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APPROXIMATION BY STANCU TYPE GENERALIZED SRIVASTAVA-GUPTA OPERATORS BASED ON CERTAIN PARAMETER

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Abstract. In the present paper, we introduce a Stancu type generalization of generalized Srivastava-Gupta operators based on certain parameter. We obtain the moments of the operators and then prove the basic convergence theorem. Next, the Voronovskaja type asymptotic formula and some direct results for the above operators are discussed. Also, weighted approximation and rate of convergence by these operators in terms of modulus of continuity are studied. Then, we obtain point-wise estimates using the Lipschitz type maximal function. Lastly, we propose a King type modification of these operators to obtain better estimates.

1. Introduction

In order to approximate Lebesgue integrable functions on $[0, \infty)$, Srivastava and Gupta [18] introduced a general family of summation-integral type operators which includes some well-known operators as special cases. They obtained the rate of convergence for functions of bounded variation. After that several researchers studied different approximation properties of these operators (see [1], [2], [14], [21], [22]).

For $f \in C^{\gamma}[0, \infty) := \{f \in C[0, \infty) : f(t) = O(t^{\gamma}), \gamma > 0\}$, Verma [20] defined the following generalization of Srivastava-Gupta operators based on certain

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parameter $\rho > 0$ in the following way:

$$L_{n,\rho,c}(f;x) = \sum_{k=1}^{\infty} p_{n,k}(x,c) \int_0^{\infty} \Theta_{n,k}^{\rho}(t,c) f(t)dt + p_{n,0}(x,c)f(0), \tag{1.1}$$

where

$$p_{n,k}(x,c) = \frac{(-x)^k}{k!}\phi_{n,c}^{(k)}(x), \tag{1.2}$$

$$\Theta_{n,k}^{\rho}(t,c) = \begin{cases} \frac{n\rho}{\Gamma(k\rho)} e^{-n\rho t}(n\rho t)^{k\rho-1}, & c = 0, \\ \frac{\Gamma(\frac{n\rho}{c}+k\rho)}{\Gamma(k\rho)\Gamma(\frac{n\rho}{c})} \frac{c^{k\rho}t^{k\rho-1}}{(1+ct)^{\frac{n\rho}{c}+k\rho}}, & c \in N. \end{cases}$$

and

$$\phi_{n,c}(x) = \begin{cases} e^{-nx}, & c = 0, \\ (1+cx)^{-n/c}, & c \in N. \end{cases}$$

For the properties of $\phi_{n,c}(x)$, we refer the readers to [18]. For $\rho = 1$ the operators (1.1) reduced to the Srivastava-Gupta operators [18]. In [20], Verma studied some results in simultaneous approximation by the operators $L_{n,\rho,c}$.
In [19], Stancu introduced the positive linear operators $P_n^{(\alpha,\beta)} : C[0,1] \to C[0,1]$ by modifying the Bernstein polynomial as

$$P_n^{(\alpha,\beta)}(f;x) = \sum_{k=0}^{n} b_{n,k}(x) f\left(\frac{k+\alpha}{n+\beta}\right),$$

where $b_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k},\ x \in [0,1]$ is the Bernstein basis function and α, β are any two real numbers which satisfy the condition that $0 \leq \alpha \leq \beta$.
In the recent years, Stancu type generalization of the certain operators introduced by several researchers and obtained different type of approximation properties of many operators, we refer some of the important papers in this direction as [1], [2], [9] and [10].

For $f \in C^{\gamma}[0,\infty),\ \ 0 \leq \alpha \leq \beta$ we introduce the following Stancu type generalization of the operators (1.1):

$$\begin{aligned} L_{n,\rho,c}^{(\alpha,\beta)}(f;x) &= \sum_{k=1}^{\infty} p_{n,k}(x,c) \int_0^{\infty} \Theta_{n,k}^{\rho}(t,c) f\left(\frac{nt+\alpha}{n+\beta}\right) dt \\ &\quad + p_{n,0}(x,c) f\left(\frac{\alpha}{n+\beta}\right). \end{aligned} \tag{1.3}$$

For $\alpha = \beta = 0$, we denote $L_{n,\rho,c}^{(\alpha,\beta)}(f;x)$ by $L_{n,\rho,c}(f;x)$.

