Khayyam J. Math. 2 (2016), no. 2, 194–200
DOI: 10.22034/kjm.2017.43707

Khayyam Journal of Mathematics
emis.de/journals/KJM
kjm-math.org

ON THE SHARP BOUNDS FOR A COMPREHENSIVE CLASS OF ANALYTIC AND UNIVALENT FUNCTIONS BY MEANS OF CHEBYSHEV POLYNOMIALS

S. BULUT[1]* AND N. MAGESH[2]

Communicated by T. Bhattacharyya

Abstract. In this paper, we obtain initial coefficient bounds for functions belong to a comprehensive subclass of univalent functions by using the Chebyshev polynomials and also we find Fekete-Szegö inequalities for this class. All results are sharp.

1. Introduction

Let $\mathbb{R} = (-\infty, \infty)$ be the set of real numbers, $\mathbb{C}$ be the set of complex numbers and

$$\mathbb{N} := \{1, 2, 3, \ldots\} = \mathbb{N}_0 \setminus \{0\}$$

be the set of positive integers.

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk

$$\Delta = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

We also denote by $\mathcal{S}$ the class of all functions in the normalized analytic function class $\mathcal{A}$ which are univalent in Δ.

Date: Received: 21 July 2016; Revised: 23 February 2017; Accepted: 27 February 2017.
* Corresponding author.
2010 *Mathematics Subject Classification.* Primary 30C45.
Key words and phrases. Analytic functions, univalent functions, coefficient bounds, Chebyshev polynomial, Fekete-Szegö problem.

For two functions f and g, analytic in Δ, we say that the function f is subordinate to g in Δ, and write

$$f(z) \prec g(z) \qquad (z \in \Delta),$$

if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \qquad \text{and} \qquad |\omega(z)| < 1 \quad (z \in \Delta)$$

such that

$$f(z) = g(\omega(z)) \quad (z \in \Delta).$$

Indeed, it is known that

$$f(z) \prec g(z) \quad (z \in \Delta) \Rightarrow f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta).$$

Furthermore, if the function g is univalent in Δ, then we have the following equivalence

$$f(z) \prec g(z) \quad (z \in \Delta) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta).$$

The significance of Chebyshev polynomial in numerical analysis is increased in both theoretical and practical points of view. Out of four kinds of Chebyshev polynomials, many researchers deal with orthogonal polynomials of Chebyshev. For a brief history of Chebyshev polynomials of first kind $T_n(t)$, the second kind $U_n(t)$ and their applications one can refer [5, 6, 9, 1]. The Chebyshev polynomials of the first and second kinds are well known and they are defined by

$$T_n(t) = \cos n\theta \quad \text{and} \quad U_n(t) = \frac{\sin(n+1)\theta}{\sin\theta} \qquad , (-1 < t < 1)$$

where n denotes the polynomial degree and $t = \cos\theta$.

It should be mentioned in passing that the functional expression used in (1.2) of Definition 1.1 is precisely the same as that used by Zhu [11] for investigating various extensions, generalizations and improvements of the starlikeness criteria which were proven by earlier authors.

Definition 1.1. For $\lambda \geq 1$, $0 \leq \mu \leq 1$ and $t \in (1/2, 1]$, a function $f(z)$ given by (1.1) is said to be in the class $\mathcal{B}(\lambda, \mu, t)$ if the following subordination holds for all $z \in \Delta$:

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} \prec H(z,t) := \frac{1}{1-2tz+z^2}. \tag{1.2}$$

For $\mu = 1$, we get the class $\mathcal{B}(\lambda, 1, t) = \mathcal{B}(\lambda, t)$ consists of functions f satisfying the condition

$$(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) \prec H(z,t) := \frac{1}{1-2tz+z^2}.$$

For $\lambda = 1$, we have a new class $\mathcal{B}(1, \mu, t) = \mathcal{B}(\mu, t)$ consists of Bazilevic functions:

$$f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} \prec H(z,t) := \frac{1}{1-2tz+z^2}.$$

