NEW INEQUALITIES OF HERMITE-HADAMARD TYPE FOR LOG-CONVEX FUNCTIONS

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ABSTRACT. Some new inequalities of Hermite-Hadamard type for log-convex functions defined on real intervals are given.

1. Introduction

A function \( f : I \to [0, \infty) \) is said to be log-convex or multiplicatively convex if \( \log f \) is convex, or, equivalently, if for all \( x, y \in I \) and \( t \in [0,1] \) one has the inequality:

\[
f (tx + (1 - t)y) \leq [f(x)]^t [f(y)]^{1-t}.
\]

We note that if \( f \) and \( g \) are convex and \( g \) is increasing, then \( g \circ f \) is convex; moreover, since \( f = \exp(\log f) \), it follows that a log-convex function is convex, but the converse may not necessarily be true. This follows directly from (1.1) because, by the arithmetic-geometric mean inequality, we have

\[
[f(x)]^t [f(y)]^{1-t} \leq tf(x) + (1 - t)f(y)
\]

for all \( x, y \in I \) and \( t \in [0,1] \).

Let us recall the Hermite-Hadamard inequality

\[
f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2},
\]

where \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) is a convex function on the interval \( I \), \( a, b \in I \) and \( a < b \). For related results, see [1]-[22], [25]-[28], [29]-[39] and [40]-[51].
Note that if we apply the above inequality for the log-convex functions \( f : I \to (0, \infty) \), we have that

\[
\ln \left[ f \left( \frac{a+b}{2} \right) \right] \leq \frac{1}{b-a} \int_a^b \ln f \left( x \right) dx \leq \frac{\ln f(a) + \ln f(b)}{2},
\]

(1.3)

from which we get

\[
f \left( \frac{a+b}{2} \right) \leq \exp \left[ \frac{1}{b-a} \int_a^b \ln f \left( x \right) dx \right] \leq \sqrt{f(a)f(b)},
\]

(1.4)

which is an inequality of Hermite-Hadamard’s type for log-convex functions.

By using simple properties of log-convex functions, Dragomir and Mond proved in 1998 the following result [31].

**Theorem 1.1.** Let \( f : I \to [0, \infty) \) be a log-convex mapping on I and \( a, b \in I \) with \( a < b \). Then one has the inequality:

\[
f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b \sqrt{f(x)f(a+b-x)}dx \leq \sqrt{f(a)f(b)}.
\]

(1.5)

The inequality between the first and the second term in (1.5) may be improved as follows [31]. A different upper bound for the middle term in (1.5) can be also provided.

**Theorem 1.2.** Let \( f : I \to (0, \infty) \) be a log-convex mapping on I and \( a, b \in I \) with \( a < b \). Then one has the inequalities:

\[
f \left( \frac{a+b}{2} \right) \leq \exp \left[ \frac{1}{b-a} \int_a^b \ln f \left( x \right) dx \right]
\leq \frac{1}{b-a} \int_a^b \sqrt{f(x)f(a+b-x)}dx \leq \sqrt{f(a)f(b)}\]

(1.6)

where \( L(p, q) \) is the logarithmic mean of the strictly positive real numbers \( p, q \), i.e.,

\[
L(p, q) := \frac{p - q}{\ln p - \ln q} \text{ if } p \neq q \text{ and } L(p, p) := p.
\]

The last inequality in (1.6) was obtained in a different context in [41].

As shown in [57], the following result also holds.

