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LINEAR PRESERVERS OF RIGHT SGUT-MAJORIZATION ON $\mathbb{R}_n$

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Abstract. A matrix R is called a *generalized row substochastic* (g-row substochastic) if the sum of entries on every row of R is less than or equal to one. For x, $y \in \mathbb{R}_n$, it is said that x is *rsgut-majorized* by y (denoted by $x \prec_{rsgut} y$) if there exists an n-by-n upper triangular g-row substochastic matrix R such that $x = yR$. In the present paper, we characterize the linear preservers and strong linear preservers of rsgut-majorization on $\mathbb{R}_n$.

1. Introduction and preliminaries

Let $\mathbf{M}_{n,m}$ be the set of all n-by-m real matrices, and let $\mathbb{R}_n$ be the set of 1-by-n real vectors. A matrix $R = [r_{ij}] \in \mathbf{M}_n = \mathbf{M}_{n,n}$ is called a generalized row substochastic matrix if $\sum_{j=1}^{n} r_{ij} \leq 1$ for all i. For vectors $x, y \in \mathbb{R}_n$, we say that x is *rsgut-majorized* by y, denoted by $x \prec_{rsgut} y$, if $x = yR$ for some n-by-n upper triangular g-row substochastic matrix R.

A linear function $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ preserves an order relation $\prec$ in $\mathbf{M}_{n,m}$, if $TX \prec TY$ whenever $X \prec Y$. Also, T is said to strongly preserve $\prec$ if for all X, $Y \in \mathbf{M}_{n,m}$

$$X \prec Y \Leftrightarrow TX \prec TY.$$

For $X, Y \in \mathbf{M}_{n,m}$, X is said to be sgut-majorized by Y (denoted by $X \prec_{sgut} Y$) if there exists an n-by-n upper triangular g-row substochastic matrix R such that $X = RY$. In [7], the second author characterized all linear preservers and strong linear preservers of $\prec_{sgut}$ on $\mathbb{R}^n$ and $\mathbf{M}_{n,m}$, respectively. In this paper, we study the right case of this relation.

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Some kinds of linear preservers of majorization were mentioned in [1], [4], and [9]. For more information about right and left matrix majorization, see [2], [3], and [5]-[8].

The following conventions will be fixed throughout the paper: The standard basis of $\mathbb{R}_n$ is denoted by $\{e_1, \ldots, e_n\}$. The submatrix of A obtained from A by rows $n_1, \ldots, n_l$ and columns $m_1, \ldots, m_k$ is denoted by $A[n_1, \ldots, n_l|m_1, \ldots, m_k]$. The abbreviation of $A[n_1, \ldots, n_l|n_1, \ldots, n_l]$ is denoted by $A[n_1, \ldots, n_l]$. The summation of all components of a vector x in $\mathbb{R}_n$ is denoted by $tr(x)$. The matrix representation of a linear function $T : \mathbb{R}_n \to \mathbb{R}_n$ with respect to the standard basis is denoted by $[T]$. We have $Tx = x[T]$. The summation of all entries of i^{th} row of $[T]$ is denoted by r_i, also, we will use the following standard function:

$$\text{sgn(x)} = \begin{cases} 1 & x > 0, \\ 0 & x = 0, \\ -1 & x < 0. \end{cases}$$

2. Main results

In this section, we pay attention to the g-row substochastic upper triangular matrices and introduce a new type of majorization. We obtain an equivalent condition for rsgut-majorization on $\mathbb{R}_n$, and we characterize linear functions $T : \mathbb{R}_n \to \mathbb{R}_n$ preserving $\prec_{rsgut}$. Also, we characterize strong linear preservers of rsgut-majorization on $\mathbb{R}_n$.

The following proposition provides a criterion for rsgut-majorization on $\mathbb{R}_n$.

Proposition 2.1. *Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}_n$. Then $x \prec_{rsgut} y$ if and only if $y = 0$, implies that $x = 0$ or $x = (0, \ldots, 0, x_t, \ldots, x_n)$, in which $t = \min\{l : y_l \neq 0\}$ and one of the following conditions holds:*
(i) $y \geq 0$ and $\text{tr(x)} \leq \text{tr(y)}$,
(ii) $y \leq 0$ and $\text{tr(x)} \geq \text{tr(y)}$,
(iii) $y \ngeq 0$ and $y \nleq 0$.

Proof. Let us first prove the sufficiency conditions. Clearly, if $x = y = 0$, then $x \prec_{rsgut} y$. If $x = (0, \ldots, 0, x_t, \ldots, x_n)$, in which $t = \min\{l : y_l \neq 0\}$; assume that (i) or (ii) holds. If $t = n$; set

$$R = (r_{ij}) = \begin{cases} \frac{x_n}{y_n} & (i, j) = (n, n), \\ 0 & \text{otherwise.} \end{cases}$$

If $t < n$; choose

$$R = (r_{ij}) = \begin{cases} 1 & (i, j) = (l, n), \quad t + 1 \leq l \leq n, \\ \frac{x_k}{y_t} & t \leq k \leq n - 1, \\ \frac{x_n - \sum_{j=t+1}^{n} y_j}{y_t} & (i, j) = (t, n), \\ 0 & \text{otherwise.} \end{cases}$$

We see that $x = yR$, where R is g-row substochastic upper triangular matrix that implies $x \prec_{rsgut} y$. Now, assume that (iii) holds. Without loss of generality, suppose that there exists some p $(t < p \leq n)$ such that $y_t > 0 > y_p$. We claim that $x \prec_{rsgut} y$. Put

$$R = (r_{ij}) = \begin{cases} \frac{x_k}{y_t} & (i,j) = (t,k), \quad t \leq k \leq n-1, \\ \leq \min\{1 - \sum_{k=t}^{n-1} r_{tk}, \frac{x_n - y_j}{y_t}\} & (i,j) = (t,n), \\ \frac{x_n - r_{tn} y_t}{y_p} & (i,j) = (p,n), \\ 0 & \text{otherwise.} \end{cases}$$

We observe that $x = yR$, where R is g-row substochastic upper triangular. This implies $x \prec_{rsgut} y$ as claimed.

Next, assume that $x \prec_{rsgut} y$ and $y \neq 0$. We observe that there exists some g-row substochastic upper triangular matrix R such that $x = yR$. Set $t = \min\{l : y_l \neq 0\}$. Then $x = (0, \ldots, 0, x_t, \ldots, x_n)$. If $y \ngeq 0$ and $y \nleq 0$, then (iii) occurs. If not; then $y \geq 0$ or $y \leq 0$.

We observe that $\mathrm{tr(x)} = r_1 y_1 + r_2 y_2 + \cdots + r_n y_n$, in which $r_i = \sum_{j=1}^{n} r_{ij}$ for each i $(1 \leq i \leq n)$. Since $r_i \leq 1$ for each i $(1 \leq i \leq n)$, we have

$$\begin{cases} \mathrm{tr(x)} \leq \mathrm{tr(y)}; & if \quad y \geq 0, \\ \mathrm{tr(x)} \geq \mathrm{tr(y)}; & if \quad y \leq 0. \end{cases}$$

□

Lemma 2.2. *Let $T : \mathbb{R}_n \to \mathbb{R}_n$ be a linear preserver of $\prec_{rsgut}$. Then $[T]$ is an upper triangular matrix.*

Proof. Let $[T] = [a_{ij}]$. The proof is by induction on n. If $n = 1$; there is nothing to prove. Suppose that $n \geq 2$. First, we claim that $a_{21} = \cdots = a_{n1} = 0$. We consider two cases.

Case I. Let $a_{11} = 0$. As $e_n, e_{n-1}, \ldots, e_2 \prec_{rsgut} e_1$, we have $Te_n, \ldots, Te_2 \prec_{rsgut} Te_1$. It follows that $a_{21} = \cdots = a_{n1} = 0$.

Case II. Let $a_{11} \neq 0$. Without loss of generality, we assume that $a_{11} = 1$. Let there be some k $(2 \leq k \leq n)$ such that $a_{k1} \neq 0$. Set $x = e_1$ and $y = e_1 - \frac{1}{a_{k1}} e_k$. We see that $x \prec_{rsgut} y$, and so $Tx \prec_{rsgut} Ty$, which is a contradiction. Hence $a_{k1} = 0$ for each k $(2 \leq k \leq n)$.

In continuing we assume that the matrix representation of every linear preserver of $\prec_{rsgut}$ on $\mathbb{R}_{n-1}$ is an upper triangular matrix. Let $S : \mathbb{R}_{n-1} \to \mathbb{R}_{n-1}$ be the linear function with $[S] = [T][2, 3, \ldots, n]$. Let $x' = (x_2, \ldots, x_n), y' = (y_2, \ldots, y_n) \in \mathbb{R}_{n-1}$ and let $x' \prec_{rsgut} y'$. Then $x := (0, x') \prec_{rsgut} y := (0, y')$ and hence we get $Tx = (0, Sx') \prec_{rsgut} Ty = (0, Sy')$ since $x_1 = 0$ and $a_{21} = \cdots = a_{n1} = 0$. This implies that $Sx' \prec_{rsgut} Sy'$. Therefore, S preserves $\prec_{rsgut}$ on $\mathbb{R}_{n-1}$. The induction hypothesis insures us that $[S]$ is an $(n-1) \times (n-1)$ upper triangular matrix. Therefore, $[T]$ is upper triangular. □

We will use the notation $I_{j,n}^{t \mapsto k}$ for the fact that t is replaced with k in $I_{j,n}$, and r_i is the summation of all entries of i^{th} row of $[T]$.

Theorem 2.3. *Let $T : \mathbb{R}_n \to \mathbb{R}_n$ be a linear function, and let $[T] = [a_{ij}]$. If $a_{11}, \dots, a_{nn} \neq 0$, then T preserves $\prec_{rsgut}$ if and only if $[T]$ is upper triangular, $\mathrm{sgn(a_{11})} = \mathrm{sgn(a_{22})} = \cdots = \mathrm{sgn(a_{nn})}$, the elements on superdiagonal are zero, or have opposite sign of the elements on diagonal, $r_1 = \cdots = r_n$, $D^{n-1}_{n-3} \geq 0$, $I_{0,n} := D^n_{n-2} \geq 0$, $n-3$ conditions of the following hold.*

$$\begin{aligned}
&(1) &:& \quad I_{1,n}(\mathrm{sgn(a_{11})})^1 \geq 0,\\
&(2) &:& \quad I_{2,n}(\mathrm{sgn(a_{11})})^2 \geq 0,\\
&\vdots\\
&(n-3) &:& \quad I_{n-3,n}(\mathrm{sgn(a_{11})})^{\mathrm{n}-3} \geq 0,
\end{aligned}$$

in which

$$\begin{aligned}
I_{j,n} &= a_{n-(j+1)n-(j+1)} I^{n-(j+1)\mapsto n-(j+2)}_{j-1,n} - a_{n-(j+2)n-(j+1)} I_{j-1,n},\\
D^j_i &= a_{ij-1}a_{j-1j} - a_{ij}a_{j-1j-1},
\end{aligned}$$

and if $Te_n = (*, \dots, *, a_{n-in}, 0, \dots, 0, a_{nn})$, where $1 \leq i \leq n-1$, then $a_{11}a_{n-in} \leq 0$.

Proof. We assume $a_{11} = 1$ without loss of generality.

