary 5.3. *Let ν be a positive measure on M. Then $T_\nu \in \mathcal{S}_1$ if and only if $\mu(M) < \infty$. In particular, if the support of μ is compact, then $T_\mu \in \mathcal{S}_p$ for each $p \geq 1$.*

Proof. Suppose that $\mu(M) < \infty$. By Lemma 5.2

$$\begin{aligned}
\operatorname{tr}(T_\mu) &= \int_M \tilde{T_\mu}(z) dv_g(z) \\
&\asymp \int_M \int_M < T_\mu s_z(w), s_z(w) > dv_g(w) dv_g(z) \\
&\asymp \int_M \int_M |s_z(w)|^2 d\mu(w) dv_g(z) \\
&\asymp \int_M \int_M |K(w,z)|^2 |dv_g(z) \Big) d\mu(w) \\
&\asymp \int_M |K(w,w)| d\mu(w) \asymp \mu(M).
\end{aligned}$$

Let $T_\mu \in \mathcal{S}_1$ and $z_0 \in M$ fixed. By diagonal bound estimates we have

$$\begin{aligned} \operatorname{tr}(T) &\asymp \int_M \tilde{T}(z)dv_g(z) \asymp \int_M \Big(\int_M |K(w,z)|^2 dv_g(z)\Big) d\mu(w) \\ &\succeq \int_M \Big(\int_{B_g(z_0,\delta)} |K(w,z)|^2 |dv_g(w)\Big) d\mu(z) \asymp vol_g(B_g(z_0,\delta)\mu(M) \\ &\succeq \mu(M). \end{aligned}$$

□

We will need the following simple lemma that is well known in the classical Fock space setting [33].

Lemma 5.4. *Let $r > 0$ and let (e_j) be any orthonormal basis for $\mathcal{F}^2(M,L)$. If (a_j) is an r-lattice of M and H is the operator on $\mathcal{F}^2(M,L)$ defined by $He_j := s_{a_j}$, then H can be extended to a bounded operator on all of $\mathcal{F}^2(M,L)$ whose operator norm is bounded above by a constant that only depends on r.*

Proof. Let $\sigma, t \in \mathcal{F}^2(M,L)$, then

$$< H\sigma, t >= \sum_{j=1}^{\infty} < \sigma, e_j >< s_{a_j}, t > .$$

Since

$$s_{a_j}(w) \otimes \frac{e(a_j)}{|e(a_j)|} = K(w, a_j),$$

where e is a frame of L around a_j. Since

$$t(a_j) = \int \int_M < t(w), K(w, a_j) > dv_g(w)$$

by Cauchy-Schwarz inequality and Proposition 3.1

$$\begin{aligned} | < A\sigma, t > | &\leq \sum_{j=1}^{\infty} | < \sigma, e_j >_{L^2} || < s_{a_j}, t >_{L^2} | \\ &= \sum_{j=1}^{\infty} | < \sigma, e_j >_{L^2} | \Big| < s_{a_j}, t >_{L^2} \frac{e(a_j)}{|e(a_j)|} \Big|_{L_{a_j}} \\ &\leq \|\sigma\|_2 \Big(\sum_{j=1}^{\infty} \Big| \int_M < t(w), K(w, a_j) > dv_g(w) \Big|^2_{L_{a_j}} \Big)^{\frac{1}{2}} \\ &\leq \|\sigma\|_2 \Big(\sum_{j=1}^{\infty} |t(a_j)|^2 \Big)^{\frac{1}{2}} \\ &\preceq \|\sigma\|_2 \Big(\sum_{j=1}^{\infty} \int_{B_g(a_j,r)} |t|^2 dv_g \Big)^{\frac{1}{2}} \\ &\preceq \|\sigma\|_2 \|t\|_2. \end{aligned}$$

□

Lemma 5.5. *Let $p \geq 1$. If $\phi \in L^p(M, dv_g)$ and T_ϕ be the Toeplitz operator with symbol ϕ*

$$T_\phi s(z) = \int_M < s(w), K(w,z) > \phi(w) dv_g(w)$$

for all $s \in \mathcal{F}^2(M, L)$, then $T_\phi \in \mathcal{S}_p$.

