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COMPOSITION OPERATORS ON WEIGHTED BERGMAN-NEVANLINNA SPACES WITH ADMISSIBLE WEIGHTS

A.K. SHARMA[1] * AND E. SUBHADARSINI[1]

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Abstract. A non-negative, non-increasing integrable function ω is an admissible weight if $\omega(r)/(1-r)^{1+\gamma}$ is non-decreasing for some $\gamma > 0$ and $\lim_{r\to 1}\omega(r) = 0$. In this paper, we characterize boundedness and compactness of composition operators on weighted Bergman-Nevanlinna spaces with admissible weights.

1. Introduction and preliminaries

Let $\mathbb{D}$ be the open unit disk in the complex plane, $H(\mathbb{D})$ the space of all holomorphic functions on $\mathbb{D}$. Let ω be a non-negative, non-increasing integrable function such that $\omega(r)(1-r)^{-(1+\gamma)}$ is nondecreasing for some $\gamma > 0$ and and $\lim_{r\to 1}\omega(r) = 0$. We extend ω on $\mathbb{D}$ by setting $\omega(z) = \omega(|z|)$, $z \in \mathbb{D}$, and call it a weight. We assume that our weights are normalized so that $\int_{\mathbb{D}} \omega(z)dA(z) = 1$, where $dA(z) = \frac{1}{\pi}dxdy = \frac{1}{\pi}rdrd\theta$, $(z = x+iy = re^{i\theta})$ stands for normalized area measure on $\mathbb{D}$. Such a weight function is called an *admissible weight*. Of course the classical weights $\omega(r) = (1-r^2)^{\alpha}$; $\alpha > -1$ are admissible weights.
The notation $a \lesssim b$ means that there is a positive constant C such that $a \leq Cb$. Moreover, if $a \lesssim b$ and $b \lesssim a$, then we write $a \asymp b$.

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* Corresponding author.

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For ω an admissible weight, the weighted *Bergman-Nevanlinna* space is the space of functions $f \in H(\mathbb{D})$ such that

$$||f||_{\mathcal{A}^0_\omega} = \int_{\mathbb{D}} \log^+ |f(z)|\omega(z)dA(z) < \infty,$$

where

$$\log^+ x = \begin{cases} \log x & \text{if } x \geq 1 \\ 0 & \text{if } \ x < 1. \end{cases}$$

Note that despite the norm notation, $||f||_{\mathcal{A}^0_\omega}$ fails to satisfy the properties of norm. However, $(f,g) \to ||f-g||_{\mathcal{A}^0_\omega}$ defines a translation invariant metric on $\mathcal{A}^0_\omega$ that turns $\mathcal{A}^0_\omega$ into a complete metric space. The space $\mathcal{A}^0_\omega$ can be viewed as the limit as $p \to 0$ of the weighted Bergman space $\mathcal{A}^p_\omega$, defined by

$$\mathcal{A}^p_\omega = \Big\{ f \in H(\mathbb{D}) : ||f||_{\mathcal{A}^p_\omega} = \Big(\int_{\mathbb{D}} |f(z)|^p \omega(z)dA(z) \Big)^{1/p} < \infty \Big\},$$

in the sense that

$$\lim_{p\to 0} \frac{t^p - 1}{p} = \log t, \qquad 0 < t < \infty.$$

The Bergman-Nevanlinna space $\mathcal{A}^0_\omega$ contains all the Bergman spaces $\mathcal{A}^p_\alpha$ for all p, $0 < p < \infty$. Obviously, the inequalities

$$\log^+ x \leq \log(1+x) \leq 1 + \log^+ x, \quad x \geq 0,$$

imply that

$$||f||_{\mathcal{A}^0_\omega} \asymp \int_{\mathbb{D}} \log(1+|f(z)|)\omega(z)dA(z) < \infty. \tag{1.1}$$

Let φ be a holomorphic self-map of $\mathbb{D}$. The composition operator C_φ induced by φ is defined by $C_\varphi f = f \circ \varphi$ for $f \in H(\mathbb{D})$. This type of operator has gained increasing attention during the last three decades, mainly due to the fact that it provide a link between classical function theory, functional analysis and operator theory. For general background on composition operators, we refer to [3, 5] and references therein. Recently, several authors have considered composition operators between different spaces of holomorphic functions, including Nevanlinna type spaces, see for example [1, 2] and [6–12].