First, we prove the sufficiency of the condition. Let x, $y \in \mathbb{R}_n$, and let $x \prec_{rsgut} y$. We should prove $Tx \prec_{rsgut} Ty$. We know that

$$Tx = (x_1, a_{12}x_1 + a_{22}x_2, \dots, a_{1n}x_1 + a_{2n}x_2 + \cdots + a_{nn}x_n),$$

and

$$Ty = (y_1, a_{12}y_1 + a_{22}y_2, \dots, a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{nn}y_n).$$

We consider three steps.

Step (I). $Ty \ngeq 0$ and $Ty \nleq 0$. If $y_1 \neq 0$, then $Tx \prec_{rsgut} Ty$. If not; we see that x_1 is also zero. If $y_2 \neq 0$, since $a_{22} \neq 0$, then $Tx \prec_{rsgut} Ty$. If $y_2 = 0$, we have $(Tx)_2 = (Ty)_2 = 0$. If $y_3 \neq 0$; by continuing this process, we observe that $Tx \prec_{rsgut} Ty$.

Step (II). $Ty \geq 0$. We claim that $y \geq 0$. If it happens, then $\mathrm{tr(Tx)} = r_n \mathrm{tr(x)}$ and $\mathrm{tr(Ty)} = r_n \mathrm{tr(y)}$. If $y = 0$, then $x = 0$. So, $Tx \prec_{rsgut} Ty$. If $y \neq 0$; in this case we consider two cases $y_1 \neq 0$ and $y_1 = 0$. In the first case, we have $\mathrm{tr(x)} \leq \mathrm{tr(y)}$, and hence $\mathrm{tr(Tx)} \leq \mathrm{tr(Ty)}$. Therefore, as $(Ty)_1 \neq 0$ and $Ty \geq 0$, we conclude that $Tx \prec_{rsgut} Ty$. If $y_1 = 0$, then $x_1 = 0$. Now, we consider two cases $y_2 = 0$ and $y_2 \neq 0$. Similarly, one shows that $Tx \prec_{rsgut} Ty$. Hence it is enough to show that $y \geq 0$.

As $Ty \geq 0$, y_1 is greater than or equal to zero, too. From $(Ty)_2 \geq 0$, $a_{12} \leq 0$, $a_{22} > 0$, and $y_1 \geq 0$, we deduce $y_2 \geq 0$. Note that

$$\begin{aligned}
&(1) : (Ty)_3 \geq 0 &\Leftrightarrow&\quad A_1 := \tfrac{a_{13}}{-a_{33}} y_1 + \tfrac{a_{23}}{-a_{33}} y_2 \leq y_3,\\
&\vdots\\
&(i) : (Ty)_{i+2} \geq 0 &\Leftrightarrow&\quad A_i := \tfrac{a_{1i+2}}{-a_{i+2i+2}} y_1 + \tfrac{a_{2i+2}}{-a_{i+2i+2}} y_2 + \cdots + \tfrac{a_{i+1i+2}}{-a_{i+2i+2}} y_{i+1} \leq y_{i+2},\\
&\vdots\\
&(n-2) : (Ty)_n \geq 0 &\Leftrightarrow&\quad A_{n-2} := \tfrac{a_{1n}}{-a_{nn}} y_1 + \tfrac{a_{2n}}{-a_{nn}} y_2 + \cdots + \tfrac{a_{n-1n}}{-a_{nn}} y_{n-1} \leq y_n.
\end{aligned}$$

If we prove that $A_1, A_2, \ldots, A_{n-2} \geq 0$, then $y_3, \ldots, y_n \geq 0$. Thus, $y \geq 0$.

By induction on n we prove it whenever $n \geq 3$. If $n = 3$; we know that $a_{23} \leq 0$. If $a_{23} = 0$, by the hypothesis, we have $a_{13} \leq 0$. Hence $A_1 \geq 0$. If $a_{23} < 0$ and $A_1 < 0$, then $\frac{a_{13}}{-a_{23}} y_1 > y_2 \geq \frac{a_{12}}{-a_{22}} y_1$. If $y_1 = 0$, we obtain a contradiction. If not; then $y_1 > 0$, and then $a_{13}a_{22} > a_{12}a_{23}$, which is a contradiction. Thus, $A_1 \geq 0$. Now assume that $n > 3$ and $A_1, A_2, \ldots, A_{n-3} \geq 0$. If $A_{n-2} < 0$; it means that $a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{n-1n}y_{n-1} > 0$. By the hypothesis, $a_{n-1n} \leq 0$. So, we should consider two steps:

Step (1). If $a_{n-1n} < 0$, then

$$\begin{aligned} \tfrac{a_{1n}}{-a_{n-1n}} y_1 + \tfrac{a_{2n}}{-a_{n-1n}} y_2 + \cdots + \tfrac{a_{n-2n}}{-a_{n-1n}} y_{n-2} &\geq y_{n-1} \\ &\geq \tfrac{a_{1n-1}}{-a_{n-1n-1}} y_1 + \cdots + \tfrac{a_{n-2n-1}}{-a_{n-1n-1}} y_{n-2}, \end{aligned}$$

which implies

$$I_{0,n}^{n-2\mapsto 1} y_1 + I_{0,n}^{n-2\mapsto 2} y_2 + \cdots + I_{0,n}^{n-2\mapsto n-3} y_{n-3} < -I_{0,n} y_{n-2}.$$

We know $I_{0,n} \geq 0$. We consider two cases.

Case (I). $I_{0,n} = 0$. Then

$$I_{0,n}^{n-2\mapsto 1} y_1 + I_{0,n}^{n-2\mapsto 2} y_2 + \cdots + I_{0,n}^{n-2\mapsto n-4} y_{n-4} < -I_{0,n}^{n-2\mapsto n-3} y_{n-3}.$$

Here we have $I_{1,n} = a_{n-2n-2} I_{0,n}^{n-2\mapsto n-3}, \ldots, I_{1,n}^{n-3\mapsto n-2} = a_{n-2n-2} I_{0,n}^{n-2\mapsto n-4}$, and it follows that

$$I_{1,n}^{n-3\mapsto 1} y_1 + I_{1,n}^{n-3\mapsto 2} y_2 + \cdots + I_{1,n}^{n-3\mapsto n-4} y_{n-4} < -I_{1,n} y_{n-3}.$$

Case (II). $I_{0,n} > 0$. In this case we see that

$$I_{1,n}^{n-3\mapsto 1} y_1 + I_{1,n}^{n-3\mapsto 2} y_2 + \cdots + I_{1,n}^{n-3\mapsto n-4} y_{n-4} < -I_{1,n} y_{n-3}.$$

Similar to the recent procedure, we obtain

$$\begin{gathered} I_{2,n}^{n-4\mapsto 1} y_1 + I_{2,n}^{n-4\mapsto 2} y_2 + \cdots + I_{2,n}^{n-4\mapsto n-5} y_{n-5} < -I_{2,n} y_{n-4}, \\ I_{3,n}^{n-5\mapsto 1} y_1 + I_{3,n}^{n-5\mapsto 2} y_2 + \cdots + I_{3,n}^{n-5\mapsto n-6} y_{n-6} < -I_{3,n} y_{n-5}, \\ \vdots \\ I_{n-4,n}^{2\mapsto 1} y_1 < -I_{n-4,n} y_2. \end{gathered}$$

We know that $I_{n-4,n} \geq 0$.

Case (I). $I_{n-4,n} = 0$. This implies that $I_{n-4,n}^{2\mapsto 1} y_1 < 0$. We see that y_1 can not be zero. So, $y_1 > 0$ and then $I_{n-4,n}^{2\mapsto 1} < 0$. It shows that $I_{n-3,n} < 0$, that is a contradiction.

Case (II). $I_{n-4,n} > 0$. We have

$$\frac{I_{n-4,n}^{2\mapsto 1}}{I_{n-4,n}} y_1 > y_2 \geq \frac{-a_{12}}{a_{22}} y_1.$$

We observe that y_1 can not be zero and so, $y_1 > 0$. Then $\frac{I_{n-4,n}^{2\mapsto 1}}{-I_{n-4,n}} > \frac{-a_{12}}{a_{22}}$, and hence $I_{n-3,n} < 0$, which is a contradiction.

Step (2). $a_{n-1n} = 0$. We have

$$a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{n-2n}y_{n-2} > 0.$$

By the hypothesis, $a_{n-2n} \leq 0$. We consider two cases.

Case (I). $a_{n-2n} < 0$. We see that

$$\begin{aligned} \tfrac{a_{1n}}{-a_{n-2n}} y_1 + \cdots + \tfrac{a_{n-3n}}{-a_{n-2n}} y_{n-3} &> y_{n-2} \\ &\geq \tfrac{a_{1n-2}}{-a_{n-2n-2}} y_1 + \cdots + \tfrac{a_{n-3n-2}}{-a_{n-2n-2}} y_{n-3}, \end{aligned}$$

which implies

$$I_{1,n}^{n-3\mapsto 1} y_1 + I_{1,n}^{n-3\mapsto 2} y_2 + \cdots + I_{1,n}^{n-3\mapsto n-4} y_{n-4} < -I_{1,n} y_{n-3}.$$

We know $I_{1,n} \geq 0$.

Case (1). $I_{1,n} = 0$. Then

$$I_{1,n}^{n-3\mapsto 1} y_1 + I_{1,n}^{n-3\mapsto 2} y_2 + \cdots + I_{1,n}^{n-3\mapsto n-5} y_{n-5} < -I_{1,n}^{n-3\mapsto n-4},$$

which implies

$$I_{2,n}^{n-4\mapsto 1} y_1 + I_{2,n}^{n-4\mapsto 2} y_2 + \cdots + I_{2,n}^{n-4\mapsto n-5} y_{n-5} < -I_{2,n} y_{n-4}.$$

Case (2). $I_{1,n} > 0$. This implies that

$$\begin{aligned} \tfrac{I_{1,n}^{n-3\mapsto 1}}{-I_{1,n}} y_1 + \cdots + \tfrac{I_{1,n}^{n-3\mapsto n-4}}{-I_{1,n}} y_{n-4} &> y_{n-3} \\ &\geq \tfrac{a_{1n-3}}{-a_{n-3n-3}} y_1 + \cdots + \tfrac{a_{n-4n-3}}{-a_{n-3n-3}} y_{n-4}, \end{aligned}$$

which implies

$$I_{2,n}^{n-4\mapsto 1} y_1 + I_{2,n}^{n-4\mapsto 2} y_2 + \cdots + I_{2,n}^{n-4\mapsto n-5} y_{n-5} < -I_{2,n} y_{n-4}.$$

So,

$$I_{2,n}^{n-4\mapsto 1} y_1 + I_{2,n}^{n-4\mapsto 2} y_2 + \cdots + I_{2,n}^{n-4\mapsto n-5} y_{n-5} < -I_{2,n} y_{n-4}.$$

Therefore, in any case

$$I_{2,n}^{n-4\mapsto 1} y_1 + I_{2,n}^{n-4\mapsto 2} y_2 + \cdots + I_{2,n}^{n-4\mapsto n-5} y_{n-5} < -I_{2,n} y_{n-4}.$$

Similarly, it can be proven

$$\begin{aligned} I_{3,n}^{n-5\mapsto 1} y_1 + I_{3,n}^{n-5\mapsto 2} y_2 + \cdots + I_{3,n}^{n-5\mapsto n-6} y_{n-6} &< -I_{3,n} y_{n-5}, \\ I_{4,n}^{n-6\mapsto 1} y_1 + I_{4,n}^{n-6\mapsto 2} y_2 + \cdots + I_{4,n}^{n-6\mapsto n-7} y_{n-7} &< -I_{4,n} y_{n-6}, \\ &\vdots \\ I_{n-4,n}^{2\mapsto 1} y_1 &< -I_{n-4,n} y_2. \end{aligned}$$

As $I_{n-4,n} \geq 0$, we come to a contradiction.

