Khayyam J. Math. 3 (2017), no. 1, 81–89
DOI: 10.22034/kjm.2017.46951

Khayyam Journal of Mathematics
emis.de/journals/KJM
kjm-math.org

PROXIMAL POINT ALGORITHMS FOR NUMERICAL RECKONING FIXED POINTS OF HYBRID-TYPE MULTIVALUED MAPPINGS IN HILBERT SPACES

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Communicated by M. Ito

ABSTRACT. In this paper, we propose a new iteration process to approximate minimizers of proper convex and lower semi-continuous functions and fixed points of λ-hybrid multivalued mappings in Hilbert spaces. We also provide an example to illustrate the convergence behavior of the proposed iteration process and numerically compare the convergence of the proposed iteration scheme with the existing schemes.

1. Introduction and basic definitions

Let H be a real Hilbert space and C be a nonempty subset of H. Let $CB(C)$ and $K(C)$ denote the families of nonempty closed bounded subsets and nonempty compact subsets of C, respectively. The *Pompeiu-Hausdorff metric* on $CB(C)$ is defined by

$$\mathcal{H}(A,B)=\max\left\{\sup_{x\in A}\operatorname{dist}(x,B),\sup_{y\in B}\operatorname{dist}(y,A)\right\} \text{ for } A,B\in CB(C),$$

where $\operatorname{dist}(x,C)=\inf\{\|x-y\|:y\in C\}$. An element $x\in C$ is called a *fixed point* of a multivalued mapping $T:C\to CB(C)$ if $x\in Tx$. The set of fixed points of T is denoted by $F(T)$.

Recall that a multivalued mapping $T:C\to CB(C)$ is said to be

- *nonexpansive* if

$$\mathcal{H}(Tx,Ty)\le\|x-y\|,\ \forall x,y\in C;$$

Date: Received: 30 March 2017; Revised: 16 June 2017 ; Accepted: 17 June 2017.

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2010 *Mathematics Subject Classification.* Primary 47H09; Secondary 47H10, 65K10, 65K15.

Key words and phrases. Proximal point algorithm, hybrid multivalued mapping, Ishikawa iteration, S-iteration, Hilbert spaces.

- *nonspreading* [17] if

$$2\mathcal{H}(Tx,Ty)^2 \leq \operatorname{dist}(y,Tx)^2 + \operatorname{dist}(x,Ty)^2, \ \ \forall x,y \in C;$$

- *λ-hybrid* [18] if there exists $\lambda \in \mathbb{R}$ such that

$$(1+\lambda)\mathcal{H}(Tx,Ty)^2 \leq (1-\lambda)\|x-y\|^2 + \lambda \operatorname{dist}(y,Tx)^2 + \lambda \operatorname{dist}(x,Ty)^2, \ \ \forall x,y \in C.$$

We note that 0-hybrid is nonexpansive and 1-hybrid is nonspreading. It is known by [18] that if T is λ-hybrid, then $F(T)$ is closed. In addition, if T satisfies the condition: $Tp = \{p\}$ for all $p \in F(T)$, then $F(T)$ is also convex.

We now give two examples for the class of λ-hybrid multivalued mappings.

Example 1.1 ([18])**.** Let $C = [1,7]$ with the usual metric and $T : C \to CB(C)$ be the multivalued mapping defined by

$$Tx = \begin{cases} \{1\}, & x \in [1,4]; \\ \left[1, \dfrac{2x^2+1}{x^2+1}\right], & x \in (4,7]. \end{cases}$$

It is easy to show that T is λ-hybrid for $\lambda \in [\frac{1}{2}, 1]$ but it is not nonexpansive.

Example 1.2 ([7])**.** Let $C = [-3,0]$ with the usual metric and $T : C \to CB(C)$ be the multivalued mapping defined by

$$Tx = \begin{cases} \{0\}, & x \in [-2,0]; \\ [-e^{x+2}, 0], & x \in [-3,-2). \end{cases}$$

It is easy to show that T is 1-hybrid but it is not nonexpansive.