**Theorem 1.3.** Let \( f : I \to (0, \infty) \) be a log-convex mapping on I and \( a, b \in I \) with \( a < b \). Then one has the inequalities:

\[
f \left( \frac{a+b}{2} \right) \leq \left( \frac{1}{b-a} \int_a^b \sqrt{f(x)}dx \right)^2 \leq \frac{1}{b-a} \int_a^b f \left( x \right) dx.
\]

(1.7)

The following result improving the classical first Hermite-Hadamard inequality for differentiable log-convex functions also hold [15].
Theorem 1.4. Let $f : I \to (0, \infty)$ be a differentiable log-convex function on the interval of real numbers $I$ (the interior of $I$) and $a, b \in I$ with $a < b$. Then the following inequalities hold:

\[
\frac{1}{b-a} \int_a^b f(x) \, dx \geq L \left( \exp \left( \frac{f'(\frac{a+b}{2})}{f'\left(\frac{a+b}{2}\right)} \frac{(b-a)}{2} \right) , \exp \left[ -\frac{f'(\frac{a+b}{2})}{f'\left(\frac{a+b}{2}\right)} \frac{(b-a)}{2} \right] \right) \geq 1.
\] (1.8)

The second Hermite-Hadamard inequality can be improved as follows [15].

Theorem 1.5. Let $f : I \to \mathbb{R}$ be as in Theorem 1.4. Then we have the inequality:

\[
\frac{f(a)+f(b)}{2} \geq 1 + \log \left( 1 - \frac{1}{b-a} \int_a^b f(x) \, dx \right) \geq 1 + \log \left( \int_a^b f(x) \, dx \right) \geq 1.
\] (1.9)

Motivated by the above results, we establish in this paper some new inequalities for log-convex functions, some of them improving earlier results. Applications for special means are also provided.

2. NEW INEQUALITIES

The following refinement of the Hermite-Hadamard inequality holds.

Lemma 2.1. Let $h : [a, b] \to \mathbb{R}$ be a convex function and $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ an arbitrary division of $[a, b]$ with $n \geq 2$. Then

\[
h \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \sum_{i=0}^{n-1} h \left( \frac{x_i + x_{i+1}}{2} \right) \left( x_{i+1} - x_i \right) \leq \frac{1}{b-a} \int_a^b h(x) \, dx \\
\leq \frac{1}{b-a} \sum_{i=0}^{n-1} \frac{h(x_i) + h(x_{i+1})}{2} \left( x_{i+1} - x_i \right) \leq \frac{h(a) + h(b)}{2}.
\] (2.1)

The inequality (2.1) was obtained in 1994 as a particular case of a more general result (see [14] and also mentioned in [34, p. 22]). For a direct proof, see the recent paper [27].

Theorem 2.2. Let $f : [a, b] \to (0, \infty)$ be a log-convex function on $[a, b]$ and $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ an arbitrary division of $[a, b]$ with $n \geq 1$. 

Then
\[
 f \left( \frac{a+b}{2} \right) \leq \prod_{i=1}^{n-1} \left[ f \left( \frac{x_i + x_{i+1}}{2} \right) \right]^{\frac{x_{i+1} - x_i}{b-a}} \leq \exp \left( \frac{1}{b-a} \int_a^b \ln f(x) \, dx \right) \leq \prod_{i=1}^{n-1} \left[ \sqrt{f(x_i) f(x_{i+1})} \right]^{\frac{x_{i+1} - x_i}{b-a}} \leq \sqrt{f(a) f(b)}.
\] (2.2)

**Proof.** If we write the inequality (2.1) for the function \( h = \ln f \), then we get
\[
 \ln f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \sum_{i=0}^{n-1} (x_{i+1} - x_i) \ln f \left( \frac{x_i + x_{i+1}}{2} \right) \leq \frac{1}{b-a} \int_a^b \ln f(x) \, dx \leq \frac{1}{b-a} \sum_{i=0}^{n-1} \ln f(x_i) + \ln f(x_{i+1}) \leq \frac{1}{2} \frac{\ln f(a) + \ln f(b)}{2}.
\]

This inequality is equivalent to
\[
 \ln f \left( \frac{a+b}{2} \right) \leq \ln \left( \prod_{i=1}^{n-1} \left[ f \left( \frac{x_i + x_{i+1}}{2} \right) \right]^{\frac{x_{i+1} - x_i}{b-a}} \right) \leq \frac{1}{b-a} \int_a^b \ln f(x) \, dx \leq \ln \left( \prod_{i=1}^{n-1} \left[ \sqrt{f(x_i) f(x_{i+1})} \right]^{\frac{x_{i+1} - x_i}{b-a}} \right) \leq \ln \sqrt{f(a) f(b)}.
\] (2.3)

This inequality is of interest in itself.