Case (II). $a_{n-2n} = 0$. According to the hypothesis, $a_{n-3n} \leq 0$. In a similar fashion, we can complete the proof. Therefore, $A_{n-2} \geq 0$.

Step (III). $Ty \leq 0$. This step can be proven in the similar manner as the proof of Step (II).

To prove the necessity of the condition, assume that T preserves $\prec_{rsgut}$. Lemma 2.2 ensures that $[T]$ is upper triangular. First, we want to prove $\mathrm{sgn}(\mathrm{a}_{11}) = \mathrm{sgn}(\mathrm{a}_{22}) = \cdots = \mathrm{sgn}(\mathrm{a}_{\mathrm{nn}})$, the elements on superdiagonal are zero, or have opposite sign of the elements on diagonal, and $r_1 = \cdots = r_n$. The proof is by induction on n. If $n = 1$; there is nothing to prove. For $n \geq 2$, assume that the matrix

representation of every linear preserver of $\prec_{rsgut}$ on $\mathbb{R}_{n-1}$ has desirable conditions. Let $S : \mathbb{R}_{n-1} \to \mathbb{R}_{n-1}$ be the linear function with $[S] = [T][2, 3, \ldots, n]$. As T preserves $\prec_{rsgut}$ on $\mathbb{R}_n$, S is also on $\mathbb{R}_{n-1}$. The induction hypothesis insures $\mathrm{sgn}(a_{22}) = \cdots = \mathrm{sgn}(a_{nn})$, the elements on superdiagonal are zero, or have opposite sign of the elements on diagonal, and $r_2 = \cdots = r_n$.
If

$$y = (y_1, y_2, \cdots, y_n), \tag{2.1}$$

in which $y_1 = 1, y_2 = \frac{-a_{12}}{a_{22}}, y_3 = \frac{a_{23}a_{12}-a_{13}a_{22}}{a_{22}a_{33}}, y_i = \frac{I_{i-3,i}}{a_{22}a_{33}\ldots a_{ii}}, \quad \forall i = 4, 5, \ldots, n.$

We claim that $Ty = (1, 0, \ldots, 0)$. We prove it by induction on n. The induction step is easy to prove. Now suppose the statement holds for $n - 1$. We want to prove it for n. We observe that $(Ty)_1 = 1, (Ty)_2 = 0, (Ty)_3 = 0, (Ty)_j = a_{1j} + a_{2j}y_j + \cdots + a_{jj}y_j, \quad \forall j = 4, 5, \ldots, n-1$. By the induction hypothesis, $(Ty)_j = 0$ for each $4 \le j \le n-1$. It is enough to show that $(Ty)_n = 0$. As $(Ty)_2 = \cdots = (Ty)_{n-1} = 0$, we have $\sum_{i=2}^n (Ty)_i = (Ty)_n$. On the other hand,

$$\begin{aligned} \textstyle\sum_{i=2}^n (Ty)_i &= (a_{12} + a_{13} + \cdots + a_{1n}) + (a_{22} + a_{23} + \cdots + a_{2n})y_2 + \cdots + a_{nn}y_n, \\ &= (r_n - 1) + r_n(y_2 + \cdots + y_n). \end{aligned}$$

Hence $(Ty)_n = (r_n - 1) + r_n(y_2 + \cdots + y_n)$. We claim that $y_2 + \cdots + y_n = \frac{1-r_n}{r_n}$, implies $(Ty)_n = 0$. We have $y_2 + \cdots + y_n = \frac{1-r_n}{r_n}$. If we prove $a_{22} > 0$, then $a_{33}, \ldots, a_{nn} > 0, \quad a_{23}, a_{34}, \ldots, a_{n-1n} \le 0$. If $a_{22} \not> 0$; then $a_{22} < 0$.

Case (I). $a_{12} < 0$. Set y as (2.1). We observe that $y \not\ge 0$, $y \not\le 0$, and $y_1 \ne 0$. So, for every $x \in \mathbb{R}_n$ we have $x \prec_{rsgut} y$. Set $x = x_n e_n$ where $x_n < \frac{1}{a_{nn}}$. Then $Tx \not\prec_{rsgut} Ty$, a contradiction.

Case (II). $a_{12} > 0$. Consider y as (2.1). Put $x = x_n e_n$ such that $x_n < \min\{\frac{1}{a_{nn}}, \mathrm{tr(y)}\}$. We conclude $x \prec_{rsgut} y$ and $Tx \not\prec_{rsgut} Ty$, which is a contradiction.

Case (III). $a_{12} = 0$. Choose $y_1 = 1$, $y_2 = -1$, and $y_3, \ldots, y_n$ such that $(Ty)_3 = \cdots = (Ty)_n = 0$. We see that $Ty = (1, -a_{22}, 0, \ldots, 0)$. Consider $x = x_n e_n$, in which $x_n < \frac{1-a_{22}}{a_{nn}}$. We deduce $x \prec_{rsgut} y$ but $Tx \not\prec_{rsgut} Ty$, a contradiction.

Therefore, $a_{22} > 0$. We claim that $a_{12} \le 0$. If $a_{12} > 0$; by choosing y as (2.1) and $x = x_n e_n$ such that $x_n < \frac{1}{a_{nn}}$ we have $x \prec_{rsgut} y$ but $Tx \not\prec_{rsgut} Ty$, which is a contradiction. Thus, $a_{12} \le 0$.

We want to show that $r_1 = r_n$. If $r_1 \ne r_n$;

Step (I). $r_1 < r_n$. Choose y as (2.1) and $x = x_n e_n$. We have $Ty = (1, 0, \ldots, 0)$ and $Tx = (0, \ldots, 0, r_n x_n)$. Note that $1 = \mathrm{tr}(Ty) = r_1 + r_n(y_2 + \cdots + y_n)$, which implies $1 - r_1 = r_n(y_2 + \cdots + y_n)$ and $\mathrm{tr}(y) = y_1 + \frac{1-r_1}{r_n}$.

Choose $\frac{1}{r_n} < x_n \le \mathrm{tr}(y)$. It shows that $x \prec_{rsgut} y$ but $Tx \not\prec_{rsgut} Ty$, that is a contradiction.

Step (II). $r_1 > r_n$. Set y as (2.1) and $x = (\frac{r_n+1}{r_n})e_1 - \frac{r_1}{r_n e_n}$. We conclude $x \prec_{rsgut} y$ but $Tx \not\prec_{rsgut} Ty$, that is a contradiction. Hence $r_1 = r_n$, and so, we have $r_1 = \cdots = r_n$.

We claim that $D^n_{n-2} \ge 0$. If $D^n_{n-2} < 0$; let $S : \mathbb{R}_3 \to \mathbb{R}_3$ be the linear function with $[S] = [T][n-2, n-1, n]$. As T preserves $\prec_{rsgut}$ on $\mathbb{R}_n$, S is also on $\mathbb{R}_3$.

Select $x, y \in \mathbb{R}_3$ such that $x = x_1e_1$, and $y = (1, \frac{-a_{n-2n-1}}{a_{n-1n-1}}, \frac{D^n_{n-2}}{a_{n-1n-1}a_{nn}})$, in which $x_1 > \frac{1}{a_{nn}}$. Observe that $x \prec_{rsgut} y$ but $Sx \not\prec_{rsgut} Sy$ is a contradiction. So, $D^n_{n-2} \geq 0$.

We want to show that $D^{n-1}_{n-3} \geq 0$. If $D^{n-1}_{n-3} < 0$; let $S : \mathbb{R}_4 \to \mathbb{R}_4$ be the linear function with $[S] = [T][n-3, n-2, n-1, n]$. Since T preserves $\prec_{rsgut}$ on $\mathbb{R}_n$, S is also on $\mathbb{R}_4$. By choosing $x, y \in \mathbb{R}_4$ such that $x = x_4e_4$, and $y = (1, \frac{-a_{n-3n-2}}{a_{n-2n-2}}, \frac{D^{n-1}_{n-3}}{a_{n-2n-2}a_{n-1n-1}}, \frac{I_{1,4}}{a_{22}a_{33}a_{44}})$, in which $x_4 > \frac{1}{a_{44}}$, we see that $x \prec_{rsgut} y$ but $Sx \not\prec_{rsgut} Sy$, which is a contradiction.

Now we prove the following $n-3$ conditions:
$(1) : I_{1,n}(\mathrm{sgn}(\mathrm{a}_{11}))^1 \geq 0$,
$(2) : I_{2,n}(\mathrm{sgn}(\mathrm{a}_{11}))^2 \geq 0$,
$\vdots$
$(n-3) : I_{n-3,n}(\mathrm{sgn}(\mathrm{a}_{11}))^{\mathrm{n}-3} \geq 0$.

We see that n must be greater than or equal to four. By induction on n, we prove it. If $n = 4$ and $I_{1,4} < 0$, we consider $x = x_4e_4$, and $y = (1, \frac{-a_{12}}{a_{22}}, \frac{a_{23}a_{12}-a_{13}a_{22}}{a_{22}a_{33}}, \frac{I_{1,4}}{a_{22}a_{33}a_{44}})$, where $x_4 < \frac{1}{a_{44}}$. See $x \prec_{rsgut} y$ but $Tx \not\prec_{rsgut} Ty$, which is a contradiction.

Assume that $n \geq 5$ and the statement holds for $5, 6, \ldots, n-4$. We prove the statement for $n-3$. If not; $I_{n-3,n} < 0$. Choose y as (2.1) and $x = x_ne_n$, where $x_n > \frac{1}{a_{nn}}$. We conclude $x \prec_{rsgut} y$ but $Tx \not\prec_{rsgut} Ty$, a contradiction.

We should prove the following statement now. If $Te_n = (*, \ldots, *, a_{n-in}, 0, \ldots, 0, a_{nn})$, where $1 \leq i \leq n-1$, then $a_{11}a_{n-in} \leq 0$. If $a_{n-in} > 0$; let $S : \mathbb{R}_{i+1} \to \mathbb{R}_{i+1}$ be the linear function with $[S] = [T][n-i, n-(i+1), \ldots, n]$. Since T preserves $\prec_{rsgut}$ on $\mathbb{R}_n$, S is also on $\mathbb{R}_{i+1}$. Set $y_1 = 1$, and choose $y_2, \ldots, y_{i+1}$ such that $Sy = (1, 0, \ldots, 0)$. Choose $x = x_ne_n$, where $x_n > \frac{1}{a_{nn}}$. This implies that $x \prec_{rsgut} y$ but $Tx \not\prec_{rsgut} Ty$, which is a contradiction. □

For each k $(1 \leq k \leq n)$ we define $j_k := \min\{l | k \leq l \leq n, a_{kl} \neq 0, a_{k+1l} = \cdots = a_{nl} = 0\}$.