Let X and Y be topological vector spaces whose topologies are induced by translation-invariant metrics d_X and d_Y, respectively. Then a linear operator $T : X \to Y$ is called *metrically bounded* if there exists a positive constant K such that

$$d_Y(Tf, 0) \leq K d_X(f, 0),$$

for all $f \in X$. When X and Y are Banach spaces, the notation of metric boundedness co-insides with that of boundedness. An operator $T : X \to Y$ is said to be *metrically compact* if it takes every metric ball in X into a relatively compact set in Y. In this paper, we consider metric boundedness and metric compactness of C_φ on weighted Bergman-Nevanlinna spaces $\mathcal{A}^0_\omega$. From now on metrically or metric will be dropped since there is no danger of confusion.

2. Main results

In this section, we characterize boundedness and compactness of composition operators on weighted Bergman-Nevanlinna spaces with admissible weight.
In what follows, we make use of the Carleson measure, so we first give a short introduction to Carleson sets and Carleson measures.

The *arcs* in the unit circle $\partial\mathbb{D}$ are sets of the form

$$I = \{z \in \partial\mathbb{D} : \theta_1 \leq \arg z < \theta_2\},$$

where θ_1, $\theta_2 \in [0, 2\pi)$ and $\theta_1 < \theta_2$. Normalized length of an arc I will be denoted by $|I|$, that is,

$$|I| = \frac{1}{2\pi}\int_I |dz|.$$

Let I be an arc in $\partial\mathbb{D}$ and let $S(I)$ be the Carleson sets defined by

$$S(I) = \{z \in \mathbb{D} : 1 - |I| \leq |z| < 1, z/|z| \in I\}.$$

A positive Borel measure μ on $\mathbb{D}$ is called an *ω-Carleson measure* if

$$||\mu||_\omega = \sup_{0<|I|<1} \frac{\mu(S(I))}{\omega(1 - |I|)|I|^2} < \infty$$

and a *vanishing ω-Carleson measure* if

$$\lim_{|I|\to 0} \frac{\mu(S(I))}{\omega(1 - |I|)|I|^2} = 0.$$

Recall that for a and z in $\mathbb{D}$, the pseudohyperbolic distance d between a and z is defined by

$$d(a, z) = |\sigma_a(z)| = \left|\frac{a - z}{1 - \bar{a}z}\right|.$$

For $r \in (0, 1)$ and $a \in \mathbb{D}$, denote by $D(a, r)$, the pseudohyperbolic disk whose pseudohyperbolic center is a and whose pseudohyperbolic radius is r, that is

$$D(a, r) = \Big\{z \in \mathbb{D} : d(a, z) < r\Big\}.$$

Since σ_a is a linear fractional transformation, the pseudohyperbolic disk $D(a, r)$ is also a Euclidean disk. Except for the special case when $D(a, r) = r\mathbb{D}$, the Euclidean center and Euclidean radius of $D(a, r)$ do not coincide with pseudohyperbolic center and pseudohyperbolic radius. The Euclidean center and Euclidean radius of $D(a, r)$ are

$$\frac{1 - r^2}{1 - r^2|a|^2}a \quad \text{and} \quad \frac{1 - |a|^2}{1 - r^2|a|^2}r$$

respectively. Moreover, for $0 < r < 1/3$, there exists a positive integer M and a sequence $(z_n)_{n\in\mathbb{N}} \subset \mathbb{D}$ such that $\inf_{n\neq m} |\sigma_{z_n}(z_m)| > 0$, $\cup_{n=1}^{\infty} D(z_n, r) = \mathbb{D}$ and every point in $\mathbb{D}$ belongs to at most M sets in the family $\{D(z_n, 3r)\}_{n\in\mathbb{N}}$. We denote by $A(D(a, r))$ the area of $D(a, r)$. It is well-known that

$$A(D(a, r)) \asymp |1 - \bar{a}z|^2 \asymp (1 - |a|^2)^2 \asymp (1 - |z|^2)^2 \asymp A(D(z, r)) \tag{2.1}$$

for $z \in D(a, r)$.

The next can be found in [4, Lemma 2.4].