Proof. Assume $p = 1$. Let $g \in L^1(M, dv_g)$ and (e_j) be an orthonormal set on $\mathcal{F}^2(M, L)$. By Fubini Theorem

$$< T_\phi e_j(z), e_j(z) >= \int_M |e_j(z)|^2 \phi(z) dv_g(z).$$

Hence

$$\begin{aligned}
\sum_{j=1}^{\infty} | < T_\phi e_j, e_j > | &= \sum_{j=1}^{\infty} \Big| \int_M |e_j(w)|^2 \phi(w) dv_g(w) \\
&\leq \int_M \sum_{j=1}^{\infty} |e_j(w)||\phi(w)| dv_g(w) \\
&= \int_m |\phi(w)||K(w,w)| d_g(w) \\
&\preceq \|\phi\|_1 \quad \text{(by diagonal estimate)}.
\end{aligned}$$

Thus, for $p = 1$, $T_\phi \in \mathcal{S}_1$ and $\|T_\mu\|_{\mathcal{S}_1} \preceq \|\phi\|_1$. Also $\|T_\mu\|_{\mathcal{S}_\infty} \preceq \|\phi\|_\infty$. By interpolation of Lemma 5.1, we can get $T_\phi \in \mathcal{S}_p$ and $\|T_\phi\|_{\mathcal{S}_p} \leq \|\phi\|_p$. □

Lemma 5.6. *Suppose that (M, g) satisfies the conditions (1), (2), (3) and (4) of Section 4. Let $r > 0$ and $0 < p < 1$. The following are equivalent:*
(a) $\tilde{\mu} \in L^p(M, dv_g)$.
(b) $\mu(B_g(., r) \in L^p(M, dv_g)$.
(c) $\mu(B_g(a_j, r)) \in \ell^p(\mathbb{N})$.

Proof. $(c) \Longrightarrow (a)$. We have

$$\begin{aligned}
\tilde{\mu}(z) &= \int_M |k_z(w)|^2 d\mu(w) \\
&= \sum_{j=1}^{\infty} \int_{B_g(a_j,r)} \frac{|K(w,z)|^2}{|K(z,z)|} d\mu(w) \\
&\leq C \sum_{j=1}^{\infty} \int_{B_g(a_j,r)} e^{-2\alpha d_g(w,z)} d\mu(w).
\end{aligned}$$

Since $d_g(w,) \geq d_g(z,a_j) - d_g(a_j,w)$ for all $w \in B_g(a_j,r)$

$$\begin{aligned}
\tilde{\mu}(z) &\leq \sum_{j=1}^{\infty} \int_{B_g(a_j,r)} e^{-2\alpha d_g(w,z)} d\mu(w) \\
&\leq C \sum_{j=1}^{\infty} \int_{B_g(a_j,r)} e^{-2\alpha(d_g(z,a_j)-r)} d\mu(w) \\
&\preceq \sum_{j=1}^{\infty} e^{-2\alpha d_g(z,a_j)} \mu(B_j(a_j,r)).
\end{aligned}$$

By Hölder inequality $\tilde{\mu}(z)^p \preceq \sum_{j=1}^{\infty} e^{-2p\alpha d_g(z,a_j)} \mu(B_j(a_j,r))^p$. Hence

$$\begin{aligned}
\int_M \tilde{\mu}(z)^p &\preceq \sum_{j=1}^{\infty} \int_M e^{-2p\alpha d_g(z,a_j)} \mu(B_j(a_j,r))^p \\
&\preceq \sum_{j=1}^{\infty} \mu(B_j(a_j,r))^p \sup_{j\in\mathbb{N}} \int_M e^{-2p\alpha d_g(z,a_j)} \\
&\preceq \sum_{j=1}^{\infty} \mu(B_j(a_j,r))^p < \infty.
\end{aligned}$$