Theorem 2.4. *Let $T : \mathbb{R}_n \to \mathbb{R}_n$ be a linear function, $n \geq 2$,*

$$[T] = \begin{pmatrix} a_{11} & * & * & * & \ldots & * \\ 0 & a_{22} & * & * & \ldots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & a_{n-1n-1} & a_{n-1n} \\ 0 & 0 & \ldots & 0 & 0 & 0 \end{pmatrix},$$

where $j_1 < j_2 < \cdots < j_{n-2}$, and $a_{n-1n-1}, a_{n-1n} \neq 0$. Then T preserves $\prec_{rsgut}$ if and only if one of the following conditions holds.
(a) $r_1 = \cdots = r_{n-1} = 0$.
(b) $\mathrm{sgn}(\mathrm{a}_{\mathrm{n-1n-1}}) \neq \mathrm{sgn}(\mathrm{a}_{\mathrm{n-1n}})$*, and for each $k = 0, 1, \ldots, n-3$, we have $a_{n-(k+2)n-(k+2)} \ldots a_{n-2n-2}a_{n-1n-1} > 0$ if and only if $I_{k,n} > 0$.*

Proof. It can be assumed without loss of generality that $j_k = k$, for each $1 \leq k \leq n-2$ and $a_{11} = 1$.

First, suppose that T preserves $\prec_{rsgut}$ and at least one of $r_1, \ldots, r_{n-1}$ is nonzero. We proceed by induction on n. Case $n = 2$ is easy to prove. Suppose that $n \geq 3$ and the assertion has been established for all linear preservers of $\prec_{rsgut}$ on $\mathbb{R}_{n-1}$.

Let $S : \mathbb{R}_{n-1} \to \mathbb{R}_{n-1}$ be the linear function with $[S] = [T][2, 3, \ldots, n]$. Since T preserves $\prec_{rsgut}$ on $\mathbb{R}_n$, S is also in $\mathbb{R}_{n-1}$. The induction hypothesis insures that $\mathrm{sgn}(\mathrm{a_{n-1n-1}}) \neq \mathrm{sgn}(\mathrm{a_{n-1n}})$, and $a_{n-(k+2)n-(k+2)} \ldots a_{n-2n-2}a_{n-1n-1} > 0$ if and only if $I_{k,n} > 0$, for each $k = 0, 1, \ldots, n-4$. We should prove $a_{11} \ldots a_{n-2n-2}a_{n-1n-1} > 0$ if and only if $I_{n-3,n} > 0$. If not; we have $(a_{22} \ldots a_{n-1n-1} > 0, I_{n-3,n} \leq 0)$, or $(a_{22} \ldots a_{n-1n-1} < 0, I_{n-3,n} \geq 0)$.
If $a_{22} \ldots a_{n-2n-2}I_{n-3,n} \leq 0$, then we choose y_{n-1} such that $\frac{I_{n-4,n-1}}{a_{22} \ldots a_{n-1n-1}} \leq y_{n-1} \leq \frac{I^*_{n-4,n-1}}{a_{22} \ldots a_{n-2n-2}a_{n-1n}}$. If $a_{22} \ldots a_{n-2n-2}I_{n-3,n} \geq 0$, then choose y_{n-1} such that $\frac{I_{n-4,n-1}}{a_{22} \ldots a_{n-1n-1}} \geq y_{n-1} \geq \frac{I^*_{n-4,n-1}}{a_{22} \ldots a_{n-2n-2}a_{n-1n}}$, where $I^*_{n-4,n-1} = I_{n-4,n}^{a_{1,n-1} \mapsto a_{1n}, a_{2,n-1} \mapsto a_{2n}, \ldots, a_{n-2,n-1} \mapsto a_{n-2n}}$.
Set $y_1 = 1, y_2 = \frac{-a_{12}}{a_{22}}, y_3 = \frac{a_{23}a_{12} - a_{13}a_{22}}{a_{22}a_{33}}, y_i = \frac{I_{i-3,i}}{a_{22}a_{33} \ldots a_{ii}}, \forall i = 4, \ldots, n-2, y_{n-1} = -1, y_n = -1$. We see $Ty = (1, 0, \ldots, 0, (Ty)_{n-1}, (Ty)_n)$, $(Ty)_{n-1}, (Ty)_n \geq 0$, and for each $x \in \mathbb{R}_n$, we have $x \prec_{rsgut} y$. Observe that $\mathrm{tr(Tx)} = r_1x_1 + \cdots + r_{n-1}x_{n-1}$, and $\mathrm{tr(Ty)} = r_1y_1 + \cdots + r_{n-1}y_{n-1}$. As at least one of the $r_1, \ldots, r_{n-1}$ is nonzero, we can select x such that $\mathrm{tr(Tx)} > \mathrm{tr(Ty)}$. For this x we see that $x \prec_{rsgut} y$ and $Tx \not\prec_{rsgut} Ty$, which is a contradiction. Thus, $a_{11} \ldots a_{n-2n-2}a_{n-1n-1} > 0$ if and only if $I_{n-3,n} > 0$.

Next, we want to prove T preserves $\prec_{rsgut}$. Let $x, y \in \mathbb{R}_n$ and let $x \prec_{rsgut} y$. If $r_1 = \cdots = r_{n-1} = 0$, as $\mathrm{tr(Tx)} = \mathrm{tr(Ty)} = 0$ and $a_{22}, \ldots, a_{n-1n-1} \neq 0$, then $Tx \prec_{rsgut} Ty$. Now suppose that at least one of $r_1, \ldots, r_{n-1}$ is nonzero. By induction on n, we prove $Tx \prec_{rsgut} Ty$. If $n = 2$, it is easy to see. Let $n \geq 3$ and our claim has been proved for $n-1$. We have $Tx = (x_1, a_{12}x_1 + a_{22}x_2, \ldots, a_{1n}x_1 + a_{2n}x_2 + \cdots + a_{n-1n}x_{n-1})$, and $Ty = (y_1, a_{12}y_1 + a_{22}y_2, \ldots, a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{n-1n}y_{n-1})$. If $y_1 = 0$, then, since $x_1 = 0$, the induction hypothesis insures that $Tx \prec_{rsgut} Ty$. If $y_1 \neq 0$; consider three following steps. We claim that only the first step happens.

Step (I). $Ty \not\geq 0$ and $Ty \not\leq 0$. Clearly, $Tx \prec_{rsgut} Ty$.

Step (II). $Ty \geq 0$.
We can assume, without loss of generality, that $a_{11}, a_{22}, \ldots, a_{n-1n-1} > 0$ and $a_{n-1n} < 0$. As $Ty \geq 0$, conclude that
$(1) : y_1 > 0$,
$(2) : \frac{a_{12}}{-a_{22}}y_1 \leq y_2$,
$\vdots$
$(i) : \frac{a_{1i}}{-a_{ii}}y_1 + \frac{a_{2i}}{-a_{ii}}y_2 + \cdots + \frac{a_{i-1i}}{-a_{ii}}y_{i-1} \leq y_i$,
$\vdots$
$(n-1) : \frac{a_{1n-1}}{-a_{n-1n-1}}y_1 + \frac{a_{2n-1}}{-a_{n-1n-1}}y_2 + \cdots + \frac{a_{n-2n-1}}{-a_{n-1n-1}}y_{n-2} \leq y_{n-1}$,
$(n) : \frac{a_{1n}}{-a_{n-1n}}y_1 + \frac{a_{2n}}{-a_{n-1n}}y_2 + \cdots + \frac{a_{n-2n}}{-a_{n-1n}}y_{n-2} \leq y_{n-2}$.
From $(n-1)$ and (n) deduce that

$$\left(\frac{a_{1n-1}}{-a_{n-1n-1}} + \frac{a_{1n}}{a_{n-1n}}\right)y_1 + \left(\frac{a_{2n-1}}{-a_{n-1n-1}} + \frac{a_{2n}}{a_{n-1n}}\right)y_2 + \cdots + \left(\frac{a_{n-3n-1}}{-a_{n-1n-1}} + \frac{a_{n-3n}}{a_{n-1n}}\right)y_{n-3} \leq$$

$(\frac{a_{n-2n}}{-a_{n-1n}}+\frac{a_{n-2n-1}}{a_{n-1n-1}})y_{n-2}$, which implies $\frac{-I_{1,n}^{n-3\mapsto 1}}{I_{1,n}}y_1+\frac{-I_{1,n}^{n-3\mapsto 2}}{I_{1,n}}y_2+\cdots+\frac{-I_{1,n}^{n-3\mapsto n-4}}{I_{1,n}}y_{n-4}\geq y_{n-3}$. By continuing this process, we obtain $\frac{-I_{n-4,n}^{2\mapsto 1}}{I_{n-4,n}}y_1\geq y_2\geq\frac{a_{12}}{-a_{22}}y_1$. It shows that $a_{22}I_{n-4,n}^{2\mapsto 1}-a_{12}I_{n-4,n}\leq 0$ and then $I_{n-3,n}\leq 0$, a contradiction. So, this does not happen.

Step (III). $Ty\leq 0$. Similar Step (II) it is proved that this step can not happen. So, Step (I) just happens. Therefore, T preserves $\prec_{rsgut}$. □

Theorem 2.5. *Let $T:\mathbb{R}_n\to\mathbb{R}_n$ be a linear function, $n\geq 3$ and*

$$[T]=\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n-2} & a_{1n-1} & a_{1n}\\ 0 & a_{22} & a_{23} & \dots & a_{2n-2} & a_{2n-1} & a_{2n}\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & 0 & a_{n-2n-2} & a_{n-2n-1} & a_{n-2n}\\ 0 & 0 & 0 & \dots & 0 & 0 & 0\\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix},$$

$i_1=\min\{l|n-2\leq l\leq n, a_{n-2l}\neq 0\}$,

$i_2=\min\{l|i_1<l\leq n, a_{n-2l}\neq 0, \mathrm{sgn(a_{n-2i_1})}\neq\mathrm{sgn(a_{n-2l})}\}$,

$i_3=\{l|n-2\leq l\leq n, a_{n-2l}=0\}$,

$j_1<j_2<\cdots<j_{n-3}<i_1$,

$J_{n-3,n}=a_{n-3i_1}a_{n-2i_2}-a_{n-2i_1}a_{n-3i_2}$,

$J_{k,n}=a_{k+1j_{k+1}}J_{k+1,n}^{k+1\mapsto k}-a_{kj_{k+1}}J_{k+1,n}$, $\forall\ 1\leq k\leq n-4$.

$(*)$ *If for some p the set*

$$\{l|j_p<l\leq n, a_{pl}\neq 0, a_{p+1l}=\cdots=a_{nl}=0, \mathrm{sgn(a_{pj_p})}=\mathrm{sgn(a_{pl})}\}$$

is nonempty, namely, $\{w_1,\dots,w_s\}$, then $J_{p-1,n}^{w_i}:=J_{p-1,n}^{j_p\mapsto w_i}$, $\forall\ 1\leq i\leq s$.

Then, T preserves $\prec_{rsgut}$ if and only if one of the following conditions occurs.
(a) $r_1=\cdots=r_{n-2}=0$.

(b) $\mathrm{sgn(J_{n-3})}=\mathrm{sgn(a_{n-3j_{n-3}}a_{n-2i_1})}$, *and for each $k=1,2,\dots,n-4$, if $(*)$ holds,* $\mathrm{sgn(J_{k,n})},\dots,$ *or* $\mathrm{sgn(J_{k,n}^{w_s})}$ *is equal to* $\mathrm{sgn(a_{kj_k}a_{k+1j_{k+1}}\dots a_{n-3j_{n-3}}a_{n-2i_1})}$, *if $(*)$ does not hold,* $\mathrm{sgn(J_{k,n})}=\mathrm{sgn(a_{kj_k}a_{k+1j_{k+1}}\dots a_{n-3j_{n-3}}a_{n-2i_1})}$.