If we take the exponential in (2.3), we get the desired result (2.2). \(\square\)

**Corollary 2.3.** Let \( f : [a, b] \to (0, \infty) \) be a log-convex function on \([a, b]\) and \( x \in [a, b] \), then
\[
 f \left( \frac{a+b}{2} \right) \leq f \left( \frac{a+x}{2} \right) \frac{f \left( \frac{b-x}{2} \right)}{f \left( \frac{b-a}{2} \right)} \leq \exp \left( \frac{1}{b-a} \int_a^b \ln f(x) \, dx \right) \leq \left[ \sqrt{f(a)} \right]^{\frac{x-a}{b-a}} \left[ \sqrt{f(b)} \right]^{\frac{b-x}{b-a}} \leq \sqrt{f(a) f(b)}.
\] (2.4)
and, equivalently

\[
\ln f \left( \frac{a + b}{2} \right) \leq \frac{x - a}{b - a} \ln f \left( \frac{a + x}{2} \right) + \frac{b - x}{b - a} \ln f \left( \frac{x + b}{2} \right)
\]  
\leq \frac{1}{b - a} \int_a^b \ln f (x) \, dx
\leq \frac{1}{2} \left[ \ln f (x) + \frac{(x - a) \ln f (a) + (b - x) \ln f (b)}{b - a} \right]
\leq \frac{\ln f (a) + \ln f (b)}{2}.
\]  

(2.5)

Remark 2.4. If we take in (2.5) \( x = \frac{a + b}{2} \), then we get

\[
\ln f \left( \frac{a + b}{2} \right) \leq \frac{1}{2} \left[ \ln f \left( \frac{3a + b}{4} \right) + \ln f \left( \frac{a + 3b}{4} \right) \right]
\leq \frac{1}{b - a} \int_a^b \ln f (x) \, dx
\leq \frac{1}{2} \left[ \ln f \left( \frac{a + b}{2} \right) + \frac{\ln f (a) + \ln f (b)}{2} \right] \leq \frac{\ln f (a) + \ln f (b)}{2}.
\]  

(2.6)

From the second inequality in (2.6) we get

\[
0 \leq \frac{1}{b - a} \int_a^b \ln f (x) \, dx - \ln f \left( \frac{a + b}{2} \right)
\leq \frac{\ln f (a) + \ln f (b)}{2} - \frac{1}{b - a} \int_a^b \ln f (x) \, dx,
\]

which shows that the integral term in (1.3) is closer to the left side than to the right side of that inequality.

We also have the particular inequalities:

\[
\ln f \left( \frac{a + b}{2} \right)
\leq \frac{1}{\sqrt{b} + \sqrt{a}} \left[ \sqrt{a} \ln f \left( \frac{\sqrt{a} (\sqrt{a} + \sqrt{b})}{2} \right) + \sqrt{b} \ln f \left( \frac{\sqrt{b} (\sqrt{a} + \sqrt{b})}{2} \right) \right]
\leq \frac{1}{b - a} \int_a^b \ln f (y) \, dy
\leq \frac{1}{2} \left[ \sqrt{b} \ln f (b) + \sqrt{a} \ln f (a) + \ln f \left( \sqrt{ab} \right) \right] \leq \frac{\ln f (a) + \ln f (b)}{2}.
\]  

(2.7)
The following reverses of the Hermite-Hadamard inequality hold [23, 24].