$(a) \implies (b)$. By diagonal bound estimate $|K(z,z)| \asymp 1$ and $|K(z,w)| \succeq |K(z,z)|$ for all $w \in B_g(z,\delta)$

$$\begin{aligned}
\tilde{\mu}(z) &= \int_M |k_z(w)|^2 d\mu(w) \\
&\geq \int_{B_g(z,r)} |k_z(w)|^2 d\mu(w) \\
&\succeq \int_{B_g(z,r)} |K(w,z)|^2 d\mu(w) \\
&\succeq \sum_{j=1}^{\infty} \int_{B_g(z,r)\cap B_g(a_j,\delta)} |K(w,z)|^2 d\mu(w) \\
&\succeq \sum_{j=1}^{\infty} \int_{B_g(z,r)\cap B_g(a_j,\delta)} d\mu(w). \succeq \mu(B(z,r))
\end{aligned}$$

$(b) \implies (c)$. We have

$$\sum_{j=1}^{\infty} \int_{B_g(a_j,\frac{r}{2})} \mu(B(z,r))^p dv_g(z) \preceq \int_M \mu(B_g(z,r))^p dv_g(z).$$

Since for any $z \in B_g(a_j,\frac{r}{2})$: $\mu(B_g(z,r)) \geq \mu(B_g(a_j,\frac{r}{2})$, then

$$\sum_{j=1}^{\infty} \mu(B_g(a_j,\frac{r}{2}))^p \preceq \int_M \mu(B_g(z,r))^p dv_g(z).$$

Thus $\mu(B_g(.,r)) \in L^p(M,dv_g)$ implies that $(\mu(B_g(a_j,r)) \in \ell^p(\mathbb{N})$. $\square$

5.1. **Proof of Theorem 1.4 for the case** $1 \leq p < \infty$**.** $(a) \Longrightarrow (b)$. Since T_μ is a positive operator, then $T_\mu \in \mathcal{S}_p$ if and only if $T_\mu^p \in \mathcal{S}_1$. By Proposition 6.3.3 in [33]

$$\begin{aligned}
\tilde{T}_\mu^p(z) &= \int_M < T_\mu^p s_z(w), s_z(w) > dv_g(w) \\
&\geq \Big(\int_M < T_\mu s_z(w), s_z(w) >\Big)^p \\
&= (\tilde{\mu}(z))^p.
\end{aligned}$$

Hence by Lemma 5.2

$$\int_M (\tilde{\mu}(z))^p dv_g(z) \leq \int_M |\tilde{T^p}_\mu(z)| \leq \mathrm{tr}(T_\mu^p) < \infty.$$

Then $\phi \in L^p(M,dv_g)$.
$(b) \Longrightarrow (c)$. Put

$$\phi_r(z) = \mu(B_g(z,r)).$$

By diagonal estimates for the Bergman kernel, for some $\epsilon > 0$ we have

$$\begin{aligned}
\mu(B_g(z,\epsilon)) &\preceq \int_{B_g(z,\epsilon)} |K(z,w)|^2 d\mu(w) \\
&\preceq \frac{1}{|K(z,z)|} \int_{B_g(z,\epsilon)} |K(z,w)|^2 d\mu(w) \\
&\preceq \tilde{\mu}(z).
\end{aligned}$$

Hence $z \to \phi_\epsilon(z) := \mu(B_g(z,\epsilon)) \in L^p(M,dv_g)$.
$(c) \Longrightarrow (a)$. Suppose that $T_\phi \in \mathcal{S}_p$. For $z_0 \in M$ fixed, write $\mu = \mu_1 + \mu_2$, where

$$\mu_1 := \mu \mid_{B_g(z_0,\epsilon)} \quad \text{and} \quad \mu_2 := \mu \mid_{M\setminus B_g(z_0,\epsilon)}$$