For $\lambda = 1$ and $\mu = 1$, we have the class $\mathcal{B}(t)$ consists of functions f satisfying the condition

$$f'(z) \prec H(z,t) := \frac{1}{1-2tz+z^2}.$$

We note that if $t = \cos\alpha$, where $\alpha \in (-\pi/3, \pi/3)$, then

$$H(z,t) = \frac{1}{1-2\cos\alpha z+z^2} = 1 + \sum_{n=1}^{\infty} \frac{\sin(n+1)\alpha}{\sin\alpha} z^n \quad (z \in \Delta).$$

Thus

$$H(z,t) = 1 + 2\cos\alpha z + (3\cos^2\alpha - \sin^2\alpha)z^2 + \dots \quad (z \in \Delta).$$

From [10], we can write

$$H(z,t) = 1 + U_1(t)z + U_2(t)z^2 + \dots \quad (z \in \Delta, \quad t \in (-1,1))$$

where

$$U_{n-1} = \frac{\sin(n \arccos t)}{\sqrt{1-t^2}} \quad (n \in \mathbb{N})$$

are the Chebyshev polynomials of the second kind and we have

$$U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t),$$

and

$$U_1(t) = 2t, \quad U_2(t) = 4t^2 - 1, \quad U_3(t) = 8t^3 - 4t, \dots. \tag{1.3}$$

The generating function of the first kind of Chebyshev polynomial $T_n(t)$, $t \in [-1,1]$, is given by

$$\sum_{n=0}^{\infty} T_n(t)z^n = \frac{1-tz}{1-2tz+z^2} \qquad (z \in \Delta).$$

The first kind of Chebyshev polynomial $T_n(t)$ and the second kind of Chebyshev polynomial $U_n(t)$ are connected by:

$$\frac{dT_n(t)}{dt} = nU_{n-1}(t); \quad T_n(t) = U_n(t) - tU_{n-1}(t); \quad 2T_n(t) = U_n(t) - U_{n-2}(t).$$

In this present paper, motivated by the earlier work of Dziok et al. [6], we use the Chebyshev polynomials expansions to provide sharp bounds for the initial coefficients of univalent functions in $\mathcal{B}(\lambda,\mu,t)$. We also solve Fekete-Szegö problem for functions in this class.

2. Coefficient bounds for the function class $\mathcal{B}(\lambda,\mu,t)$

Theorem 2.1. *For $\lambda \geq 1$, $\mu \geq 0$ and $t \in (1/2, 1]$, let the function $f(z)$ given by (1.1) be in the class $\mathcal{B}(\lambda,\mu,t)$. Then*

$$|a_2| \leq \frac{2t}{\lambda+\mu}, \tag{2.1}$$

$$|a_3| \leq \frac{2t}{2\lambda+\mu} \max\left\{1\ ,\ \left|\frac{4t^2-1}{2t} - \frac{(\mu-1)(2\lambda+\mu)}{(\lambda+\mu)^2}t\right|\right\} \tag{2.2}$$

and

$$\left|a_3-\eta a_2^2\right| \leq \begin{cases} \frac{2t}{2\lambda+\mu} & , \quad \eta \in [\eta_1,\eta_2] \\ \frac{2t}{2\lambda+\mu}\left|\frac{4t^2-1}{2t}-(\mu-1+2\eta)\frac{2\lambda+\mu}{(\lambda+\mu)^2}t\right| & , \quad \eta \notin [\eta_1,\eta_2] \end{cases}, \tag{2.3}$$

where

$$\eta_1 = \frac{1-\mu}{2}+\frac{(\lambda+\mu)^2}{4(2\lambda+\mu)}\frac{4t^2-2t-1}{t^2} \tag{2.4}$$

and

$$\eta_2 = \frac{1-\mu}{2}+\frac{(\lambda+\mu)^2}{4(2\lambda+\mu)}\frac{4t^2+2t-1}{t^2}. \tag{2.5}$$

All of the inequalities are sharp.