**Lemma 2.5.** Let \( h : [a, b] \to \mathbb{R} \) be a convex function on \([a, b]\). Then

\[
0 \leq \frac{1}{8} \left[ h_+ \left( \frac{a + b}{2} \right) - h_- \left( \frac{a + b}{2} \right) \right] (b - a) \leq \frac{h(a) + h(b)}{2} - \frac{1}{b - a} \int_a^b h(x) \, dx \leq \frac{1}{8} \left[ h_- (b) - h_+ (a) \right] (b - a)
\]

and

\[
0 \leq \frac{1}{8} \left[ h_+ \left( \frac{a + b}{2} \right) - h_- \left( \frac{a + b}{2} \right) \right] (b - a) \leq \frac{1}{b - a} \int_a^b h(x) \, dx - h \left( \frac{a + b}{2} \right) \leq \frac{1}{8} \left[ h_- (b) - h_+ (a) \right] (b - a).
\]

The constant \( \frac{1}{8} \) is best possible in all inequalities from (2.9) and (2.10).

In the case of log-convex functions we have:

**Theorem 2.6.** Let \( f : [a, b] \to (0, \infty) \) be a log-convex function on \([a, b]\). Then

\[
1 \leq \exp \left( \frac{1}{8} \left[ \frac{f_+ \left( \frac{a+b}{2} \right) - f_- \left( \frac{a+b}{2} \right)}{f \left( \frac{a+b}{2} \right)} \right] (b - a) \right)
\]

\[
\leq \frac{\sqrt{f(a) f(b)}}{\exp \left( \frac{1}{b-a} \int_a^b \ln f(x) \, dx \right)}
\]

\[
\leq \exp \left( \frac{1}{8} \left[ \frac{f_- (b) - f_+ (a)}{f(b) - f(a)} \right] (b - a) \right)
\]
\[1 \leq \exp \left( \frac{1}{8} \left[ \frac{f_+ \left( \frac{a+b}{2} \right) - f_- \left( \frac{a+b}{2} \right)}{f \left( \frac{a+b}{2} \right)} \right] (b-a) \right) \] (2.12)

\[\leq \exp \left( \frac{1}{b-a} \int_a^b \ln f(x) \, dx \right) \]

\[\leq \exp \left( \frac{1}{8} \left[ \frac{f_- (b)}{f(b)} - \frac{f_+ (a)}{f(a)} \right] (b-a) \right) .\]

**Proof.** If we write the inequality (2.9) for the convex function \(h = \ln f\)

\[0 \leq \frac{1}{8} \left[ \frac{f_+ \left( \frac{a+b}{2} \right) - f_- \left( \frac{a+b}{2} \right)}{f \left( \frac{a+b}{2} \right)} \right] (b-a) \]

\[\leq \frac{\ln f(a) + \ln f(b)}{2} - \frac{1}{b-a} \int_a^b \ln f(x) \, dx \]

\[\leq \frac{1}{8} \left[ \frac{f_- (b)}{f(b)} - \frac{f_+ (a)}{f(a)} \right] (b-a) \]

that is equivalent to

\[0 \leq \ln \left[ \exp \left( \frac{1}{8} \left[ \frac{f_+ \left( \frac{a+b}{2} \right) - f_- \left( \frac{a+b}{2} \right)}{f \left( \frac{a+b}{2} \right)} \right] (b-a) \right) \right] \]

\[\leq \ln \left( \frac{\sqrt{f(a) f(b)}}{\exp \left( \frac{1}{b-a} \int_a^b \ln f(x) \, dx \right)} \right) \]

\[\leq \ln \left[ \exp \left( \frac{1}{8} \left[ \frac{f_- (b)}{f(b)} - \frac{f_+ (a)}{f(a)} \right] (b-a) \right) \right] .\]

By taking the exponential in this inequality, we get the desired result (2.11).

The inequality (2.12) follows from (2.10). \[\square\]

We also have the following result.