Let $g : H \to (-\infty, \infty]$ be a proper and convex function. One of the major problems for optimization is to find a point $x \in H$ such that

$$g(x) = \min_{y \in H} g(y).$$

We denote the set of all minimizers of g on H by $\operatorname{argmin}_{y\in H} g(y)$.

The proximal point algorithm (shortly, the PPA) is an important tool in solving optimization problem which was initiated by Martinet [14] in 1970. Later, Rockafellar [16] studied the convergence of PPA for finding a solution of the unconstrained convex minimization problem in H as follows. Let g be a proper, convex and lower semi-continuous function on H. The PPA is defined by $x_1 \in H$ and

$$x_{n+1} = \operatorname*{argmin}_{u \in H} \left[g(y) + \frac{1}{2\lambda_n}\|u - x_n\|^2 \right], \quad \forall n \geq 1,$$

where $\lambda_n > 0$ for all $n \geq 1$. It was shown that if g has a minimizer and $\sum_{n=1}^{\infty} \lambda_n = \infty$, then the sequence $\{x_n\}$ converges weakly to a minimizer of g. However, PPA does not necessarily converges strongly in general (see [9]). Recently, several authors proposed modifications of Rochafellar's PPA to have strong convergence, for example [11, 12].

In the recent years, the problem of finding a common element of the set of solutions of various convex minimization problems and the set of fixed points for a single-valued mapping in the framework of Hilbert spaces and Banach spaces have been intensively studied by many authors, for instance, see [3, 4, 9, 13] and the references therein.

Recently, Chang et al. [6] combined PPA with Ishikawa iteration process (called *PPA-Ishikawa iteration process*) $\{x_n\}$ which is given by

$$\begin{cases} y_n = \operatorname*{argmin}_{u\in C}\left[g(u)+\dfrac{1}{2\lambda_n}\|u-x_n\|^2\right], \\ z_n=(1-\beta_n)x_n+\beta_n w_n,\ \ w_n\in Ty_n, \\ x_{n+1}=(1-\alpha_n)x_n+\alpha_n v_n,\ \ v_n\in Tz_n,\ \ \forall n\ge 1, \end{cases} \tag{1.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$, C is a nonempty closed convex subset of a Hilbert space H, $g: C\to(-\infty,\infty]$ is a proper convex and lower semi-continuous function, and $T: C\to K(C)$ is a nonspreading multivalued mapping. By using the iteration process (1.1), they proved weak convergence and strong convergence theorems for minimizers of proper convex and lower semi-continuous functions and fixed points of nonspreading multivalued mappings in Hilbert spaces.

In 2007, Agarwal et al. [1] introduced the S-iteration process for finding a fixed point of single-valued nonlinear mappings in Banach spaces. They also showed, theoretically as well as numerically, that the S-iteration process is faster than the Ishikawa iteration process for some nonlinear mappings.

Motivated by the above recorded studies, in this work, we propose a new iteration process called *PPA-S-iteration* which is a modification of PPA and S-iteration process. We prove weak convergence theorems for the proposed iteration process for minimizers of proper convex and lower semi-continuous functions and fixed points of λ-hybrid multivalued mappings in Hilbert spaces. Our results are refinements and generalizations of many recent results from the current literature. We also provide numerical examples to illustrate the convergence behavior of the proposed algorithm.

2. Some useful lemmas

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. We denote the strong convergence and the weak convergence of the sequence $\{x_n\}$ to a point $x\in H$ by $x_n\to x$ and $x_n\rightharpoonup x$, respectively. It is known in [15] that a Hilbert space H satisfies *Opial's condition*, that is, for any sequence $\{x_n\}$ with $x_n\rightharpoonup x$, the inequality

$$\limsup_{n\to\infty}\|x_n-x\|<\limsup_{n\to\infty}\|x_n-y\|$$

holds for every $y\in H$ with $y\neq x$.