Proof. First, we prove the suficiency of the condition, by induction on n. Case $n=3$ is easy to see. Now suppose that $n\geq 4$. Let $x,y\in\mathbb{R}_n$ such that $x\prec_{rsgut}y$.

We want to prove $Tx\prec_{rsgut}Ty$. If $y_1=0$, then x_1 is zero, too. The induction hypothesis ensures that $Tx\prec_{rsgut}Ty$. If $y_1\neq 0$, we have three following cases.

Case (I). $Ty\ngeq 0$ and $Ty\nleq 0$, we see that $Tx\prec_{rsgut}Ty$.

Case (II). $Ty\geq 0$, we know that $a_{1j_1},a_{2j_2},\dots,a_{n-3j_{n-3}}$ can be positive or negative also, we have $a_{n-2i_1}a_{n-2i_2}<0$. Consider without loss of generality

$a_{1j_1}, a_{2j_2}, \ldots, a_{n-3j_{n-3}} > 0$, and $a_{n-2i_1} > 0 > a_{n-2i_2}$. Since $Ty \geq 0$, we have

$$\begin{array}{lll} (1) & : & a_{1j_1}y_1 \geq 0, \\ (2) & : & a_{1j_2}y_1 + a_{2j_2}y_2 \geq 0, \\ \vdots & & \\ (n-3) & : & a_{1j_{n-3}}y_1 + a_{2j_{n-3}}y_2 + \cdots + a_{n-3j_{n-3}}y_{n-3} \geq 0, \\ (i_1) & : & a_{1i_1}y_1 + a_{2i_1}y_2 + \cdots + a_{n-3i_1}y_{n-3} + a_{n-2i_1}y_{n-2} \geq 0, \\ (i_2) & : & a_{1i_2}y_1 + a_{2i_2}y_2 + \cdots + a_{n-3i_2}y_{n-3} + a_{n-2i_2}y_{n-2} \geq 0. \end{array}$$

First, assume that $(*)$ does not hold. From (i_1) and (i_2), we conclude that

$$-J_{n-3}^{n-3\mapsto 1}y_1 + -J_{n-3}^{n-3\mapsto 2}y_2 + \cdots + -J_{n-3}^{n-3\mapsto n-4}y_{n-4} \geq J_{n-3}y_{n-3},$$

which implies

$$\frac{J_{n-4}^{n-4\mapsto 1}}{J_{n-4}}y_1 + \frac{J_{n-4}^{n-4\mapsto 2}}{J_{n-4}}y_2 + \cdots + \frac{J_{n-4}^{n-4\mapsto n-5}}{J_{n-4}}y_{n-5} \geq y_{n-4}.$$

By continuing this process we observe that $\frac{-J_{2,n}^{2\mapsto 1}}{J_{2,n}}y_1 \geq y_2 \geq \frac{a_{1j}}{-a_{2j}}y_1$. Therefore, we get $a_{1j_1}J_{2,n} - a_{2j_2}J_{2,n}^{2\mapsto 1} \leq 0$, which implies $J_{1,n} \leq 0$, that is a contradiction. So, in this case $Ty \geq 0$ can not be happen. If $(*)$ holds, then since $\mathrm{sgn}(\mathrm{a_{pj_p}}) = \mathrm{sgn}(\mathrm{a_{pw_i}})$ for each $1 \leq i \leq s$, the claim can be proved in a similar fashion.

Case (III). $Ty \leq 0$. Similar to Case (II) one can prove it.

For the converse, assume that T preserves $\prec_{rsgut}$. If $r_1 = \cdots = r_{n-2} = 0$, there is nothing to prove. Otherwise, at least one of them is nonzero. We proceed by induction on n. Case $n = 3$ is easy to see. Suppose that $n \geq 4$ and that the assertion has been established for all linear preservers of $\prec_{rsgut}$ on $\mathbb{R}_{n-1}$. Let $S : \mathbb{R}_{n-1} \to \mathbb{R}_{n-1}$ be the linear function with $[S] = [T][2, 3, \ldots, n]$. Since T preserves $\prec_{rsgut}$ on $\mathbb{R}_n$, we see that S preserves $\prec_{rsgut}$ on $\mathbb{R}_{n-1}$. We should just to prove the statement for $\mathrm{sgn}(\mathrm{J_{1,n}})$.

Case (I). If $(*)$ does not hold. We prove $\mathrm{sgn}(\mathrm{J_{1,n}}) = \mathrm{sgn}(\mathrm{a_{1j_1} \ldots a_{n-3j_{n-3}} a_{n-2i_1}})$. Without loss of generality, $a_{1j_1}, a_{2j_2}, \cdots, a_{n-3j_{n-3}} > 0$, and $a_{n-2i_1} > 0 > a_{n-2i_2}$. So, we should prove $\mathrm{sgn}(\mathrm{J_{1,n}}) > 0$. If not; $\mathrm{sgn}(\mathrm{J_{1,n}}) \leq 0$. It shows that $a_{2j_2}J_{2,n}^{2\mapsto 1} - a_{1j_2}J_{2,n} \leq 0$, which implies $\frac{a_{1i_1}y_1 + a_{2i_2}y_2 + \cdots + a_{n-3i_1}y_{n-3}}{-a_{2j_2}a_{n-2i_1}} \leq \frac{a_{1i_2}y_1 + a_{2i_2}y_2 + \cdots + a_{n-3i_2}y_{n-3}}{-a_{2j_2}a_{n-2i_2}}$. Choose y_{n-2} such that $\frac{a_{1i_1}y_1 + a_{2i_2}y_2 + \cdots + a_{n-3i_1}y_{n-3}}{-a_{2j_2}a_{n-2i_1}} \leq y_{n-2} \leq \frac{a_{1i_2}y_1 + a_{2i_2}y_2 + \cdots + a_{n-3i_2}y_{n-3}}{-a_{2j_2}a_{n-2i_2}}$. So, $(Ty)_{i_1}, (Ty)_{i_2} \geq 0$. Set $y_1 = 1, y_{n-1} = -1, y_n = 0$. One can select $y_2, \ldots, y_{n-3}$ such that $Ty \geq 0$. Since at least one of $r_1, \ldots, r_{n-2}$ is nonzero, we can choose x such that $x \prec_{rsgut} y$ but $Tx \not\prec_{rsgut} Ty$, which is a contradiction. Therefore, $J_{1,n} > 0$.

Case (II). If $(*)$ holds. The proof is similar to the first case. $\square$

Theorem 2.6. *Let $T:\mathbb{R}_n\to\mathbb{R}_n$ be a linear function, $n\geq 3$ and*

$$[T]=\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n-3} & a_{1n-2} & a_{1n-1} & a_{1n}\\ 0 & a_{22} & a_{23} & \dots & a_{2n-3} & a_{2n-2} & a_{2n-1} & a_{2n}\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & 0 & a_{n-3n-3} & a_{n-3n-2} & a_{n-3n-1} & a_{n-3n}\\ 0 & 0 & \dots & 0 & 0 & a_{n-2n-2} & a_{n-2n-1} & a_{n-2n}\\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{aligned} i_1 &= \min\{l|n-2\leq l\leq n, a_{n-2l}\neq 0\},\\ i_2 &= \min\{l|i_1<l\leq n, a_{n-2l}\neq 0, \mathrm{sgn(a_{n-2i_1})}\neq \mathrm{sgn(a_{n-2l})}\},\\ i_3 &= \{l|n-2\leq l\leq n, a_{n-2l}=0\}, \end{aligned}$$

$$\mathrm{sgn(a_{n-3j_{n-3}})}\neq \mathrm{sgn(a_{n-3i_3})},\quad \mathrm{j_1<j_2<\dots<j_{n-3}<i_1},$$

$$\begin{aligned} J_{n-3,n} &= a_{n-3i_1}a_{n-2i_2}-a_{n-2i_1}a_{n-3i_2},\\ J_{n-(k+1),n} &= a_{n-kj_{n-k}}J_{n-k,n}^{n-k\mapsto n-(k+1)}-a_{n-(k+1)j_{n-k}}J_{n-k,n},\quad \forall k=3,4,\dots,n-2,\\ T_{n-4,n} &= a_{n-4i_3}a_{j_{n-3}j_{n-3}}-a_{n-4j_{n-3}}a_{j_{n-3}i_3},\\ T_{n-(k+1),n} &= a_{n-kj_{n-k}}T_{n-k,n}^{n-k\mapsto n-(k+1)}-a_{n-(k+1)j_{n-k}}T_{n-k,n},\quad \forall k=4,\dots,n-2. \end{aligned}$$

If one of the following holds, then T preserves $\prec_{rsgut}$.
(*a*) $r_1=r_2=\dots=r_n=0$.
(*b*) $a_{1j_1}a_{2j_2}\dots a_{n-3j_{n-3}}a_{n-2i_1}J_{1,n}>0,\dots,a_{n-4j_{n-4}}a_{n-3j_{n-3}}a_{n-2i_1}J_{n-4,n}>0$, $a_{n-3j_{n-3}}a_{n-2i_1}J_{n-3,n}\geq 0$.
(*c*) $a_{1j_1}a_{2j_2}\dots a_{n-3j_{n-3}}a_{n-2i_1}J_{1,n}^{j_{n-3}\mapsto n-2}<0,\dots,a_{n-4j_{n-4}}a_{n-3j_{n-3}}a_{n-2i_1}$ $J_{n-4,n}^{j_{n-3}\mapsto n-2}<0, a_{n-3j_{n-3}}a_{n-2i_1}J_{n-3,n}\leq 0$.
(*d*) $a_{1j_1}a_{2j_2}\dots a_{n-4j_{n-4}}a_{n-3i_3}T_{1,n}>0, a_{2j_2}\dots a_{n-4j_{n-4}}a_{n-3i_3}T_{2,n}>0,\dots,$ $a_{n-4j_{n-4}}a_{n-3i_3}T_{n-4,n}>0$.

Proof. If (a) holds, then T clearly preserves $\prec_{rsgut}$. Assume that $x\prec_{rsgut}y$. We should prove that $Tx\prec_{rsgut}Ty$. If $n=3$; then the result is trivial. We proceed by induction on n. Suppose that $n\geq 4$ and the assertion has been established for all linear preservers of $\prec_{rsgut}$ on $\mathbb{R}^{n-1}$.

We can assume without loss of generality that $j_k=k$, $\forall\ 1\leq k\leq n-3$, $a_{11},\dots,a_{n-3n-3}>0$, $i_1=n-2$, $i_2=n-1$, $i_3=n$, $a_{n-2n-2}>0>a_{n-2n-1}$, $a_{n-3n-3}>0>a_{n-3n}$, $a_{n-2n}=0$.

We claim that $Ty\geq 0$ or $Ty\leq 0$ can not be happen. So, $Ty\ngeq 0$ and $Ty\nleq 0$ imply that $Tx\prec_{rsgut}Ty$. Suppose that $Ty\geq 0$.