In the present paper, we study the basic convergence theorem, Voronovskaja type asymptotic formula, local approximation, rate of convergence, weighted approximation and pointwise estimation of the operators (1.3). Further, to obtain better approximation we also modify the operators (1.3) by using King type approach.

2. Moment estimates

Lemma 2.1. [20] *For $L_{n,\rho,c}(t^m;x)$, $m=0,1,2$, we have*

(1) $L_{n,\rho,c}(1;x)=1$;
(2) $L_{n,\rho,c}(t;x)=\frac{n\rho x}{(n\rho-c)}$;
(3) $L_{n,\rho,c}(t^2;x)=\left\{\frac{n(n+c)\rho^2}{(n\rho-c)(n\rho-2c)}\right\}x^2+\left\{\frac{n\rho(1+\rho)}{(n\rho-c)(n\rho-2c)}\right\}x$.

Lemma 2.2. *For the operators $L^{(\alpha,\beta)}_{n,\rho,c}(f;x)$ as defined in (1.3), the following equalities hold:*

(1) $L^{(\alpha,\beta)}_{n,\rho,c}(1;x)=1$;
(2) $L^{(\alpha,\beta)}_{n,\rho,c}(t;x)=\frac{n^2\rho x+\alpha(n\rho-c)}{(n\rho-c)(n+\beta)}$;
(3) $L^{(\alpha,\beta)}_{n,\rho,c}(t^2;x)=\left\{\frac{n^3\rho^2(n+c)}{(n\rho-c)(n\rho-2c)(n+\beta)^2}\right\}x^2+\left\{\frac{n^3\rho(1+\rho)+2n^2\alpha\rho(n\rho-2c)}{(n\rho-c)(n\rho-2c)(n+\beta)^2}\right\}x+\frac{\alpha^2}{(n+\beta)^2}$.

Proof. For $x\in[0,\infty)$, in view of Lemma 2.1, we have

$$L^{(\alpha,\beta)}_{n,\rho,c}(1;x) \;=\; 1.$$

Next, for $f(t)=t$, again applying Lemma 2.1, we get

$$\begin{aligned}L^{(\alpha,\beta)}_{n,\rho,c}(t;x) &= \sum_{k=1}^{\infty}p_{n,k}(x,c)\int_0^{\infty}\Theta^{\rho}_{n,k}(t,c)\left(\frac{nt+\alpha}{n+\beta}\right)dt+p_{n,0}(x,c)\left(\frac{\alpha}{n+\beta}\right)\\ &= \frac{n}{n+\beta}L_{n,\rho,c}(t,x)+\frac{\alpha}{n+\beta}=\frac{n^2\rho x+\alpha(n\rho-c)}{(n\rho-c)(n+\beta)}.\end{aligned}$$

Proceeding similarly, we have

$$\begin{aligned}L^{(\alpha,\beta)}_{n,\rho,c}(t^2;x) &= \sum_{k=1}^{\infty}p_{n,k}(x,c)\int_0^{\infty}\Theta^{\rho}_{n,k}(t,c)\left(\frac{nt+\alpha}{n+\beta}\right)^2dt+p_{n,0}(x,c)\left(\frac{\alpha}{n+\beta}\right)^2\\ &= \left(\frac{n}{n+\beta}\right)^2L_{n,\rho,c}(t^2,x)+\frac{2n\alpha}{(n+\beta)^2}L_{n,\rho,c}(t,x)+\left(\frac{\alpha}{n+\beta}\right)^2\\ &= \left\{\frac{n^3\rho^2(n+c)}{(n\rho-c)(n\rho-2c)(n+\beta)^2}\right\}x^2\\ &\quad+\left\{\frac{n^3\rho(1+\rho)+2n^2\alpha\rho(n\rho-2c)}{(n\rho-c)(n\rho-2c)(n+\beta)^2}\right\}x+\frac{\alpha^2}{(n+\beta)^2}.\end{aligned}$$