Proof. Let the function $f(z)$ is given by (1.1) be in the class $\mathcal{B}(\lambda,\mu,t)$. From (1.2), we have

$$\begin{aligned} &(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} \\ = \quad &1+U_1(t)w(z)+U_2(t)w^2(z)+\cdots \qquad (z\in\Delta) \end{aligned} \tag{2.6}$$

for some analytic function

$$w(z)=c_1z+c_2z^2+c_3z^3+\cdots \qquad (z\in\Delta), \tag{2.7}$$

such that $w(0)=0$ and $|w(z)|<1$. It is well-known that (see [7]) if $|w(z)|<1$, $z\in\Delta$, then

$$|c_j|\leq 1 \quad \text{for all} \quad j\in\mathbb{N} \tag{2.8}$$

and

$$\left|c_2-\delta c_1^2\right|\leq \max\{1,|\delta|\} \quad \text{for all} \quad \delta\in\mathbb{R}. \tag{2.9}$$

From (2.6) and (2.7), we have

$$\begin{aligned} &(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} \\ = \quad &1+U_1(t)c_1z+\left[U_1(t)c_2+U_2(t)c_1^2\right]z^2+\cdots \ (z\in\Delta). \end{aligned} \tag{2.10}$$

From (2.10), we have

$$\begin{aligned} &1+(\lambda+\mu)a_2z+(2\lambda+\mu)\left(a_3+\frac{\mu-1}{2}a_2^2\right)z^2 \\ &+(3\lambda+\mu)\left(a_4+a_2a_3(\mu-1)+\frac{(\mu-1)(\mu-2)}{6}a_2^3\right)z^3+\ldots \\ = \quad &1+U_1(t)c_1z+\left[U_1(t)c_2+U_2(t)c_1^2\right]z^2+\cdots \qquad (z\in\Delta). \end{aligned} \tag{2.11}$$

Equating the coefficients in (2.11), we get

$$(\lambda+\mu)a_2=U_1(t)c_1 \tag{2.12}$$

and

$$(2\lambda+\mu)\left(a_3+\frac{\mu-1}{2}a_2^2\right)=U_1(t)c_2+U_2(t)c_1^2. \tag{2.13}$$

The inequality (2.1) is clear. By using (2.12) and (2.13) for some $\eta \in \mathbb{R}$, we get

$$\left|a_3-\eta a_2^2\right| \leq \frac{U_1(t)}{2\lambda+\mu}\left|c_2-\left\{-\frac{U_2(t)}{U_1(t)}+\left(\frac{\mu-1}{2}+\eta\right)\frac{2\lambda+\mu}{(\lambda+\mu)^2}U_1(t)\right\}c_1^2\right|.$$

From (2.9), it follows that

$$\left|a_3-\eta a_2^2\right| \leq \frac{U_1(t)}{2\lambda+\mu}\max\left\{1,\ \left|-\frac{U_2(t)}{U_1(t)}+\left(\frac{\mu-1}{2}+\eta\right)\frac{2\lambda+\mu}{(\lambda+\mu)^2}U_1(t)\right|\right\}.$$

Next, using (1.3) in the above equation, we have

$$\left|a_3-\eta a_2^2\right| \leq \frac{2t}{2\lambda+\mu}\max\left\{1,\ \left|\frac{4t^2-1}{2t}-(\mu-1+2\eta)\frac{2\lambda+\mu}{(\lambda+\mu)^2}t\right|\right\}.$$

Since $t>0$, we get

$$\left|\frac{4t^2-1}{2t}-(\mu-1+2\eta)\frac{2\lambda+\mu}{(\lambda+\mu)^2}t\right|\leq 1$$

if and only if $\eta_1 \leq \eta \leq \eta_2$ where η_1 and η_2 are given in (2.4) and (2.5). So we obtain (2.3). If we take $\eta=0$, then we obtain the inequality (2.2).