If $y_1 = 0$, by the induction hypothesis $Tx \prec_{rsgut} Ty$. Let $y_1 \neq 0$. We have

$$\begin{array}{lll} (1) & : & y_1 > 0, \\ (2) & : & \frac{a_{12}}{-a_{22}} y_1 \leq y_2, \\ (3) & : & \frac{a_{13}}{-a_{33}} y_1 + \frac{a_{23}}{-a_{33}} y_2 \leq y_3, \\ \vdots & & \\ (n-4) & : & \frac{a_{1n-4}}{-a_{n-4n-4}} y_1 + \frac{a_{2n-4}}{-a_{n-4n-4}} y_2 + \cdots + \frac{a_{n-5n-4}}{-a_{n-4n-4}} y_{n-5} \leq y_{n-4}, \\ (n-3) & : & \frac{a_{1n-3}}{-a_{n-3n-3}} y_1 + \frac{a_{2n-3}}{-a_{n-3n-3}} y_2 + \cdots + \frac{a_{n-4n-3}}{-a_{n-3n-3}} y_{n-4} \leq y_{n-3}, \\ (n-2) & : & \frac{a_{1n-2}}{-a_{n-2n-2}} y_1 + \frac{a_{2n-2}}{-a_{n-2n-2}} y_2 + \cdots + \frac{a_{n-3n-2}}{-a_{n-2n-2}} y_{n-3} \geq y_{n-2}, \\ (n-1) & : & \frac{a_{1n-1}}{-a_{n-2n-1}} y_1 + \frac{a_{2n-1}}{-a_{n-2n-1}} y_2 + \cdots + \frac{a_{n-3n-1}}{-a_{n-2n-1}} y_{n-3} \geq y_{n-2}, \\ (n) & : & \frac{a_{1n}}{-a_{n-3n}} y_1 + \frac{a_{2n}}{-a_{n-3n}} y_2 + \cdots + \frac{a_{n-4n}}{-a_{n-3n}} y_{n-4} \geq y_{n-3}. \end{array}$$

If (b) happens; by using the relation $(n-2)$ and $(n-1)$, we conclude $J_{n-3,n}^{n-3 \mapsto 1} y_1 + J_{n-3,n}^{n-3 \mapsto 2} y_2 + \cdots + J_{n-3,n}^{n-3 \mapsto n-4} y_{n-4} \leq -J_{n-3,n} y_{n-3}$. Also, we know that $J_{n-3,n} \geq 0$. First consider $J_{n-3,n} > 0$ and observe that $\frac{J_{n-3,n}^{n-3 \mapsto 1}}{-J_{n-3,n}} y_1 + \cdots + \frac{J_{n-3,n}^{n-3 \mapsto n-4}}{-J_{n-3,n}} y_{n-4} \geq y_{n-3}$. By using $(n-3)$, we get $J_{n-4,n}^{n-4 \mapsto 1} y_1 + J_{n-4,n}^{n-4 \mapsto 2} y_2 + \cdots + J_{n-4,n}^{n-4 \mapsto n-5} y_{n-5} \leq -J_{n-4,n} y_{n-4}$, and as $J_{n-4,n} > 0$, we have $\frac{J_{n-4,n}^{n-4 \mapsto 1}}{-J_{n-4,n}} y_1 + \cdots + \frac{J_{n-4,n}^{n-4 \mapsto n-5}}{-J_{n-4,n}} y_{n-5} \geq y_{n-4}$. By using $(n-4)$ and continuing this process, we deduce $J_{1,n} \leq 0$, which is a contradiction. If $J_{n-3,n} = 0$, we have $J_{n-3,n}^{n-3 \mapsto 1} y_1 + J_{n-3,n}^{n-3 \mapsto 2} y_2 + \cdots + J_{n-3,n}^{n-3 \mapsto n-5} y_{n-5} \leq -J_{n-3,n}^{n-3 \mapsto n-4} y_{n-4}$. By multiply this relation with $a_{n-3n-3} > 0$, we get $J_{n-4,n}^{n-4 \mapsto 1} y_1 + J_{n-4,n}^{n-4 \mapsto 2} y_2 + \cdots + J_{n-4,n}^{n-4 \mapsto n-5} y_{n-5} \leq -J_{n-4,n} y_{n-4}$, a contradiction. Therefore, in this case, $Ty \geq 0$ can not be happen.

If (c) holds; from $(n-2)$ and $(n-1)$ we deduce $\frac{J_{n-3,n}^{n-3 \mapsto 1}}{-J_{n-3,n}} y_1 + \cdots + \frac{J_{n-3,n}^{n-3 \mapsto n-4}}{-J_{n-3,n}} y_{n-4} \leq y_{n-3}$, whenever $J_{n-3,n} < 0$. Now, if use (n) and continue as before, we obtain some contradiction. If $J_{n-3,n} = 0$, similarly, one can prove.

If (d) holds; from (n) and $(n-3)$ conclude $\frac{T_{n-4,n}^{n-4 \mapsto 1}}{-T_{n-4,n}} y_1 + \cdots + \frac{T_{n-4,n}^{n-4 \mapsto n-5}}{-T_{n-4,n}} y_{n-5} \leq y_{n-4}$. By continuing the same process, we have that $T_{1,n} \geq 0$, a contradiction. Thus, $Ty \geq 0$ can not be happen, too. Similarly, the case $Ty \leq 0$ can be proved. □

Theorem 2.7. *Let $T : \mathbb{R}_n \to \mathbb{R}_n$ be a linear function, $n \geq 4$ and*

$$[T] = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n-3} & a_{1n-2} & a_{1n-1} & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n-3} & a_{2n-2} & a_{2n-1} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & a_{n-3n-3} & a_{n-3n-2} & a_{n-3n-1} & a_{n-3n} \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{pmatrix},$$

$i_1 = \min\{l | n-3 \leq l \leq n, a_{n-3l} \neq 0\}$,

$i_2 = \min\{l | i_1 < l \leq n, a_{n-3l} \neq 0, \mathrm{sgn}(\mathrm{a}_{n-3i_1}) \neq \mathrm{sgn}(\mathrm{a}_{n-3l})\}$,

$\{l_1, l_2\} = \{n-3, n-2, n-1, n\} \setminus \{i_1, i_2\}, \text{where} \ \ \mathrm{l_1 < l_2},$

$j_1 < j_2 < \ldots < j_{n-4} < i_1,$

$$\begin{aligned} I^{i,j}_{n-4,n} &= a_{n-4i}a_{n-3j} - a_{n-4j}a_{n-3i}, \\ I^{i,j}_{n-(k+1),n} &= a_{n-kj_{n-k}} I^{i,jn-k \mapsto n-(k+1)}_{n-k,n} - a_{n-(k+1)j_{n-k}} I^{i,j}_{n-k,n}, \quad \forall k = 4, 5, \ldots, n-2. \end{aligned}$$

Then T preserves $\prec_{rsgut}$ if one of the following holds.
(a) $r_1 = r_2 = \cdots = r_n = 0.$
(b)
• *If* $\mathrm{sgn}(\mathrm{a_{n-3l_1}}) \neq \mathrm{sgn}(\mathrm{a_{n-3l_2}})$, *one of the following holds.*
(1)$a_{1j_1}a_{2j_2}\ldots a_{n-4j_{n-4}}a_{n-3i_1}I^{i_1,i_2}_{1,n} > 0, \ldots, a_{n-4j_{n-4}}a_{n-3i_1}I^{i_1,i_2}_{n-4,n} > 0.$

(2) $a_{1j_1}a_{2j_2}\ldots a_{n-4j_{n-4}}a_{n-3i_1}I^{l_1,l_2}_{1,n} > 0, \ldots, a_{n-4j_{n-4}}a_{n-3i_1}I^{l_1,l_2}_{n-4,n} > 0.$

(3) *If* $a_{n-3i_1}a_{n-3l_i} < 0,$

$a_{1j_1}a_{2j_2}\ldots a_{n-4j_{n-4}}a_{n-3i_1}I^{i_1,l_i}_{1,n} > 0, \ldots, a_{n-4j_{n-4}}a_{n-3i_1}I^{i_1,l_i}_{n-4,n} > 0.$

(4) *If* $a_{n-3i_2}a_{n-3l_i} < 0,$

$a_{1j_1}a_{2j_2}\ldots a_{n-4j_{n-4}}a_{n-3i_1}I^{i_2,l_i}_{1,n} > 0, \ldots, a_{n-4j_{n-4}}a_{n-3i_1}I^{i_2,l_i}_{n-4,n} > 0.$

• *If* $\mathrm{sgn}(\mathrm{a_{n-3l_1}}) = \mathrm{sgn}(\mathrm{a_{n-3l_2}}),$

$a_{1j_1}a_{2j_2}\ldots a_{n-4j_{n-4}}a_{n-3i_k}I^{i_k,t}_{1,n} > 0, \ldots, a_{n-4j_{n-4}}a_{n-3i_k}I^{i_k,t}_{n-4,n} > 0,$

where $\mathrm{sgn}(\mathrm{a_{n-3i_k}}) \neq \mathrm{sgn}(\mathrm{a_{n-3l_1}})$ *and* $t = \{l_1, l_2, i_1, i_2\} \setminus \{i_k\}.$

• *If* $a_{n-3l_1} = a_{n-3l_2} = 0$; *one of the following holds.*

$n - t_0 = \max\{t | 1 \leq t \leq n-4, a_{tn-1} or a_{tn} \neq 0\}$, $t_1 = \{l_i | a_{n-t_0 l_i} > 0\},$

$t_2 = \{l_i | a_{n-t_0 l_i} = 0\}$, $a_{n-t_0 j_{n-t_0}}a_{n-t_0 l_1}, a_{n-t_0 j_{n-t_0}}a_{n-t_0 l_2} \geq 0$, *$t_2$ may be empty.*

(1) $a_{1j_1}a_{2j_2}\ldots a_{n-t_0 j_{n-t_0}}a_{n-3i_1}I^{i_1,i_2 j_{n-t_0} \mapsto t_1}_{1,n} > 0, \ldots, a_{n-(t_0+1)j_{n-(t_0+1)}}a_{n-t_0 j_{n-t_0}}$

$a_{n-3i_1}I^{i_1,i_2 j_{n-t_0} \mapsto t_1}_{n-(t_0+1),n} > 0$, $a_{n-t_0 j_{n-t_0}}a_{n-3i_1}I^{i_1,i_2}_{n-t_0,n} > 0.$

(2) $a_{1j_1}a_{2j_2}\ldots a_{n-t_0 j_{n-t_0}}a_{n-3i_1}I^{i_1,i_2 j_{n-k} \mapsto t_2}_{1,n} > 0, \ldots, a_{n-(k+1)j_{n-(k+1)}}\ldots a_{n-t_0 j_{n-t_0}}$

$a_{n-3i_1}I^{i_1,i_2 j_{n-k} \mapsto t_2}_{n-(k+1),n} > 0, a_{n-kj_{n-k}}\ldots a_{n-t_0 j_{n-t_0}}a_{n-3i_1}I^{i_1,i_2}_{n-k,n} > 0, \ldots, a_{n-t_0 j_{n-t_0}}$

$a_{n-3i_1}I^{i_1,i_2}_{n-t_0,n} > 0.$

(3) $a_{1j_1}a_{2j_2}\ldots a_{n-t_0 j_{n-t_0}}a_{n-3i_1}I^{i_1,i_2}_{1,n} > 0, \ldots, a_{n-t_0 j_{n-t_0}}a_{n-3i_1}I^{i_1,i_2}_{n-t_0,n} > 0.$