Lemma 2.1. *Let ω be an admissible weight and let $a \in \mathbb{D}$. Then there is some $\gamma > 0$ such that*

$$\int_{\mathbb{D}} \frac{\omega(z)}{|1-\overline{a}z|^{4+2\gamma}} dA(z) \asymp \frac{\omega(a)}{(1-|a|^2)^{2+2\gamma}}.$$

Lemma 2.2. *Let $\gamma > 0$. Let ω be an admissible weight and let $a \in \mathbb{D}$. Then there is some $\gamma > 0$ such that*

$$f_a(z) = \exp\left\{\frac{(1-|a|^2)^{2+2\gamma}}{\omega(a)(1-\overline{a}z)^{4+2\gamma}}\right\} \tag{2.2}$$

is in $\mathcal{A}^0_\omega$ for every $a \in \mathbb{D}$. Moreover, $\sup_{a\in\mathbb{D}} ||f_a||_{\mathcal{A}^0_\omega} \lesssim 1$.

Proof. Let $a \in \mathbb{D}$ and f_a be as in (2.2). Then by Lemma 2.1, we have that

$$\begin{aligned} ||f_a||_{\mathcal{A}^0_\omega} &= \int_{\mathbb{D}} \log^+ |f_a(z)|\omega(z)dA(z) \\ &= \int_{\mathbb{D}} \Re\Big(\frac{(1-|a|^2)^{2+2\gamma}}{\omega(a)(1-\overline{a}z)^{4+2\gamma}}\Big)\omega(z)dA(z) \\ &\le \int_{\mathbb{D}} \frac{(1-|a|^2)^{2+2\gamma}}{\omega(a)|1-\overline{a}z|^{4+2\gamma}}\omega(z)dA(z) \\ &\lesssim 1. \end{aligned}$$

Thus we have that $f_a \in \mathcal{A}^0_\omega$ and $\sup_{a\in\mathbb{D}} ||f_a||_{\mathcal{A}^0_\omega} \lesssim 1$. □

Theorem 2.3. *Let ω be an admissible weight. Then the following statements are equivalent:*

(a) *μ is an ω-Carleson measure on $\mathbb{D}$.*
(b) *There is a constant $C(\omega, \mu) > 0$ such that*

$$\int_{\mathbb{D}} \log(1 + |f(z)|)d\mu(z) \le C(\omega, \mu)||f||_{\mathcal{A}^0_\omega}.$$

Proof. Suppose that (b) holds. Let I be and arc in $\partial\mathbb{D}$ such that $0 < |I| < 1$ and $a = (1-|I|)e^{i\theta}$. Then $a \in \mathbb{D}$ and $|a| = 1-|I|$. Consider the function f_a as in (2.2), where $a = (1-|I|)e^{i\theta}$. Then by Lemma 2.1, $\sup_{a\in\mathbb{D}} ||f_a||_{\mathcal{A}^0_\omega} \lesssim 1$. Thus by (b), we have

$$\int_{\mathbb{D}} \log(1 + |f_a(z)|)d\mu(z) \lesssim C(\omega, \mu).$$

That is,

$$C(\omega, \mu) \gtrsim \int_{\mathbb{D}} \log^+ |f_a(z)|d\mu(z) = \int_{\mathbb{D}} \Re\Big(\frac{(1-|a|^2)^{2+2\gamma}}{\omega(a)(1-\overline{a}z)^{4+2\gamma}}\Big)d\mu(z).$$

Now

$$\Re\left(\frac{(1-|a|^2)^{2+2\gamma}}{\omega(a)(1-\overline{a}z)^{4+2\gamma}}\right) = \frac{(1-|a|^2)^{2+2\gamma}}{\omega(a)(1-|a|)^{4+2\gamma}}\Re\left(\left(\frac{1-|a|}{1-\overline{a}z}\right)^{4+2\gamma}\right)$$
$$= \frac{(1-|a|^2)^{2+2\gamma}}{\omega(a)(1-|a|)^{4+2\gamma}}\Re\left(\left(1+\frac{|a|(1-ze^{-i\theta})}{1-|a|}\right)^{-(4+2\gamma)}\right)$$
$$\gtrsim \frac{(1-|a|^2)^{2+2\gamma}}{\omega(a)(1-|a|)^{4+2\gamma}}, \quad \text{if} \quad z \in S(I).$$

So we have that

$$\Re\left(\frac{(1-|a|^2)^{2+2\gamma}}{\omega(a)(1-\overline{a}z)^{4+2\gamma}}\right) \gtrsim \frac{1}{\omega(1-|I|)|I|^2}, \quad z \in S(I).$$