Let $g: H\to(-\infty,\infty]$ be a proper convex and lower semi-continuous function. For any $\lambda>0$, define the Moreau-Yosida resolvent of g in a real Hilbert space H as follows:

$$J_\lambda x=\operatorname*{argmin}_{u\in H}\left[g(u)+\frac{1}{2\lambda}\|u-x\|^2\right],$$

for all $x\in H$. It was shown in [9] that the set of fixed points of the resolvent associated with g coincides with the set of minimizers of g. Also, the resolvent J_λ of g is nonexpansive for all $\lambda>0$ (see [10]).

Lemma 2.1 (The resolvent identity, [10]). *Let H be a real Hilbert space and $g : H \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. For each $x \in H$ and $\lambda > \mu > 0$, the following identity holds:*

$$J_\lambda x = J_\mu \left(\frac{\lambda - \mu}{\lambda} J_\lambda x + \frac{\mu}{\lambda} x \right).$$

Lemma 2.2 (Sub-differential inequality, [2]). *Let H be a real Hilbert space and $g : H \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Then, for all $x, y \in H$ and $\lambda > 0$, the following sub-differential inequality holds:*

$$\frac{1}{2\lambda}\|J_\lambda x - y\|^2 - \frac{1}{2\lambda}\|x - y\|^2 + \frac{1}{2\lambda}\|x - J_\lambda x\|^2 \leq g(y) - g(J_\lambda x). \tag{2.1}$$

In order to prove our main results, we need the following lemmas.

Lemma 2.3 ([5]). *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ be a nonexpansive single-valued mapping. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ with $x_n - Tx_n \to 0$, then $x = Tx$.*

Lemma 2.4 ([18]). *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to K(C)$ be a λ-hybrid multivalued mapping. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $y_n \in Tx_n$ with $x_n - y_n \to 0$, then $x \in Tx$.*

3. Weak convergence theorems

In this section, we prove some weak convergence theorem for minimizers of proper convex and lower semi-continuous functions and fixed points of λ-hybrid multivalued mappings in Hilbert spaces.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to K(C)$ be a λ-hybrid multivalued mapping and $g : C \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Suppose that $\mathcal{F} = F(T) \cap \operatorname{argmin}_{u \in C} g(u)$ is nonempty and $Tq = \{q\}$ for all $q \in F(T)$. For $x_1 \in C$, let the PPA-S-iteration process $\{x_n\}$ be defined by*

$$\begin{cases} y_n = \underset{u \in C}{\operatorname{argmin}} \left[g(u) + \dfrac{1}{2\lambda_n}\|u - x_n\|^2 \right], \\ z_n = (1 - \beta_n)x_n + \beta_n w_n, \ \ w_n \in Ty_n, \\ x_{n+1} = (1 - \alpha_n)w_n + \alpha_n v_n, \ \ v_n \in Tz_n, \ \ \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0, 1]$ such that $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for all $n \geq 1$, and $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ for all $n \geq 1$ and some λ. Then the sequence $\{x_n\}$ converges weakly to an element of $\mathcal{F}$.

Proof. Let $q \in \mathcal{F}$. Then we have $q \in Tq$ and $g(q) \leq g(u)$ for all $u \in C$. It follows that

$$g(q) + \frac{1}{2\lambda_n}\|q - q\|^2 \leq g(u) + \frac{1}{2\lambda_n}\|u - q\|^2, \ \ \forall u \in C,$$

and hence $q = J_{\lambda_n} q$ for all $n \geq 1$. Since $y_n = J_{\lambda_n} x_n$, it implies by nonexpansiveness of J_{λ_n} that

$$\|y_n - q\| = \|J_{\lambda_n} x_n - J_{\lambda_n} q\| \leq \|x_n - q\|. \tag{3.2}$$