(4) $a_{1j_1}a_{2j_2}\ldots a_{n-t_0 j_{n-t_0}}a_{n-3i_1}I^{i_1,i_2 j_{n-t_0} \mapsto t_1 j_{n-k} \mapsto t_2}_{1,n} > 0, \ldots, a_{n-(k+1)j_{n-(k+1)}}\ldots$

$a_{n-t_0 j_{n-t_0}}a_{n-3i_1}I^{i_1,i_2 j_{n-t_0} \mapsto t_1 j_{n-k} \mapsto t_2}_{n-(k+1),n} > 0, a_{n-kj_{n-k}}\ldots a_{n-t_0 j_{n-t_0}}a_{n-3i_1}$

$I^{i_1,i_2 j_{n-t_0} \mapsto t_1}_{n-k,n} > 0, \ldots, a_{n-(t_0+1)j_{n-(t_0+1)}}a_{n-t_0 j_{n-t_0}}a_{n-3i_1}I^{i_1,i_2 j_{n-t_0} \mapsto t_1}_{n-(t_0+1),n} > 0,$

$a_{n-t_0 j_{n-t_0}}a_{n-3i_1}I^{i_1,i_2}_{n-t_0,n} > 0.$

• *If* $a_{n-3l_1} = 0, a_{n-3l_2} \neq 0$, *one of the following holds.*

$n-k = \max\{t|1 \leq t \leq n-4, a_{tl_1} \neq 0, a_{t+1l_1} = \cdots = a_{n-3l_1=0}\}$, $a_{n-kj_{n-k}}a_{n-kl_1} \geq 0$,

(1) $a_{1j_1}a_{2j_2}\ldots a_{n-4_0j_{n-4}}a_{n-3i_1}I^{i_1,i_2}_{1,n} > 0, \ldots, a_{n-4j_{n-4}}a_{n-3i_1}I^{i_1,i_2}_{n-4,n} > 0$.

(2) $a_{1j_1}a_{2j_2}\ldots a_{n-4j_{n-4}}a_{n-3i_1}I^{i_1,i_2j_{n-k}\mapsto l_1}_{1,n} > 0, \ldots, a_{n-(k+1)j_{n-(k+1)}}\ldots a_{n-4j_{n-4}}$
$a_{n-3i_1}I^{i_1,i_2j_{n-k}\mapsto l_1}_{n-(k+1),n} > 0, a_{n-kj_{n-k}}\ldots a_{n-4j_{n-4}}a_{n-3i_1}I^{i_1,i_2}_{n-k,n} > 0, \ldots, a_{n-4j_{n-4}}a_{n-3i_1}$
$I^{i_1,i_2}_{n-4,n} > 0$.

(3) *If* $a_{n-3i_1}a_{n-3l_2} > 0$,
$a_{1j_1}a_{2j_2}\ldots a_{n-4_0j_{n-4}}a_{n-3i_1}I^{i_2,l_2}_{1,n} < 0, \ldots, a_{n-4j_{n-4}}a_{n-3i_1}I^{i_2,l_2}_{n-4,n} < 0$,
or $a_{1j_1}a_{2j_2}\ldots a_{n-4_0j_{n-4}}a_{n-3i_1}I^{i_2,l_2j_{n-k}\mapsto l_1}_{1,n} < 0, \ldots, a_{n-(k+1)j_{n-(k+1)}}\ldots a_{n-4j_{n-4}}$
$a_{n-3i_1}I^{i_2,l_2j_{n-k}\mapsto l_1}_{n-(k+1),n} > 0, a_{n-kj_{n-k}}\ldots a_{n-4j_{n-4}}a_{n-3i_1}I^{i_2,l_2}_{n-k,n} > 0, a_{n-4j_{n-4}}a_{n-3i_1}$
$I^{i_2,l_2}_{n-4,n} < 0$.

(4) *If* $a_{n-3i_1}a_{n-3l_2} < 0$,
$a_{1j_1}a_{2j_2}\ldots a_{n-4_0j_{n-4}}a_{n-3i_1}I^{i_1,l_2}_{1,n} < 0, \ldots, a_{n-4j_{n-4}}a_{n-3i_1}I^{i_1,l_2}_{n-4,n} < 0$,
or $a_{1j_1}a_{2j_2}\ldots a_{n-4_0j_{n-4}}a_{n-3i_1}I^{i_1,l_2j_{n-k}\mapsto l_1}_{1,n} < 0, \ldots, a_{n-(k+1)j_{n-(k+1)}}\ldots a_{n-4j_{n-4}}$
$a_{n-3i_1}I^{i_1,l_2j_{n-k}\mapsto l_1}_{n-(k+1),n} > 0, a_{n-kj_{n-k}}\ldots a_{n-4j_{n-4}}a_{n-3i_1}I^{i_1,l_2}_{n-k,n} > 0, a_{n-4j_{n-4}}$
$a_{n-3i_1}I^{i_1,l_2}_{n-4,n} < 0$.

- *If* $a_{n-3l_1} \neq 0, a_{n-3l_2} = 0$; *in the case before replacing* $l_1 \leftrightarrow l_2$.

Proof. If (a) happens, we see that T preserves $x \prec_{rsgut} y$. Let $x, y \in \mathbb{R}_n$ be such that $x \prec_{rsgut} y$. We prove $Tx \prec_{rsgut} Ty$, by induction on n. Clearly, if $n = 4$, then $Tx \prec_{rsgut} Ty$. Suppose that $n \geq 5$ and the statement holds for $n = 5, 6, \ldots, n-1$. We should prove it for n.

We claim that $Ty \geq 0$ and $Ty \leq 0$ can not be happen. So, $Ty \ngeq 0$ and $Ty \nleq 0$. In this case, $Tx \prec_{rsgut} Ty$. Without loss of generality, assume that $Ty \geq 0$, and $j_k = k$ for each $k = 1, 2, \ldots, n-4$, $i_1 = n-3$, $i_2 = n-2$, $l_1 = n-1$, $l_2 = n$, $a_{11}, a_{22}, \ldots, a_{n-3n-3} > 0 > a_{n-3n-2}$. If $y_1 = 0$, by the hypothesis induction, we have $Tx \prec_{rsgut} Ty$. If $y_1 \neq 0$; as $Ty \geq 0$, we see

$$\begin{aligned}
(1) &\quad : \quad y_1 > 0,\\
(2) &\quad : \quad \tfrac{a_{12}}{-a_{22}}y_1 \leq y_2,\\
(3) &\quad : \quad \tfrac{a_{13}}{-a_{33}}y_1 + \tfrac{a_{23}}{-a_{33}}y_2 \leq y_3,\\
&\vdots\\
(n-4) &\quad : \quad \tfrac{a_{1n-4}}{-a_{n-4n-4}}y_1 + \tfrac{a_{2n-4}}{-a_{n-4n-4}}y_2 + \cdots + \tfrac{a_{n-5n-4}}{-a_{n-4n-4}}y_{n-5} \leq y_{n-4},\\
(n-3) &\quad : \quad \tfrac{a_{1n-3}}{-a_{n-3n-3}}y_1 + \tfrac{a_{2n-3}}{-a_{n-3n-3}}y_2 + \cdots + \tfrac{a_{n-4n-3}}{-a_{n-3n-3}}y_{n-4} \leq y_{n-3},\\
(n-2) &\quad : \quad \tfrac{a_{1n-2}}{-a_{n-3n-2}}y_1 + \tfrac{a_{2n-2}}{-a_{n-3n-2}}y_2 + \cdots + \tfrac{a_{n-4n-2}}{-a_{n-3n-2}}y_{n-4} \geq y_{n-3}.
\end{aligned}$$

Step (I). If $\mathrm{sgn}(\mathrm{a_{n-3l_1}}) \neq \mathrm{sgn}(\mathrm{a_{n-3l_2}})$. Let $a_{n-3n-1} > 0 > a_{n-3n}$. So,

$$\begin{aligned}(n-1) \quad &: \quad \tfrac{a_{1n-1}}{-a_{n-3n-1}}y_1 + \tfrac{a_{2n-1}}{-a_{n-3n-1}}y_2 + \cdots + \tfrac{a_{n-4n-1}}{-a_{n-3n-1}}y_{n-4} \leq y_{n-3},\\ (n) \quad &: \quad \tfrac{a_{1n}}{-a_{n-3n}}y_1 + \tfrac{a_{2n}}{-a_{n-3n}}y_2 + \cdots + \tfrac{a_{n-4n}}{-a_{n-3n}}y_{n-4} \geq y_{n-3}.\end{aligned}$$

If (1) holds; we have $I^{i_1,i_2}_{1,n}, I^{i_1,i_2}_{2,n}, \ldots, I^{i_1,i_2}_{n-4,n} > 0$. From $(n-3)$ and $(n-2)$ we conclude $\frac{I^{i_1,i_2 n-4\mapsto 1}_{n-4,n}}{-I^{i_1,i_2}_{n-4,n}}y_1 + \cdots + \frac{I^{i_1,i_2 n-4\mapsto n-5}_{n-4,n}}{-I^{i_1,i_2}_{n-4,n}}y_{n-5} \geq y_{n-4}$. By using $(n-4)$ we observe that $\frac{I^{i_1,i_2 n-5\mapsto 1}_{n-5,n}}{-I^{i_1,i_2}_{n-5,n}}y_1 + \cdots + \frac{I^{i_1,i_2 n-5\mapsto n-6}_{n-5,n}}{-I^{i_1,i_2}_{n-4,n}}y_{n-6} \geq y_{n-5}$. By continuing this process we get $\frac{I^{i_1,i_2 2\mapsto 1}_{2,n}}{-I^{i_1,i_2}_{n-4,n}}y_1 \geq y_2 \geq \frac{a_{12}}{-a_{22}}y_1$, and then $I^{i_1,i_2}_{1,n} \leq 0$, a contradiction.

Similarly about $(2), (3)$ and (4) by using $(n-3,n)$, $(n-2,n-1)$, or $(n-1,n)$, we obtain contradiction. Thus, $Ty \ngeq 0$.

Step (II). Suppose $\mathrm{sgn}(\mathrm{a_{n-3l_1}}) = \mathrm{sgn}(\mathrm{a_{n-3l_2}})$. Let $a_{n-3n-1}, a_{n-3n} > 0$. By the hypothesis, for $t = \{l_1, l_2, i_2\}$, we have $I^{i_2,t}_{1,n} < 0, I^{i_2,t}_{2,n} < 0, \ldots, I^{i_2,t}_{n-4,n} < 0$. So,

$$\begin{aligned}(n-1) \quad &: \quad \tfrac{a_{1n-1}}{-a_{n-3n-1}}y_1 + \tfrac{a_{2n-1}}{-a_{n-3n-1}}y_2 + \cdots + \tfrac{a_{n-4n-1}}{-a_{n-3n-1}}y_{n-4} \leq y_{n-3},\\ (n) \quad &: \quad \tfrac{a_{1n}}{-a_{n-3n}}y_1 + \tfrac{a_{2n}}{-a_{n-3n}} + \cdots + \tfrac{a_{n-4n}}{-a_{n-3n}}y_{n-4} \leq y_{n-3}.\end{aligned}$$

First, we use $(n-3)$ and $(n-2)$, and see that $I^{i_2,i_1}_{1,n} \geq 0$, which is a contradiction. Similarly, in cases $(n-1, n-2)$ and $(n, n-2)$ we obtain a contradiction. So, in this case $Ty \geq 0$ can not be happen, too.