Therefore,

$$C(\omega,\mu) \gtrsim \int_{S(I)} \frac{1}{\omega(1-|I|)|I|^2} d\mu(z) = \frac{\mu(S(I))}{\omega(1-|I|)|I|^2}.$$

Thus μ is an ω-Carleson measure on $\mathbb{D}$.
Conversely, suppose that (a) holds, that is, μ is an ω-Carleson measure. Let $\{a_n\}$ be a sequence in $\mathbb{D}$ $\inf_{n\neq m}|\sigma_{a_n}(a_m)| > 0$, $\cup_{n=1}^{\infty} D(a_n, r) = \mathbb{D}$ and every point in $\mathbb{D}$ belongs to at most M sets in the family $\{D(a_n, 3r)\}_{n\in\mathbb{N}}$. For each $a_n \in \mathbb{D}$, and a fixed $r \in (0, 1/3)$ there is an arc I_n such that $0 < |I_n| < 1$, $D(a_n, r) \in S(I_n)$ and $|I_n| = 1 - |a_n|$. Using (2.2) and the fact that ω is an admissible weight, we get

$$\begin{aligned}\int_{\mathbb{D}} \log(1+|f(z)|)d\mu(z) &\leq \sum_{n=1}^{\infty}\int_{D(a_n,r)} \log(1+|f(z)|)d\mu(z)\\ &\leq \sum_{n=1}^{\infty}\mu(D(a_n,r)) \sup_{z\in D(a_n,r)} \log(1+|f(z)|)\\ &\leq C\sum_{n=1}^{\infty}\frac{\mu(D(a_n,r))}{w(a_n)(1-|a_n|^2)}\int_{D(a_n,3r)} \log(1+|f(z)|)\omega(z)dA(z)\\ &\leq C\sum_{n=1}^{\infty}\frac{\mu(S(I_n))}{w(1-|I_n|)|I_n|^2}\int_{D(a_n,3r)} \log(1+|f(z)|)\omega(z)dA(z)\\ &\leq C||\mu||_{\omega}\int_{\mathbb{D}} \log(1+|f(z)|)\omega(z)dA(z).\end{aligned}$$

Thus by (2.1), (b) holds. For ω an admissible weight, let

$$d\nu_{\omega}(z) = \omega(z)dA(z), \quad z \in \mathbb{D}.$$

□

Theorem 2.4. *Let ω be an admissible weight and φ be a holomorphic self-map of $\mathbb{D}$. Then $C_{\varphi} : \mathcal{A}^0_{\omega} \to \mathcal{A}^0_{\omega}$ is bounded if and only if the pull-back measure $\mu_{\omega,\varphi} = \nu_{\omega} \circ \varphi^{-1}$ of ν_{ω} induced by φ is an ω-Carleson measure.*

Proof. Let $f \in \mathcal{A}^0_\omega$. Then

$$||C_\varphi f||_{\mathcal{A}^0_\omega} = \int_{\mathbb{D}} \log(1 + |(f \circ \varphi)(z)|)\omega(z)dA(z) = \int_{\mathbb{D}} \log(1 + |(f(z)|)d\mu_{\omega,\varphi}(z).$$

Thus in view of Theorem 2.3, we have that C_φ is bounded on $\mathcal{A}^0_\omega$ if and only if $\mu_{\omega,\varphi}$ is an ω-Carleson measure. □

To prove the main result of this section, we need the following lemma which follows on similar lines as the proof of [6, Lemma 2.1]. We omit the details.

Lemma 2.5. *Let ω be an admissible weight and φ be a holomorphic self-map of $\mathbb{D}$. Then $C_\varphi : \mathcal{A}^0_\omega \to \mathcal{A}^0_\omega$ is compact if and only for every sequence $\{f_n\}$ which is bounded in $\mathcal{A}^0_\omega$ and converges to zero uniformly on compact subsets of $\mathbb{D}$, we have that $||C_\varphi f_n||_{\mathcal{A}^0_\omega} \to 0$.*

Theorem 2.6. *Let ω be an admissible weight and φ be a holomorphic self-map of $\mathbb{D}$. Then $C_\varphi : \mathcal{A}^0_\omega \to \mathcal{A}^0_\omega$ is compact if and only if the pull-back measure $\mu_{\omega,\varphi} = \nu_\omega \circ \varphi^{-1}$ of ν_ω induced by φ is a vanishing ω-Carleson measure, where $d\nu_\omega(z) = \omega(z)dA(z)$.*