For $q \in \mathcal{F}$, by $Tq = \{q\}$, we obtain that

$$\begin{aligned}\|z_n - q\| &\leq (1-\beta_n)\|x_n - q\| + \beta_n\|w_n - q\| \\ &= (1-\beta_n)\|x_n - q\| + \beta_n \text{dist}(w_n, Tq) \\ &\leq (1-\beta_n)\|x_n - q\| + \beta_n \mathcal{H}(Ty_n, Tq) \\ &\leq (1-\beta_n)\|x_n - q\| + \beta_n\|y_n - q\| \\ &\leq \|x_n - q\|. \end{aligned} \tag{3.3}$$

By (3.2) and (3.3), we have

$$\begin{aligned}\|x_{n+1} - q\| &\leq (1-\alpha_n)\|w_n - q\| + \alpha_n\|v_n - q\| \\ &= (1-\alpha_n)\text{dist}(w_n, Tq) + \alpha_n \text{dist}(v_n, Tq) \\ &\leq (1-\alpha_n)\mathcal{H}(Ty_n, Tq) + \alpha_n \mathcal{H}(Tz_n, Tq) \\ &\leq (1-\alpha_n)\|y_n - q\| + \alpha_n\|z_n - q\| \qquad (3.4) \\ &\leq \|x_n - q\|. \end{aligned}$$

This implies that $\{\|x_n - q\|\}$ is decreasing and bounded below, thus $\lim_{n\to\infty}\|x_n - q\|$ exists for all $q \in \mathcal{F}$. Without loss of generality, we can assume that

$$\lim_{n\to\infty}\|x_n - q\| = d \text{ for some } d. \tag{3.5}$$

It follows by (3.2) and (3.4) that

$$\|x_{n+1} - q\| \leq (1-\alpha_n)\|x_n - q\| + \alpha_n\|z_n - q\|.$$

By simplifying we have

$$\begin{aligned}\|x_n - q\| &\leq \frac{1}{\alpha_n}(\|x_n - q\| - \|x_{n+1} - q\|) + \|z_n - q\| \\ &\leq \frac{1}{a}(\|x_n - q\| - \|x_{n+1} - q\|) + \|z_n - q\|. \end{aligned}$$

This implies that

$$d = \liminf_{n\to\infty}\|x_n - q\| \leq \liminf_{n\to\infty}\|z_n - q\|. \tag{3.6}$$

It follows from (3.3) and (3.5) that

$$\limsup_{n\to\infty}\|z_n - q\| \leq \limsup_{n\to\infty}\|x_n - q\| = d.$$

This together with (3.8) shows that

$$\lim_{n\to\infty}\|z_n - q\| = d. \tag{3.7}$$

Also, by (3.3), we get

$$\|z_n - q\| \leq (1-\beta_n)\|x_n - q\| + \beta_n\|y_n - q\|,$$

which can be rewritten as

$$\begin{aligned}\|x_n - q\| &\leq \frac{1}{\beta_n}(\|x_n - q\| - \|z_n - q\|) + \|y_n - q\| \\ &\leq \frac{1}{a}(\|x_n - q\| - \|z_n - q\|) + \|y_n - q\|. \end{aligned}$$

This implies by (3.5) and (3.7) that

$$d = \liminf_{n\to\infty} \|x_n - q\| \le \liminf_{n\to\infty} \|y_n - q\|. \tag{3.8}$$

From (3.2) and (3.5), we have

$$\limsup_{n\to\infty} \|y_n - q\| \le \limsup_{n\to\infty} \|x_n - q\| = d.$$

This together with (3.8) shows that

$$\lim_{n\to\infty} \|y_n - q\| = d. \tag{3.9}$$

By the sub-differential inequality (2.1), we have

$$\frac{1}{2\lambda_n}\|y_n - q\|^2 - \frac{1}{2\lambda_n}\|x_n - q\|^2 + \frac{1}{2\lambda_n}\|x_n - y_n\|^2 \le g(q) - g(y_n).$$

Since $g(q) \le g(y_n)$ for all $n \ge 1$, we have

$$\|x_n - y_n\|^2 \le \|x_n - q\|^2 - \|y_n - q\|^2.$$

It implies by (3.5) and (3.9) that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.10}$$