**Theorem 2.7.** Let \(f : [a, b] \to (0, \infty)\) be a log-convex function on \([a, b]\) and \(a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b\) be an arbitrary division of \([a, b]\) with \(n \geq 1\). Then

\[\exp \left[ \frac{1}{b-a} \int_a^b \ln f(x) \, dx \right] \leq \frac{1}{b-a} \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \sqrt{f(x) f(x_i + x_{i+1} - x)} \, dx \] (2.13)

\[\leq \frac{1}{b-a} \int_a^b f(x) \, dx.\]
Proof. Observe that we have
\[
\exp \left[ \frac{1}{b-a} \int_a^b \ln f(x) \, dx \right]
\]
\[
= \exp \left[ \frac{1}{b-a} \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \ln f(x) \, dx \right]
\]
\[
= \exp \left[ \sum_{i=0}^{n-1} \frac{x_{i+1} - x_i}{b-a} \left( \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} \ln f(x) \, dx \right) \right]
\]

Since \( \sum_{i=0}^{n-1} \frac{x_{i+1} - x_i}{b-a} = 1 \), by Jensen’s inequality for the convex function \( \exp \) we have
\[
\exp \left[ \sum_{i=0}^{n-1} \frac{x_{i+1} - x_i}{b-a} \left( \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} \ln f(x) \, dx \right) \right] \leq \sum_{i=0}^{n-1} \frac{x_{i+1} - x_i}{b-a} \exp \left( \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} \ln f(x) \, dx \right)
\]

Utilising the inequality (1.6) on each of the intervals \([x_i, x_{i+1}]\) for \( i \in \{0, \ldots, n-1\}\) we have
\[
\exp \left[ \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} \ln f(x) \, dx \right] \leq \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} \sqrt{f(x) f(x_i + x_{i+1} - x)} \, dx
\]
\[
\leq \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} f(x) \, dx,
\]
for any \( i \in \{0, \ldots, n-1\} \).

If we multiply the inequality (2.16) by \( \frac{x_{i+1} - x_i}{b-a} \) and sum over \( i \) from 0 to \( n-1 \), then we get
\[
\sum_{i=0}^{n-1} \frac{x_{i+1} - x_i}{b-a} \exp \left( \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} \ln f(x) \, dx \right) \leq \frac{1}{b-a} \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \sqrt{f(x) f(x_i + x_{i+1} - x)} \, dx \leq \frac{1}{b-a} \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x) \, dx
\]
\[
= \frac{1}{b-a} \int_a^b f(x) \, dx.
\]
Making use of (2.14), (2.15) and (2.17) we get the desired result (2.13). □
Corollary 2.8. Let $f : [a, b] \to (0, \infty)$ be a log-convex function on $[a, b]$ and $y \in [a, b]$, then

$$
\exp \left[ \frac{1}{b-a} \int_a^b \ln f(x) \, dx \right] \leq \frac{1}{b-a} \left[ \int_a^y \sqrt{f(x)} f(a+y-x) \, dx + \int_y^b \sqrt{f(x)} f(b+y-x) \, dx \right] \leq \frac{1}{b-a} \int_a^b f(x) \, dx.
$$

We define the $p$-logarithmic mean as

$$
L_p(a, b) := \begin{cases} 
\left[ \frac{p^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^\frac{1}{p}, & \text{with } a \neq b \\
1, & \text{if } a = b
\end{cases}
$$

for $p \neq 0, -1$ and $a, b > 0$.

The following result also holds.

Theorem 2.9. Let $f : [a, b] \to (0, \infty)$ be a log-convex function on $[a, b]$. Then for any $p > 0$ we have the inequality

$$
f \left( \frac{a + b}{2} \right) \leq \exp \left[ \frac{1}{b-a} \int_a^b \ln f(x) \, dx \right] \leq \left( \frac{1}{b-a} \int_a^b f^p(x) f^p(a+b-x) \, dx \right)^\frac{1}{2p} \leq \left( \frac{1}{b-a} \int_a^b f^{2p}(x) \, dx \right)^\frac{1}{2p} \leq \begin{cases} 
[L_{2p-1}(f(a), f(b))]^{1-\frac{1}{2p}} [L(f(a), f(b))]^{\frac{1}{2p}}, & p \neq \frac{1}{2}; \\
L(f(a), f(b)), & p = \frac{1}{2}.
\end{cases}
$$