By Corollary 5.5 $T_{\mu_1} \in \mathcal{S}_p$. Hence it suffices to show that $T_{\mu_2} \in \mathcal{S}_p$. If $\sigma \in \mathcal{F}^2(M,L)$ we have

$$\begin{aligned}
< T_{\phi_\epsilon}\sigma, \sigma > &= \int_M |\sigma(w)| \phi_\epsilon(w) dv_g(w) \\
&= \int_M |\sigma(w)|^2 \mu(B_g(w,\epsilon)) dv_g(w) \\
&\geq \int_{z\in M} \int_{B_g(z,\epsilon)} |\sigma(w)|^2 dv_g(w) d\mu(z) \\
&\succeq \int_{M\setminus B_g(z_0,\epsilon)} |\sigma(z)|^2 d\mu(z) \quad \text{(Prop. 3.1)} \\
&\succeq < T_{\mu_2}\sigma, \sigma > .
\end{aligned}$$

Hence $T_{\mu_2} \preceq T_{\phi_\epsilon}$ so that $\|T_{\mu_2}\|_p \preceq \|T_{\phi_\epsilon}\|_p$ and then $T_{\mu_2} \in \mathcal{S}_p$.

5.2. **Proof of Theorem 1.4 for the case $0 < p < 1$.** By Lemma 5.6, it suffices to prove $(a) \Longrightarrow (d)$ and $(b) \Longrightarrow (a)$.
$(a) \Longrightarrow (d)$. Suppose that $T_\mu \in \mathcal{S}_p$. By near diagonal uniform estimate for the Bergman kernel there exists $\delta > 0$ such that

$$\forall\ z \in M,\ \forall\ w \in B_g(z,\delta)\ :\ |K(w,z)| \succeq 1. \tag{5.2}$$

Let $r \geq 2\delta$ and (a_j) be an r-lattice. Let $(a_{k_j}) \subset (a_j)$ such that $d_g(a_{k_j}, a_{k_l}) > r$ if $j \neq l$ so that

$$d_g(w, a_{k_j}) \leq r/2 \Longrightarrow d_g(w, a_{k_l}) \geq r/2 \tag{5.3}$$

and

$$d_g(w, a_{k_j}) \leq r/2 \Longrightarrow d_g(w, a_{k_l}) \geq \frac{1}{2} d_g(a_{k_j}, a_{k_l}). \tag{5.4}$$

Let ν be the positive measure

$$\nu := \sum_j \mathbf{1}_{B_g(a_j,\delta)} \mu.$$

Then $T_\nu \leq T_\mu$ so that $\|T_\nu\|_p \leq \|T_\mu\|_p$. Let (e_t) be an orthonormal basis of $\mathcal{F}^2(M,L)$ and $H : \mathcal{F}^2(M,L) \to \mathcal{F}^2(M,L)$ the operator defined by

$$He_m = s_{a_{k_m}},$$

where $s_{a_{k_m}} \in \mathcal{F}^2(M,L)$ defined as

$$s_{a_{k_m}}(w) \otimes \frac{e(a_{k_m})}{|e(a_{k_m})|} = K(w, a_{k_m}),$$

where e is a frame of L around a_{k_m}. By off-diagonal estimate for the Bergman kernel

$$\forall\ w \in M\ :\ |s_{a_{k_m}}(w)| \preceq e^{-\alpha d_g(w,a_{k_m})}.$$

By Lemma 5.4, H can be extended to a bounded operator on all of $\mathcal{F}^2(M,L)$ whose operator norm is bounded above by a constant that only depends of (a_{k_m}). If $R = H^* T_\nu H$ then

$$\|R\|_p \leq \|T_\nu\|_p \leq \|T_\mu\|_p.$$

Consider the operators Δ and E defined by

$$\Delta s := \sum_m < He_m, e_m >< s, e_m > e_m \quad \text{and} \quad E = R - \Delta.$$

By (5.1) we have

$$\frac{1}{2}\|\Delta\|_p^p - \|E\|_p^p \leq \|H\|_p^p \leq \|T_\mu\|_p^p. \tag{5.5}$$