Since T is λ-hybrid, $q \in Tq$, and $Tq = \{q\}$, it follows from (3.1) that

$$\begin{aligned}
\|z_n - q\|^2 &= (1-\beta_n)\|x_n - q\|^2 + \beta_n\|w_n - q\|^2 - \beta_n(1-\beta_n)\|x_n - w_n\|^2 \\
&= (1-\beta_n)\|x_n - q\|^2 + \beta_n \mathrm{dist}(w_n, Tq)^2 - \beta_n(1-\beta_n)\|x_n - w_n\|^2 \\
&\le (1-\beta_n)\|x_n - q\|^2 + \beta_n \mathcal{H}(Ty_n, Tq)^2 - \beta_n(1-\beta_n)\|x_n - w_n\|^2 \\
&\le (1-\beta_n)\|x_n - q\|^2 + \beta_n\|y_n - q\|^2 - \beta_n(1-\beta_n)\|x_n - w_n\|^2 \\
&\le \|x_n - q\|^2 - \beta_n(1-\beta_n)\|x_n - w_n\|^2.
\end{aligned}$$

This implies that

$$\beta_n(1-\beta_n)\|x_n - w_n\|^2 \le \|x_n - q\|^2 - \|z_n - q\|^2.$$

Thus, by (3.5), (3.7), and the condition $0 < a \le \alpha_n, \beta_n \le b < 1$, we have

$$\lim_{n\to\infty} \|x_n - w_n\| = 0. \tag{3.11}$$

Since $\|y_n - w_n\| \le \|y_n - x_n\| + \|x_n - w_n\|$, it implies by (3.10) and (3.11) that

$$\lim_{n\to\infty} \|y_n - w_n\| = 0. \tag{3.12}$$

It follows from Lemma 2.1, nonexpansiveness of J_λ, and $\lambda_n \ge \lambda > 0$ that

$$\begin{aligned}
\|x_n - J_\lambda x_n\| &\le \|x_n - y_n\| + \|y_n - J_\lambda x_n\| \\
&= \|x_n - y_n\| + \|J_{\lambda_n} x_n - J_\lambda x_n\| \\
&= \|x_n - y_n\| + \left\| J_\lambda \left(\frac{\lambda_n - \lambda}{\lambda_n} J_{\lambda_n} x_n + \frac{\lambda}{\lambda_n} x_n \right) - J_\lambda x_n \right\| \\
&\le \|x_n - y_n\| + \left\| \left(\frac{\lambda_n - \lambda}{\lambda_n} \right) J_{\lambda_n} x_n + \frac{\lambda}{\lambda_n} x_n - x_n \right\| \\
&= \|x_n - y_n\| + \left(1 - \frac{\lambda}{\lambda_n} \right) \|J_{\lambda_n} x_n - x_n\| \\
&= \|x_n - y_n\| + \left(1 - \frac{\lambda}{\lambda_n} \right) \|y_n - x_n\| \\
&= \left(2 - \frac{\lambda}{\lambda_n} \right) \|x_n - y_n\|.
\end{aligned}$$

This together with (3.10) shows that

$$\lim_{n\to\infty} \|x_n - J_\lambda x_n\| = 0. \tag{3.13}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup q_1 \in C$. By (3.10), it implies that $y_{n_i} \rightharpoonup q_1$. This implies by (3.12) and Lemma 2.4 that $q_1 \in F(T)$. Since J_λ is a nonexpansive single-valued mapping, by (3.13) and Lemma 2.3, we get $q_1 \in F(J_\lambda) = \operatorname{argmin}_{u\in C} g(u)$. Hence, we have $q_1 \in \mathcal{F}$. We will show that $x_n \rightharpoonup q_1$. To show this, suppose not. So, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup q_2 \in C$ and $q_2 \ne q_1$. Again, as above, we can conclude that $q_2 \in \mathcal{F}$. Since $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in \mathcal{F}$, by the Opial's condition, we have

$$\begin{aligned}
\limsup_{i\to\infty} \|x_{n_i} - q_1\| &< \limsup_{i\to\infty} \|x_{n_i} - q_2\| = \lim_{n\to\infty} \|x_n - q_2\| \\
&= \limsup_{j\to\infty} \|x_{n_j} - q_2\| < \limsup_{j\to\infty} \|x_{n_j} - q_1\| \\
&= \lim_{n\to\infty} \|x_n - q_1\| = \limsup_{i\to\infty} \|x_{n_i} - q_1\|.
\end{aligned}$$