The equality (2.6) with $w(z)=z$ generates the function $\hat{f} \in \mathcal{B}(\lambda,\mu,t)$ such that

$$\hat{f}(z)=z+\frac{2t}{\lambda+\mu}z^2+\left\{\frac{4t^2-1}{2\lambda+\mu}-\frac{2(\mu-1)}{(\lambda+\mu)^2}t^2\right\}z^3+\cdots \qquad (z\in\Delta),$$

which shows the sharpness of (2.1) and (2.2) when the maximum value is bigger than 1. Also, in this case

$$\left|a_3-\eta a_2^2\right|=\left|\frac{4t^2-1}{1+2\lambda}-\eta\frac{4t^2}{(1+\lambda)^2}\right|$$

which shows the sharpness of (2.3) for $\eta \notin [\eta_1,\eta_2]$. On the other hand, the equality (2.6) with $w(z)=z^2$ generates the function $\check{f} \in \mathcal{B}(\lambda,\mu,t)$ such that

$$\check{f}(z)=z+\frac{2t}{2\lambda+\mu}z^3+\cdots \qquad (z\in\Delta),$$

which shows the sharpness of (2.2) when the maximum value is equal to 1, and (2.3) for $\eta \in [\eta_1,\eta_2]$. This completes the proof of Theorem 2.1. □

Taking $\mu=1$ in Theorem 2.1, we get the following consequence.

Corollary 2.2. *For $\lambda \geq 1$ and $t \in (1/2,1]$, let the function $f(z)$ given by (1.1) be in the class $\mathcal{B}(\lambda,t)$. Then*

$$|a_2| \leq \frac{2t}{\lambda+1},$$

$$|a_3| \leq \begin{cases} \frac{2t}{2\lambda+1} & , \quad \frac{1}{2}<t\leq\frac{1+\sqrt{5}}{4} \\ \frac{4t^2-1}{2\lambda+1} & , \quad \frac{1+\sqrt{5}}{4}\leq t\leq 1 \end{cases}$$

and

$$\left|a_3-\eta a_2^2\right| \le \begin{cases} \frac{2t}{2\lambda+1} & , \quad \eta \in [\eta_1,\eta_2] \\ \frac{2t}{2\lambda+1}\left|\frac{4t^2-1}{2t}-2\eta\frac{2\lambda+1}{(\lambda+1)^2}t\right| & , \quad \eta \notin [\eta_1,\eta_2] \end{cases},$$

where

$$\eta_1=\frac{(\lambda+1)^2\left(4t^2-2t-1\right)}{4\left(2\lambda+1\right)t^2} \qquad \text{and} \qquad \eta_2=\frac{(\lambda+1)^2\left(4t^2+2t-1\right)}{4\left(2\lambda+1\right)t^2}.$$

All of the inequalities are sharp.

Taking $\lambda=1$ in Theorem 2.1, we get the following consequence.

Corollary 2.3. *For $\mu \ge 0$ and $t\in(1/2,1]$, let the function $f(z)$ given by (1.1) be in the class $\mathcal{B}(\mu,t)$. Then*

$$|a_2| \le \frac{2t}{\mu+1},$$

$$|a_3| \le \frac{2t}{\mu+2}\max\left\{1\ ,\ \left|\frac{4t^2-1}{2t}-\frac{(\mu-1)(\mu+2)}{(\mu+1)^2}t\right|\right\}$$

and

$$\left|a_3-\eta a_2^2\right| \le \begin{cases} \frac{2t}{\mu+2} & , \quad \eta \in [\eta_1,\eta_2] \\ \frac{2t}{\mu+2}\left|\frac{4t^2-1}{2t}-(\mu-1+2\eta)\frac{\mu+2}{(\mu+1)^2}t\right| & , \quad \eta \notin [\eta_1,\eta_2] \end{cases},$$

where

$$\eta_1=\frac{1-\mu}{2}+\frac{(\mu+1)^2}{4(\mu+2)}\frac{4t^2-2t-1}{t^2}$$

and

$$\eta_2=\frac{1-\mu}{2}+\frac{(\mu+1)^2}{4(\mu+1)}\frac{4t^2+2t-1}{t^2}.$$

All of the inequalities are sharp.