We estimate $\|\Delta\|_p$ from below,

$$\begin{aligned}
\|\Delta\|_p^p &= \sum_m < De_m, e_m >^p \\
&= \sum_m < T_\nu a_{k_m}, a_{k_m} >^p \\
&= \sum_m \Big(\int_M |s_{a_{k_m}}(w)|^2 d\nu(w)\Big)^p \\
&= \sum_m \Big(\int_M |K(w, a_{k_m})|^2 d\nu(w)\Big)^p \\
&\geq \sum_m \Big(\int_{B_g(a_{k_m},\delta)} |K(w, a_{k_m})|^2 d\nu(w)\Big)^p \\
&\succeq \sum_m (\mu(B_g(a_{k_m}, \delta))^p.
\end{aligned}$$

Thus

$$\|\Delta\|_p^p \succeq \sum_m (\mu(B_g(a_{k_m}, \delta))^p. \tag{5.6}$$

We estimate $\|E\|_p$ from above,

$$\begin{aligned}
\|E\|_p^p &\leq \sum_{l\neq m} < Re_m, e_k >^p \\
&= \sum_{l\neq m} < T_\nu e_m, e_k >^p \\
&\leq \sum_{l\neq m} < T_\nu s_{a_{k_m}}, s_{a_{k_l}} >^p \\
&\leq \sum_{l\neq m} \Big(\int_M |s_{a_{k_m}}(w)||s_{a_{k_l}}(w)| d\nu(w)\Big)^p \\
&\leq \sum_{l\neq m} \Big(\int_M e^{-\alpha d_g(w,a_{k_m})} e^{-\alpha d_g(w,a_{k_l})} d\nu(w)\Big)^p \\
&\preceq e^{\frac{-\alpha pr}{2}} \sum_{m\neq l} \Big(\int_M e^{-\frac{\alpha}{2} d_g(w,a_{k_m})} e^{-\frac{\alpha}{2} d_g(w,a_{k_l})} d\nu(w)\Big)^p \ (5.3) \\
&\preceq e^{\frac{-\alpha pr}{2}} \sum_{m\neq l} \Big(\sum_j \int_{B_g(a_{k_j},\delta)} e^{-\frac{\alpha}{2} d_g(w,a_{k_m})} e^{-\frac{\alpha}{2} d_g(w,a_{k_l})} d\nu(w)\Big)^p \\
&\preceq e^{\frac{-\alpha pr}{2}} \sum_{m\neq l} \Big(\sum_j \mu(B_g(a_{k_j}, \delta)) e^{-\frac{\alpha}{4} d_g(a_{k_m},a_{k_j})} e^{-\frac{\alpha}{4} d_g(a_{k_l},a_{k_j})} d\nu(w)\Big)^p. \ (5.4)
\end{aligned}$$

Since $0 < p < 1$

$$\begin{aligned}\|E\|_p^p &\preceq e^{\frac{-\alpha pr}{2}} \sum_j \mu(B_g(a_{k_j}, \delta))^p \sum_{m \neq k} e^{-\frac{\alpha}{4} d_g(a_{k_m}, a_{k_j})} e^{-\frac{\alpha}{4} d_g(a_{k_l}, a_{k_j})} \\ &\preceq e^{\frac{-\alpha pr}{2}} \sum_j \mu(B_g(a_{k_j}, \delta))^p \Big(\sum_l e^{-\frac{\alpha}{4} d_g(a_{k_l}, a_{k_j})} \Big)^2 \\ &\preceq e^{\frac{-\alpha pr}{2}} \sum_j \mu(B_g(a_{k_j}, \delta))^p.\end{aligned}$$

Thus

$$\|E\|_p^p \preceq e^{\frac{-\alpha pr}{2}} \sum_j \mu(B_g(a_{k_j}, \delta))^p. \tag{5.7}$$