If $p \in (0, \frac{1}{2})$, then we have

$$
f \left( \frac{a + b}{2} \right) \leq \exp \left[ \frac{1}{b-a} \int_a^b \ln f(x) \, dx \right] \leq \left( \frac{1}{b-a} \int_a^b f^p(x) f^p(a+b-x) \, dx \right)^\frac{1}{2p} \leq \left( \frac{1}{b-a} \int_a^b f^{2p}(x) \, dx \right)^\frac{1}{2p} \leq \frac{1}{b-a} \int_a^b f(x) \, dx.
$$
Proof. If $f$ is a log-convex function on $[a, b]$, then $f^{2p}$ is log-convex on $[a, b]$ for $p > 0$ and by (1.6) we have

$$f^{2p} \left( \frac{a + b}{2} \right) \leq \exp \left[ \frac{1}{b - a} \int_a^b \ln f^{2p} (x) \, dx \right] \tag{2.21}$$

Taking the power $\frac{1}{2p}$ in (2.21) we get

$$f \left( \frac{a + b}{2} \right) \leq \exp \left[ \frac{1}{b - a} \int_a^b \ln f (x) \, dx \right] \tag{2.22}$$

Observe that, for $p \neq \frac{1}{2}$,

$$\left[ L \left( f^{2p} (a) , f^{2p} (b) \right) \right]^{\frac{1}{2p}} = \left[ \frac{f^{2p} (a) - f^{2p} (b)}{\ln f^{2p} (a) - \ln f^{2p} (b)} \right]^{\frac{1}{2p}}$$

and by (2.22) we get the desired result (2.19).

The last inequality in (2.20) follows by the following integral inequality for power $q \in (0, 1)$, namely

$$\frac{1}{b - a} \int_a^b f^q (x) \, dx \leq \left( \frac{1}{b - a} \int_a^b f (x) \, dx \right)^q.$$
Remark 2.10. If we take in (2.19) \( p = 1 \), then we get
\[
f \left( \frac{a+b}{2} \right) \leq \exp \left[ \frac{1}{b-a} \int_{a}^{b} \ln f(x) \, dx \right]
\tag{2.23}
\]
If we take in (2.20) \( p = \frac{1}{4} \), then we get
\[
f \left( \frac{a+b}{2} \right) \leq \exp \left[ \frac{1}{b-a} \int_{a}^{b} \ln f(x) \, dx \right]
\tag{2.24}
\]
This improves the inequality (1.7).

3. Related Inequalities

In this section, we establish some related results for log-convex functions.

Theorem 3.1. Let \( f : [a, b] \to (0, \infty) \) be a log-convex function on \([a, b]\). Then for any \( x \in [a, b] \) we have
\[
f(b)(b-x) + f(a)(x-a) - \int_{a}^{b} f(y) \, dy \geq \int_{a}^{b} f(y) \ln f(y) \, dy - \ln f(x) \int_{a}^{b} f(y) \, dy.
\tag{3.1}
\]
In particular,
\[
\frac{f(b) + f(a)}{2} - \frac{1}{b-a} \int_{a}^{b} f(y) \, dy \geq \frac{1}{b-a} \int_{a}^{b} f(y) \ln f(y) \, dy - \ln \left( \frac{a+b}{2} \right) \frac{1}{b-a} \int_{a}^{b} f(y) \, dy,
\tag{3.2}
\]
\[
\frac{f(b) \sqrt{b} + f(a) \sqrt{a}}{\sqrt{b} + \sqrt{a}} - \frac{1}{b-a} \int_{a}^{b} f(y) \, dy \geq \frac{1}{b-a} \int_{a}^{b} f(y) \ln f(y) \, dy - \ln \left( \sqrt{ab} \right) \frac{1}{b-a} \int_{a}^{b} f(y) \, dy.
\tag{3.3}
\]
and
\[
\frac{f(b)b + f(a)a}{a+b} - \frac{1}{b-a} \int_a^b f(y) \, dy \geq \frac{1}{b-a} \int_a^b f(y) \ln f(y) \, dy - \ln f\left(\frac{2ab}{a+b}\right) - \frac{1}{b-a} \int_a^b f(y) \, dy.
\] (3.4)

Proof. Since the function \(\ln f\) is convex on \([a, b]\), by the gradient inequality we have
\[
\ln f(x) - \ln f(y) \geq f'_+(y) (x - y),
\] (3.5)
for any \(x \in [a, b]\) and \(y \in (a, b)\).