Taking $\lambda=1$ and $\mu=1$ in Theorem 2.1, we get the following consequence.

Corollary 2.4. *For $t\in(1/2,1]$, let the function $f(z)$ given by (1.1) be in the class $\mathcal{B}(t)$. Then*

$$|a_2| \le t,$$

$$|a_3| \le \begin{cases} \frac{2t}{3} & , \quad \frac{1}{2}<t\le\frac{1+\sqrt{5}}{4} \\ \frac{4t^2-1}{3} & , \quad \frac{1+\sqrt{5}}{4}\le t\le 1 \end{cases}$$

and

$$\left|a_3-\eta a_2^2\right| \le \begin{cases} \frac{2t}{3} & , \quad \eta \in [\eta_1,\eta_2] \\ \left|\frac{(4+3\eta)t^2-1}{3}\right| & , \quad \eta \notin [\eta_1,\eta_2] \end{cases},$$

where

$$\eta_1=\frac{4t^2-2t-1}{3t^2} \quad \text{and} \quad \eta_2=\frac{4t^2+2t-1}{3t^2}.$$

All of the inequalities are sharp.

References

1. Ş. Altınkaya and S. Yalçın, *On the Chebyshev polynomial bounds for classes of univalent functions*, Khayyam J. Math., **2** (2016), no. 1, 1–5.
2. M. Chen, *On the regular functions satisfying* $\Re(f(z)/z) > \alpha$, Bull. Inst. Math. Acad. Sinica, **3** (1975), 65–70.
3. P.N. Chichra, *New subclasses of the class of close-to-convex functions*, Proc. Amer. Math. Soc., **62** (1977), 37–43.
4. S.S. Ding, Y. Ling, G.J. Bao, *Some properties of a class of analytic functions*, J. Math. Anal. Appl., **195** (1995), no. 1, 71–81.
5. E.H. Doha, *The first and second kind Chebyshev coefficients of the moments of the general-order derivative of an infinitely differentiable function*, Int. J. Comput. Math., **51** (1994), 21–35.
6. J. Dziok, R. K. Raina and J. Sokół, *Application of Chebyshev polynomials to classes of analytic functions*, C. R. Math. Acad. Sci. Paris, **353** (2015), no. 5, 433–438.
7. F.R. Keogh and E.P. Merkes, *A coefficient inequality for certain classes of analytic functions*, Proc. Amer. Math. Soc., **20** (1969), 8-12.
8. T.H. MacGregor, *Functions whose derivative has a positive real part*, Trans. Amer. Math. Soc., **104** (1962), 532–537.
9. J.C. Mason, *Chebyshev polynomial approximations for the L-membrane eigenvalue problem*, SIAM J. Appl. Math., **15** (1967), 172–186.
10. T. Whittaker and G.N. Watson, *A Course of Modern Analysis*, reprint of the fourth (1927) edition, Cambridge Mathematical Library, Cambridge Univ. Press, Cambridge, 1996.
11. Y. Zhu, *Some starlikeness criterions for analytic functions*, J. Math. Anal. Appl., **335** (2007), 1452–1459.

[1] Kocaeli University, Faculty of Aviation and Space Sciences, Arslanbey Campus, 41285 Kartepe-Kocaeli, TURKEY.

E-mail address: serap.bulut@kocaeli.edu.tr

[2] P. G. and Research Department of Mathematics, Govt Arts College for Men, Krishnagiri-635001, India.

E-mail address: nmagi_2000@yahoo.co.in