By (5.5), (5.6) and (5.7), for r large enough

$$\begin{aligned}\|T_\mu\|_p^p &\geq \Big(\frac{c_1}{2} - c_2 e^{\frac{-\alpha pr}{2}} \Big) \sum_j \mu(B_g(a_{k_j}, \delta))^p \\ &\succeq \sum_j \mu(B_g(a_{k_j}, \delta))^p,\end{aligned}$$

for each sub-lattice (a_{k_j}) of the r-lattice (a_j). Thus

$$\sum_j \mu(B_g(a_j, \delta))^p \preceq \|T_\mu\|_p^p.$$

$(b) \Longrightarrow (a)$. Suppose that $\tilde{\mu} \in L^p(M, dv_g)$. By Lemma 5.6 it suffice to show

$$\mu(B_g(., \delta)) \in L^p(M dv_g) \Longrightarrow T_\mu \in \mathcal{S}_p.$$

Let $\phi_r(z) := \mu(B_g(z, \delta))$. If $s \in \mathcal{F}^2(M, L)$ we have

$$\begin{aligned}< T_{\phi_r} s, s > &= \int_M |s(z)|^2 \mu(B_g(z, \delta)) dv_g(z) \\ &= \int_M |s(z)|^2 dv_g(z) \int_M 1_{B_g(w,\delta)} d\mu(w) \\ &= \int_M d\mu(w) \int_M |s(z)|^2 1_{B_g(w,\delta)} dv_g(z) \\ &= \int_M d\mu(w) \int_{B_g(w,\delta)} |s(z)|^2 dv_g(z) \\ &\succeq < T_\mu s, s > .\end{aligned}$$

Thus $T_\mu \preceq T_{\phi_r}$. Since $T_{\phi_r} \in \mathcal{S}_p$ (Lemma 5.5) we get $T_\mu \in \mathcal{S}_p$.

5.3. **Proof of Theorem 1.5.** For the proof of Theorem 1.5, we need some preliminary lemmas.

Let (M, g) be a Kähler manifold and $(L, h) \to M$ be a holomorphic Hermitian line bundle. Let (N, ω_N) be a a Hermitian manifold. For a holomorphic map $\Phi : N \to M$, let $(\Phi^*L, \Phi^*h) \to N$ the holomorphic Hermitian line bundle, called the pull back of L, whose fibers are $(\Phi^*L)_x = L_{\Phi(x)}$ with metrics $(\Phi^*h)(x) = h(\Phi(x))$, where $x \in N$. We define the composition operator

$$\begin{aligned} C_\Phi : \mathcal{F}^2(M, L) &\longrightarrow \mathcal{F}^2(N, \Phi^*L) \\ s &\longrightarrow s \circ \Phi. \end{aligned}$$

The transform B_Φ (related to the usual Berezin transform) associated to Φ is the function on M defined as follows.

$$B_\Phi(z)^2 := \int_M |K(z, w)|^2 d\nu_\Phi(w),$$

where ν_Φ is the pull-back measure defined as follows: for all Borel set $E \subset M$

$$\nu_\Phi(E) = \int_N \mathbf{1}_{\Phi^{-1}(E)}(w) dv_{\omega_N}(w).$$

Let $z \in M$. Fix a frame e in a neighborhood U of the point z and consider an orthonormal basis $(s_j)_{j=1}^d$ of $\mathcal{F}^2(X, L)$ (where $1 \leq d \leq \infty$). In U each s_i is represented by a holomorphic function f_i such that $s_i(x) = f_i(x)e(x)$. Let

$$s_z(w) := |e(z)| \sum_{i=1}^d \overline{f_i(z)} s_i(w).$$

Then s_z is a holomorphic section and

$$\begin{aligned} |s_z(w)| &= \Big| \Big(\sum_{i=1}^d \overline{f_i(z)} s_i(w) \Big) \otimes \overline{e(z)} \Big| \\ &= \Big| \sum_{i=1}^d s_i(w) \otimes \overline{s_i(z)} \Big| \\ &= |K(w, z)|. \end{aligned}$$

By Proposition 3.3

$$\begin{aligned} \int_M |s_z|^2 dv_g(w) &= \int_M |K(w, z)|^2 dv_g(w) \\ &= |K(z, z)| \asymp 1. \end{aligned}$$