If we multiply (3.5) by \(f(y) > 0\) and integrate on \([a, b]\) over \(y\) we get
\[
\ln f(x) \int_a^b f(y) \, dy - \int_a^b f(y) \ln f(y) \, dy 
\geq \int_a^b f'_+(y) (x - y) \, dy = f(y) (x - y)\bigg|_a^b + \int_a^b f(y) \, dy 
= f(b) (x - b) + f(a) (a - x) + \int_a^b f(y) \, dy,
\]
which is equivalent to (3.1).

The inequality (3.2) follows by (3.1) on taking \(x = \frac{a+b}{2}\).

If we take in (3.1) \(x = \sqrt{ab}\), then we get
\[
f(b) \sqrt{b} \left(\sqrt{b} - \sqrt{a}\right) + f(a) \sqrt{a} \left(\sqrt{b} - \sqrt{a}\right) - \int_a^b f(y) \, dy 
\geq \int_a^b f(y) \ln f(y) \, dy - \ln f\left(\sqrt{ab}\right) \int_a^b f(y) \, dy,
\]
which is equivalent to (3.3).

If we take in (3.1) \(x = \frac{2ab}{a+b}\), then we get
\[
f(b) b \left(\frac{b-a}{a+b}\right) + f(a) a \left(\frac{b-a}{a+b}\right) - \int_a^b f(y) \, dy
\geq \int_a^b f(y) \ln f(y) \, dy - \ln f\left(\frac{2ab}{a+b}\right) \int_a^b f(y) \, dy,
\]
which is equivalent to (3.4). \(\square\)

**Corollary 3.2.** Let \(f : [a, b] \to (0, \infty)\) be a log-convex function on \([a, b]\). Then
\[
\frac{f(b) + f(a)}{2} - \frac{1}{b-a} \int_a^b f(y) \, dy
\geq \int_a^b f(y) \ln f(y) \, dy - \int_a^b f(y) \, dy \frac{1}{b-a} \int_a^b \ln f(y) \, dy \geq 0.
\] (3.6)
Proof. If we take the integral mean over $x$ in (3.1), then we get

\[
\frac{1}{b-a} \int_a^b [f(b)(b-x) + f(a)(x-a)] \, dx - \int_a^b f(y) \, dy \\
\geq \int_a^b f(y) \ln f(y) \, dy - \int_a^b f(y) \, dy \frac{1}{b-a} \int_a^b \ln f(x)
\]

and since

\[
\frac{1}{b-a} \int_a^b [f(b)(b-x) + f(a)(x-a)] \, dx = \frac{f(b) + f(a)}{2} - \frac{1}{b-a} \int_a^b f(y) \, dy,
\]

then the first inequality in (3.6) is proved.

Since $\ln$ is an increasing function on $(0, \infty)$, we have

\[
(f(x) - f(y))(\ln f(x) - \ln f(y)) \geq 0,
\]

for any $x, y \in [a,b]$, showing that the functions $f$ and $\ln f$ are synchronous on $[a,b]$.

By making use of the Čebyšev integral inequality for synchronous functions $g, h : [a,b] \to \mathbb{R}$, namely

\[
\frac{1}{b-a} \int_a^b g(x) h(x) \, dx \geq \frac{1}{b-a} \int_a^b g(x) \, dx \frac{1}{b-a} \int_a^b h(x) \, dx,
\]

we have

\[
\frac{1}{b-a} \int_a^b f(x) \ln f(x) \, dx \geq \frac{1}{b-a} \int_a^b f(x) \, dx \frac{1}{b-a} \int_a^b \ln f(x) \, dx,
\]

which proves the last part of (3.6). \qed

The inequality (3.6) improves the well known result for convex functions

\[
\frac{f(b) + f(a)}{2} \geq \frac{1}{b-a} \int_a^b f(y) \, dy.
\]