This is a contradiction. Therefore, $q_1 = q_2$ and so $\{x_n\}$ converges weakly to an element of $\mathcal{F}$. □

Recall that a multivalued mapping $T : C \subseteq H \to CB(C)$ is said to satisfy *Condition (A)* if $\|x - p\| = \operatorname{dist}(x, Tp)$ for all $x \in H$ and $p \in F(T)$ (see [17]). We see that T satisfies Condition (A) if and only if $Tp = \{p\}$ for all $p \in F(T)$. Then the following result can be obtained from Theorem 3.1 immediately.

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to K(C)$ be a λ-hybrid multivalued mapping and $g : C \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Suppose that $\mathcal{F} = F(T) \cap$ $\operatorname{argmin}_{u\in C} g(u)$ is nonempty and T satisfies Condition (A). For $x_1 \in C$, let the PPA-S-iteration process $\{x_n\}$ be defined by (3.1) where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0, 1]$ such that $0 < a \le \alpha_n, \beta_n \le b < 1$ for all $n \ge 1$, and $\{\lambda_n\}$ is a sequence* *such that $\lambda_n \geq \lambda > 0$ for all $n \geq 1$ and some λ. Then the sequence $\{x_n\}$ converges weakly to an element of $\mathcal{F}$.*

Remark 3.3. It is know that the class of λ-hybrid multivalued mappings contains the classes of nonexpansive multivalued mappings and nonspreading multivalued mappings. Thus, Theorems 3.1 and 3.2 can be applied to these classes of mappings.

4. Numerical example

In this section, using Example 4.1, we will compare the convergence of the PPA-S-iteration process (3.1) with the PPA-Ishikawa iteration process (1.1).

Example 4.1. Let $H = \mathbb{R}$ and $C = [1,7]$. For each $x \in C$, we define $g : C \to (-\infty, \infty]$ by $g(x) = \frac{1}{2}\|x-1\|^2$ and define a mapping $T : C \to K(C)$ by

$$Tx = \begin{cases} \{1\}, & x \in [1,4]; \\ \left[1, \dfrac{2x^2+1}{x^2+1}\right], & x \in (4,7]. \end{cases}$$

It is easy to check that T is λ-hybrid for $\lambda \in [\frac{1}{2}, 1]$ and g is proper convex and lower semi-continuous with $F(T) \cap \operatorname{argmin}_{u\in C} g(u) = \{1\}$. Using the proximity operator [8], we know that

$$\operatorname*{argmin}_{u\in C}\left[g(u) + \frac{1}{2}\|u-x\|^2\right] = \operatorname{prox}_g x = \frac{x+1}{2}.$$

We choose $w_n = \frac{2y_n^2+1}{y_n^2+1} \in Ty_n$, $v_n = \frac{2z_n^2+1}{z_n^2+1} \in Tz_n$, $\alpha_n = \frac{n}{2n+1}$, and $\beta_n = \frac{5n}{50n+1}$ for all $n \geq 1$. By using SciLab, we computed the iterates of (1.1) and (3.1) for the initial point $x_1 = 7$. The numerical experiments of all iterations for approximating the point 1 are given in Table 1.

Table 1. Iterates of PPA-S-iteration process and PPA-Ishikawa iteration process for $x_1 = 7$.