Lemma 5.7. *We have*

$$< C_\Phi^* C_\Phi s_z, s_z >= B_\Phi(z)^2,$$

$$B_\Phi(z)^2 = \int_M |s_z(w)|^2 d\nu_\Phi(w)$$

and

$$\int_M |B_\Phi(z)|^p dv_g(z) = \int_M < C_\Phi^* C_\Phi s_z, s_z >^{\frac{p}{2}} dv_g(z),$$

where ν_Φ is the pull-back measure defined as follows: for all Borel set $E \subset M$

$$\nu_\Phi(E) = \int_N \mathbf{1}_{\Phi^{-1}(E)}(w) dv_{\omega_N}(w).$$

Proof. We have

$$\begin{aligned} < C_\Phi^* C_\Phi s_z, s_z > &= < C_\Phi s_z, C_\Phi s_z > \\ &= \int_N |s_z(\Phi(w))|^2 dv_{\omega_N}(w) \\ &= \int_M |s_z(w)|^2 d\nu_\Phi(w) \\ &= \int_M |K(z,w)|^2 d\nu_\Phi(w) \\ &= \int_M |K(z,\Phi(w))|^2 dv_g(w) \\ &= B_\Phi(z)^2. \end{aligned}$$

□

The following lemma presents a desired connection between composition operators and Toeplitz operators.

Lemma 5.8. *Let (M,g) be a Kähler manifold and let $\Phi : N \to M$ be a holomorphic map such that C_Φ is bounded. Then*

$$C_\Phi^* C_\Phi = T_{\nu_\Phi},$$

where

$$T_{\nu_\Phi} s(z) = \int_M < s(w), K(w,z) > d\nu_\Phi(w).$$

Proof. Since C_Φ is bounded, for all $s, \sigma \in \mathcal{F}^2(M,L)$

$$\begin{aligned} < C_\Phi^* C_\Phi s, \sigma > &= < C_\Phi s, C_\Phi \sigma > \\ &= \int_N < s(\Phi(w)), \sigma(\Phi(w)) > dv_{\omega_N}(w) \\ &= \int_M < s(w), \sigma(w) > d\nu_\Phi(w). \end{aligned}$$

Since

$$\sigma(w) = \int_M < \sigma(z), K(z,w) > dv_g(z)$$

by Fubini Theorem

$$\begin{aligned} < C_\Phi^* C_\Phi s, \sigma > &= \int_M < s(w), \int_M K(w,z).\sigma(t) dv_g(z) > d\nu_\Phi(w) \\ &= \int_M \int_M < s(w), K(w,z).\sigma(t) > dv_g(z) d\nu_\Phi(w) \\ &= \int_M \int_M < K(z,w).s(w), \sigma(t) > dv_g(z) d\nu_\Phi(w) \\ &= \int_M < \int_M K(z,w).s(w) d\nu_\Phi(w), \sigma(z) > dv_g(z) \\ &= < \int_M K(.,w).s(w) d\nu_\Phi(w), \sigma > . \end{aligned}$$

Hence we get

$$C_{\Phi}^{*}C_{\Phi}s(z)=\int_{M}<s(w),K(w,z)>d\nu_{\Phi}(w).$$

$\square$

Corollary 5.9. *Let (M,g) be a Kähler manifold and let $\Phi: N\to M$ be a holomorphic map such that $C_{\Phi}:\mathcal{F}^2(M,L)\to\mathcal{F}^2(N,\Phi^*L)$ is bounded. If $0<p<\infty$, then $C_{\Phi}\in\mathcal{S}_p$ if and only if $T_{\nu_{\Phi}}\in\mathcal{S}_{p/2}$.*

Since $|K(z,z)|\asymp 1$ and

$$\begin{aligned}\tilde{\nu}_{\Phi}(z) &= \frac{1}{|K(z,z)|}\int_{M}|K(z,w)|^2 d\nu_{\Phi}(w)\asymp\int_{M}|K(z,w)|^2 d\nu_{\Phi}(w)\\ &\asymp B_{\Phi}(z)^2,\end{aligned}$$

then the proof of Theorem 1.5 follows from Theorems 1.2, 1.3 and 1.4.

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