Step (III). Suppose $a_{n-3l_1} = a_{n-3l_2} = 0$.
We see $a_{n-4l_1}, a_{n-4l_2} \geq 0$. Here, we consider four cases.

Case (I). $a_{n-4n-1}, a_{n-4n} > 0$.
In this case, $t_0 = 4, t_1 = \{n, n-1\}$ and $t_2 = \emptyset$. The hypothesis implies that $(I^{i_1,i_2 j_{n-4}\mapsto n}_{1,n} > 0, \ldots, I^{i_1,i_2 j_{n-4}\mapsto n}_{n-5,n} > 0, I^{i_1,i_2}_{n-4,n} > 0)$, $(I^{i_1,i_2 j_{n-4}\mapsto n-1}_{1,n} > 0, \ldots, I^{i_1,i_2}_{n-4,n} > 0)$, or $(I^{i_1,i_2}_{1,n} > 0, \ldots, I^{i_1,i_2}_{n-5,n} > 0, I^{i_1,i_2}_{n-4,n} > 0)$,

$$\begin{aligned}(n-1) \quad &: \quad \tfrac{a_{1n-1}}{-a_{n-4n-1}}y_1 + \tfrac{a_{2n-1}}{-a_{n-4n-1}}y_2 + \cdots + \tfrac{a_{n-5n-1}}{-a_{n-4n-1}}y_{n-5} \leq y_{n-4},\\ (n) \quad &: \quad \tfrac{a_{1n}}{-a_{n-4n}}y_1 + \tfrac{a_{2n}}{-a_{n-4n}}y_2 + \cdots + \tfrac{a_{n-5n}}{-a_{n-4n}}y_{n-5} \leq y_{n-4}.\end{aligned}$$

Since $I^{i_1,i_2}_{n-4,n} > 0$, from $(n-2)$ and $(n-3)$ we conclude $\frac{I^{i_1,i_2 n-4\mapsto 1}_{n-4,n}}{-I^{i_1,i_2}_{n-4,n}}y_1 + \cdots + \frac{I^{i_1,i_2 n-4\mapsto n-5}_{n-4,n}}{-I^{i_1,i_2}_{n-4,n}}y_{n-5} \geq y_{n-4}$. Now, we can use (n), $(n-1)$ or $(n-4)$.

If $I^{i_1,i_2 j_{n-4}\mapsto n}_{1,n} > 0, \ldots, I^{i_1,i_2 j_{n-4}\mapsto n}_{n-5,n} > 0, I^{i_1,i_2}_{n-4,n} > 0$, we use (n). Hence, $\frac{I^{i_1,i_2 n-5\mapsto 1 j_{n-4}\mapsto n}_{n-5,n}}{-I^{i_1,i_2 j_{n-4}\mapsto n}_{n-5,n}}y_1 + \cdots + \frac{I^{i_1,i_2 n-5\mapsto n-6 j_{n-4}\mapsto n}_{n-5,n}}{-I^{i_1,i_2 j_{n-4}\mapsto n}_{n-5,n}}y_{n-6} \geq y_{n-5}$. By applying $(n-5)$ and continuing the process, we obtain $I^{i_1,i_2 j_{n-4}\mapsto n}_{1,n} \leq 0$, which is a contradiction.

If $(I^{i_1,i_2 j_{n-4}\mapsto n-1}_{1,n} > 0, \ldots, I^{i_1,i_2}_{n-4,n} > 0)$, or $(I^{i_1,i_2}_{1,n} > 0, \ldots, I^{i_1,i_2}_{n-5,n} > 0, I^{i_1,i_2}_{n-4,n} > 0)$, then it is easy to prove.

Case (II). $a_{n-4n-1} = 0$ and $a_{n-4n} > 0$.

Case (III). $a_{n-4n-1} > 0$ and $a_{n-4n} = 0$.

Case (IV). $a_{n-4n-1} = a_{n-4n} = 0$. By the same reasoning to arrive at the same paradox.

Step (IV). $a_{n-3l_1} = 0$, $a_{n-3l_2} \neq 0$.

Step (V). $a_{n-3l_1} \neq 0$, $a_{n-3l_2} = 0$.

Similarly to the previous steps, Step (IV) and Step (V) are investigated. □

Lemma 2.8. *Let $T : \mathbb{R}_n \to \mathbb{R}_n$ be a linear function and*

$$[T] = \begin{pmatrix} 0 & \dots & 0 & * & * & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & * & * & * & \dots & * \\ 0 & \dots & 0 & a_{kl} & * & * & \dots & * \\ 0 & \dots & 0 & 0 & * & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & * & * & \dots & * \\ 0 & \dots & 0 & 0 & a_{l+1l+1} & * & \dots & * \\ 0 & \dots & 0 & 0 & 0 & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 & a_{nn} \end{pmatrix},$$

$a_{kl} \neq 0$, $2 \leq k \leq n-1$, and $k < l$. Then T does not preserve $\prec_{rsgut}$.

Proof. Set $x = e_k$ and $y = e_1 + (\frac{a_{1l}}{-a_{kl}})e_k - e_{k+1}$. We observe that $x \prec_{rsgut} y$ but $Tx \not\prec_{rsgut} Ty$. Thus, T does not preserve $\prec_{rsgut}$. □

Remark Theorems 2.6 and 2.7 give both sufficient and necessary conditions for $n = 4$. As we know, if $T : \mathbb{R}_n \to \mathbb{R}_n$ preserves $\prec_{rsgut}$, and $S : \mathbb{R}_{n-i} \to \mathbb{R}_{n-i}$ is the linear function with $[S] = [T][i+1, \dots, n]$, then S preserves $\prec_{rsgut}$ on $\mathbb{R}_{n-i}$, too. Pay attention to Lemma 2.8. So only states may for that T preserves $\prec_{rsgut}$ on $\mathbb{R}_n$, states are listed in the above cases.

Lemma 2.9. *Let $T : \mathbb{R}_n \to \mathbb{R}_n$ be a linear function. If T strongly preserves $\prec_{rsgut}$, then T is invertible.*

Proof. Suppose that $x \in \mathbb{R}_n$ and let $Tx = 0$. As $Tx = T0$ and T strongly preserves $\prec_{rsgut}$, this implies that $x \prec_{rsgut} 0$. Hence $x = 0$ and T is invertible. □

In the following theorem the structure of linear functions $T : \mathbb{R}_n \to \mathbb{R}_n$ strongly preserving rsgut-majorization will be characterized.

Theorem 2.10. *Let $T : \mathbb{R}_n \to \mathbb{R}_n$ be a linear function. Then, T strongly preserves $\prec_{rsgut}$ if and only if $[T] = \alpha I_n$, for some $\alpha \in \mathbb{R} \setminus \{0\}$.*

Proof. As the sufficiency of the condition is easy to see, we only prove the necessity of the condition.

Assume that T strongly preserves $\prec_{rsgut}$. So, T is invertible, by Lemma 2.9. Then T and T^{-1} preserve $\prec_{rsgut}$. It shows that $[T]$ have the form Theorem 2.3. If $n = 1$; then the result is trivial.

We proceed by induction on n. Suppose that $n \geq 2$ and the assertion has been established for all linear preservers of $\prec_{rsgut}$ on $\mathbb{R}_{n-1}$. Let $S : \mathbb{R}_{n-1} \to \mathbb{R}_{n-1}$ be the linear function with $[S] = [T][2, 3, \dots, n]$.

First, we prove S strongly preserves $\prec_{rsgut}$ on $\mathbb{R}_{n-1}$. As T preserves $\prec_{rsgut}$ on $\mathbb{R}_n$, we observe that S preserves $\prec_{rsgut}$ on $\mathbb{R}_{n-1}$. Let x', $y' \in \mathbb{R}_{n-1}$, and let $Sx' \prec_{rsgut} Sy'$. Set $x = (0, x')$, $y = (0, y') \in \mathbb{R}_n$. Observe that $Tx = (0, Sx')$ and $Ty = (0, Sy')$. Since $Sx' \prec_{rsgut} Sy'$, we have $Tx \prec_{rsgut} Ty$. The hypothesis ensures that $x \prec_{rsgut} y$, and thus $x' \prec_{rsgut} y'$. Therefore, S strongly preserves $\prec_{rsgut}$ on $\mathbb{R}_{n-1}$. According to the induction hypothesis, $[S] = \alpha I_{n-1}$, for some $\alpha \in \mathbb{R} \setminus \{0\}$. So, it is enough to show that $a_{12} = \cdots = a_{1n} = 0$. Consider $x = (0, 1 + \frac{1}{\alpha}, 0, \ldots, 0)$, and $y = (1, \frac{-a_{12}}{\alpha}, \ldots, \frac{-a_{1n}}{\alpha})$. Without loss of generality, assume that $a_{11} = 1$. Since T preserves $\prec_{rsgut}$, Theorem 2.3 implies that $\alpha > 0$. Thus, $Tx \not\prec_{rsgut} Ty$. If at least one of the $a_{12}, \ldots, a_{1n}$ is positive, we see that $x \prec_{rsgut} y$ is a contradiction. Hence $a_{12}, \ldots, a_{1n} \leq 0$. Now set $x = (0, 1 + \alpha, 0, \ldots, 0)$ and $y = (1, a_{12}, \ldots, a_{1n})$. If at least one of the $a_{12}, \ldots, a_{1n}$ is nonzero, we have $x \prec_{rsgut} y$ and $T^{-1}x \not\prec_{rsgut} T^{-1}y$, which is a contradiction. Thus, $a_{12} = \cdots = a_{1n} = 0$.

On the other hand, since T preserves $\prec_{rsgut}$, Theorem 2.3 ensures that $\alpha = 1$. Therefore, $T = I_n$. □

References

1. T. Ando, *Majorization, doubly stochastic matrices, and comparision of eigenvalues*, Linear Algebra and its Applications, **118** (1989), 163–248.
2. A. Armandnejad, *Right gw-majorization on $\boldsymbol{M}_{n,m}$*, Bulletin of the Iranian Mathmatical Society, **35** (2009), no. 2, 69–76.
3. A. Armandnejad and A. Ilkhanizadeh Manesh, *Gut-majorization on $\boldsymbol{M}_{n,m}$ and its linear preservers*, Electronic Journal of Linear Algebra, **23** (2012), 646–654.
4. H. Chiang and C. K. Li, *Generalized doubly stochastic matrices and linear preservers*, Linear and Multilinear Algebra, **53** (2005), 1–11.
5. A. M. Hasani and M. Radjabalipour, *The structure of linear operators strongly preserving majorizations of matrices*, Electronic Journal of Linear Algebra, **15** (2006), 260–268.
6. A. M. Hasani and M. Radjabalipour, *On linear preservers of (right) matrix majorization*, Linear Algebra and its Applications, **423** (2007), 255–261.
7. A. Ilkhanizadeh Manesh, *On linear preservers of sgut-majorization on $\boldsymbol{M}_{n,m}$*, Bulletin of the Iranian Mathmatical Society, **42** (2016), no. 2, 470–481.
8. A. Ilkhanizadeh Manesh, *Right gut-Majorization on $\boldsymbol{M}_{n,m}$*, Electronic Journal of Linear Algebra, **31** (2016), no. 1, 13–26.
9. A. W. Marshall, I. Olkin, and B. C. Arnold, *Inequalities: Theory of majorization and its applications*, Springer, New York, 2011.

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