Proof. First suppose that $C_\varphi : \mathcal{A}^0_\omega \to \mathcal{A}^0_\omega$ is compact. Let $\{I_n\}$ be a sequence of arc in $\partial\mathbb{D}$ such that $0 < |I_n| < 1/2$ for all n and $|I_n| \to 0$ as $n \to \infty$. Consider the family of functions

$$f_n(z) = (1 - |a_n|)^2 \omega(a_n) \exp\left\{\frac{(1 - |a_n|)^{2+2\gamma}}{\omega(a_n)(1 - \overline{a_n}z)^{4+2\gamma}}\right\},$$

where $\gamma > 0$ is as in Lemma 2.1 and $a_n = (1 - |I_n|)e^{i\theta}$. Clearly, $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$. By Lemma 2.2, there exists a positive constant C such that $\sup_n \|f_n\|_{\mathcal{A}^0_\omega} \lesssim 1$. Again as in the proof of Theorem 2.3, if $z \in S(I_n)$, then

$$\Re\left(\frac{(1 - |a_n|)^{2+2\gamma}}{\omega(a_n)(1 - \overline{a_n}z)^{4+2\gamma}}\right) \gtrsim \frac{1}{\omega(1 - |I_n|)|I_n|^2}$$

and so

$$\begin{aligned}\log^+ |f_n(z)| &\geq \log^+ \left\{(1 - |a_n|)^2 \omega(a_n) \exp\left\{\Re\left(\frac{(1 - |a_n|)^{2+2\gamma}}{\omega(a_n)(1 - \overline{a_n}z)^{4+2\gamma}}\right)\right\}\right\} \\ &\geq \log^+ \left\{\omega(1 - |I_n|)|I_n|^2 \exp\left\{\frac{C}{\omega(1 - |I_n|)|I_n|^2}\right\}\right\}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\log^+\left\{\omega(1 - |I_n|)|I_n|^2 \exp\left\{\frac{C}{\omega(1 - |I_n|)|I_n|^2}\right\}\right\}\mu_{\omega,\varphi}(S(I_n)) \\ &\leq \int_{S(I_n)} \log^+ |f_n(z)| d\mu_{\omega,\varphi}(z) \\ &\leq \int_{\mathbb{D}} \log^+ |f_n(\varphi(z))|\omega(z)dA(z) \\ &= ||C_\varphi f_n||_{\mathcal{A}^0_\omega}.\end{aligned}$$

By Lemma 2.5, the compactness of $C_\varphi : \mathcal{A}^0_\omega \to \mathcal{A}^0_\omega$ forces $||C_\varphi f_n||_{\mathcal{A}^0_\omega} \to 0$ as $n \to \infty$. Thus we have that

$$\lim_{|I_n|\to 0} \log^+ \left\{ \omega(1-|I_n|)|I_n|^2 \exp\left\{ \frac{C}{\omega(1-|I_n|)|I_n|^2} \right\} \right\} \mu_{\omega,\varphi}(S(I_n)) = 0.$$

But

$$\begin{aligned} &\lim_{|I_n|\to 0} \omega(1-|I_n|)|I_n|^2 \log^+ \left\{ \omega(1-|I_n|)|I_n|^2 \exp\left\{ \frac{C}{\omega(1-|I_n|)|I_n|^2} \right\} \right\} \\ &\qquad = \lim_{t\to\infty} \frac{1}{t} \log^+ \left\{ \frac{1}{t} \exp\{Ct\} \right\} \\ &\qquad = \lim_{t\to\infty} \frac{1}{t} \left\{ Ct - \log t \right\} \\ &\qquad = C > 0. \end{aligned}$$

Therefore, it follows that

$$\lim_{|I_n|\to 0} \frac{\mu_{\omega,\varphi}(S(I_n))}{\omega(1-|I_n|)|I_n|^2} = 0.$$

Hence $\mu_{\omega,\varphi}$ is a vanishing ω-Carleson measure on $\mathbb{D}$. □

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[1] Department of Mathematics, Shri Mata Vaishno Devi University, Kakryal, Katra-182320, J&K, India.
E-mail address: aksju_76@yahoo.com
E-mail address: elinamaths@gmail.com