Iterate	The PPA-S-iteration process	The PPA-Ishikawa iteration process
x_1	7.0000000	7.0000000
x_2	1.3254177	5.3254177
x_3	1.0000000	3.9792392
x_4	1.0000000	2.7024224
x_5	1.0000000	1.9457902
$\vdots$	$\vdots$	$\vdots$
x_{26}	1.0000000	1.0000010
x_{27}	1.0000000	1.0000005
x_{28}	1.0000000	1.0000003
x_{29}	1.0000000	1.0000001
x_{30}	1.0000000	1.0000001

References

1. R.P. Agarwal, D. ORegan, D.R. Sahu, *Iterative construction of fixed points of nearly asymptotically nonexpansive mappings*, J. Nonlinear Convex. Anal., **8** (2007), no. 1, 61–79.
2. L. Ambrosio, N. Gigli, G. Savare, *Gradient ows in metric spaces and in the space of probability measures*, Second edition, Lectures in Mathematics ETH Zurich, Birkhauser Verlag, Basel, (2008).
3. M. Bačák, *The proximal point algorithm in metric spaces*, Isr. J. Math., **194** (2013), 689–701.
4. O.A. Boikanyo, G. Morosanu, *A proximal point algorithm converging strongly for general errors*, Optim. Lett.,**4** (2010), 635-641.
5. F.E. Browder, *Fixed point theorems for noncompact mappings in Hilbert spaces*, Proc. Nat. Acad. Sci. USA., **53** (1965), 1272–1276.
6. S.S. Chang, D.P. Wu, L. Wang, G. Wang, *Proximal point algorithms involving fixed point of nonspreading-type multivalued mappings in Hilbert spaces*, J. Nonlinear Sci. Appl., **9** (2016), 5561–5569.
7. W. Cholamjiak, *Shrinking projection methods for a split equilibrium problem and a nonspreading-type multivalued mapping*, to appear in J. Nonlinear Sci. Appl.
8. P.L. Combettes, J.C. Pesquet, *Proximal splitting methods in signal processing*, In: H.H. Bauschke, R. Burachik, P.L. Combettes, V. Elser, D.R. Luke and H. Wolkowicz, (eds.) Fixed-Point Algorithms for Inverse Problems in Science and Engineering, pp. 185-212 Springer, New York, 2011.
9. O. Güler, *On the convergence of the proximal point algorithm for convex minimization*, SIAM J. Control Optim., **29** (1991), 403–419.
10. J. Jost, *Convex functionals and generalized harmonic maps into spaces of nonpositive curvature*, Comment. Math. Helv., **70** (1995), 659–673.
11. S. Kamimura, F. Kohsaka and W. Takahashi, *Weak and strong convergence theorems for maximal monotone operators in a Banach space*, Set-Valued Anal., **12** (2004), 417–429.
12. G. Kassay, *The proximal point algorithm for reflexive Banach spaces*, Studia Univ. Babes Bolyai Math., **30** (1985), 9–17.
13. G. Marino, H.K. Xu, *Convergence of generalized proximal point algorithms*, Commun. Pure Appl. Anal., **3** (2004), 791–808.
14. B. Martinet, *Regularisation d'inequations variationnelles par approximations successives*, Rev. Fr. Inform. Rech. Oper., **4** (1970), 154–158.
15. Z. Opial, *Weak convergence of the sequence of successive approximation for nonexpansive mappings*, Bull. Amer. Math. Soc., **73** (1967), 591–597.
16. R.T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM J. Control Optim., **14** (1976), 877–898.
17. S. Suantai, P. Cholamjiak, Y.J. Cho, W. Cholamjiak, *On solving split equilibrium problems and fixed point problems of nonspreading multi-valued mappings in Hilbert spaces*, Fixed Point Theory Appl., **35** (2016), 1–16.
18. S. Suantai, W. Phuengrattana, *Existence and convergence theorems for λ-hybrid mappings in Hilbert spaces, Dynamics of Continuous*, Discrete and Impulsive Systems Series A: Mathematical Analysis, **22** (2015), 177–188.

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