We have:

**Corollary 3.3.** Let $f : [a,b] \to (0, \infty)$ be a log-convex function on $[a,b]$. If $f(a) \neq f(b)$ and

\[
\alpha_f := \frac{\int_a^b f'(y) \, y \, dy}{\int_a^b f'(y) \, dy} = \frac{bf(b) - af(a) - \int_a^b f(y) \, dy}{f(b) - f(a)} \in [a,b], \quad (3.7)
\]

then

\[
\ln f(\alpha_f) \geq \frac{\int_a^b f(y) \ln f(y) \, dy}{\int_a^b f(y) \, dy}. \quad (3.8)
\]

**Proof.** Follows from (3.1) by observing that

\[
f(b)(b - \alpha_f) + f(a)(\alpha_f - a) = \int_a^b f(y) \, dy.
\]

\qed
Remark 3.4. We observe that if $f : [a, b] \to (0, \infty)$ is nondecreasing with $f(a) \neq f(b)$, the condition (3.7) is satisfied.

We also have:

Corollary 3.5. Let $f : [a, b] \to (0, \infty)$ be a log-convex function on $[a, b]$. Then

\[
 f(b) \left( b - \int_a^b yf(y) \, dy \right) + f(a) \left( \int_a^b yf(y) \, dy - a \right) - \int_a^b f(y) \, dy \geq 0.
\]

Proof. The first inequality follows by (3.1) on taking $x = \int_a^b yf(y) \, dy \in [a, b]$ since $f(y) > 0$ for any $y \in [a, b]$.

By Jensen’s inequality for the convex function $\ln f$ and the positive weight $f$ we have

\[
 \frac{\int_a^b f(y) \ln f(y) \, dy}{\int_a^b f(y) \, dy} \geq f \left( \frac{\int_a^b f(y) \, dy}{\int_a^b f(y) \, dy} \right),
\]

which proves the second inequality in (3.9). \qed

4. Applications

The function $f : (0, \infty) \to (0, \infty)$, $f(t) = \frac{1}{t}$ is log-convex on $(0, \infty)$. If we use the inequality (2.2) for this function, then we have

\[
 A(a, b) \geq \prod_{i=1}^{n-1} \left[ A(x_i, x_{i+1}) \right]^{\frac{x_{i+1}-x_i}{b-a}} \geq I(a, b) \geq G(a, b),
\]

for any $a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ an arbitrary division of $[a, b]$ with $n \geq 1$.

In particular, we have

\[
 A(a, b) \geq \left[ A(a, x) \right]^{\frac{x-a}{b-a}} \left[ A(x, b) \right]^{\frac{b-x}{b-a}} \geq I(a, b) \geq G(a, b),
\]

for any $x \in [a, b]$.

If we use the inequalities (2.11) and (2.12) for $f : (0, \infty) \to (0, \infty)$, $f(t) = \frac{1}{t}$, then we have

\[
 (1 \leq) \frac{I(a, b)}{G(a, b)} \leq \exp \left( \frac{1}{8} \left( b-a \right)^2 \right).
\]
and
\[ (1 \leq) \frac{A(a,b)}{I(a,b)} \leq \exp \left( \frac{1}{8} \frac{(b-a)^2}{ab} \right). \] (4.4)

If we use the inequality (3.6) for \( f : (0, \infty) \to (0, \infty), f(t) = \frac{1}{t} \), then we have
\[ L(a,b) - H(a,b) \geq (b-a) H(a,b) \ln \left( \frac{I(a,b)}{G(a,b)} \right) (\geq 0). \] (4.5)

The interested reader may apply the above inequalities for other log-convex functions such as \( f(t) = \frac{1}{t^p}, p > 0, t > 0 \), \( f(t) = \exp g(t) \), with \( g \) any convex function on an interval, etc. The details are omitted.

**References**


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