□

Lemma 2.3. *For $f\in C_B[0,\infty)$ (space of all real valued bounded and uniformly continuous functions on $[0,\infty)$ endowed with norm $\| f \|_{C_B[0,\infty)}=\sup_{x\in[0,\infty)}|f(x)|$),*

$$\| L^{(\alpha,\beta)}_{n,\rho,c}(f) \|\leq\| f \| .$$

Proof. In view of (1.3) and Lemma 2.2, the proof of this lemma easily follows. □

Remark 2.4. For every $x \in [0,\infty)$ and $n\rho > 2c$ we have

$$\begin{aligned} L_{n,\rho,c}^{(\alpha,\beta)}((t-x);x) &= \left\{\frac{c(n+\beta)-n\rho\beta}{(n\rho-c)(n+\beta)}\right\}x+\frac{\alpha}{(n+\beta)} \\ &= \mu_{n,\rho,c}^{(\alpha,\beta)}(x) \end{aligned}$$

and

$$L_{n,\rho,c}^{(\alpha,\beta)}\left((t-x)^2;x\right) = \left\{\frac{n^3\rho^2(n+c)-(n\rho-2c)(n+\beta)(n^2\rho-n\rho\beta+nc+c\beta)}{(n\rho-c)(n\rho-2c)(n+\beta)^2}\right\}x^2$$
$$+\left\{\frac{n^3\rho(1+\rho)+2\alpha(n\rho-2c)(nc+c\beta-n\rho\beta)}{(n\rho-c)(n\rho-2c)(n+\beta)^2}\right\}x+\frac{\alpha^2}{(n+\beta)^2}=\gamma_{n,\rho,c}^{(\alpha,\beta)}(x).$$

3. Main results

Theorem 3.1. *(Voronovskaja type theorem) Let f be bounded and integrable on $[0,\infty)$, second derivative of f exists at a fixed point $x \in [0,\infty)$, then*

$$\lim_{n\to\infty} n\left(L_{n,\rho,c}^{(\alpha,\beta)}(f;x)-f(x)\right)=\left(\alpha+\left(\frac{c}{\rho}-\beta\right)x\right)f'(x)+\frac{x(1+cx)}{2}\left(1+\frac{1}{\rho}\right)f''(x).$$

Proof. Let $x \in [0,\infty)$ be fixed. From the Taylor's theorem, we may write

$$f(t)=f(x)+(t-x)f'(x)+\frac{1}{2}f''(x)(t-x)^2+\xi(t,x)(t-x)^2, \tag{3.1}$$

where $\xi(t,x)$ is the Peano form of the remainder and $\lim_{t\to x}\xi(t,x)=0$.
Applying $L_{n,\rho,c}^{(\alpha,\beta)}(f,x)$ on both sides of (3.1), we have

$$\begin{aligned} n\left(L_{n,\rho,c}^{(\alpha,\beta)}(f;x)-f(x)\right) &= nf'(x)L_{n,\rho,c}^{(\alpha,\beta)}((t-x);x)+\frac{1}{2}nf''(x)L_{n,\rho,c}^{(\alpha,\beta)}\left((t-x)^2;x\right) \\ &\quad +nL_{n,\rho,c}^{(\alpha,\beta)}\left(\xi(t,x)(t-x)^2;x\right). \end{aligned}$$

In view of Remark 2.4, we have

$$\lim_{n\to\infty} nL_{n,\rho,c}^{(\alpha,\beta)}((t-x);x)=\alpha+\left(\frac{c}{\rho}-\beta\right)x \tag{3.2}$$

and

$$\lim_{n\to\infty} nL_{n,\rho,c}^{(\alpha,\beta)}\left((t-x)^2;x\right)=x(1+cx)\left(1+\frac{1}{\rho}\right). \tag{3.3}$$

Now, we shall show that

$$\lim_{n\to\infty} nL_{n,\rho,c}^{(\alpha,\beta)}\left(\xi(t,x)(t-x)^2;x\right)=0.$$

By using Cauchy-Schwarz inequality, we have

$$L_{n,\rho,c}^{(\alpha,\beta)}\left(\xi(t,x)(t-x)^2;x\right)\le\left(L_{n,\rho,c}^{(\alpha,\beta)}(\xi^2(t,x);x)\right)^{1/2}\times\left(L_{n,\rho,c}^{(\alpha,\beta)}((t-x)^4;x)\right)^{1/2}. \tag{3.4}$$

We observe that $\xi^2(x,x)=0$ and $\xi^2(.,x)\in C_B[0,\infty)$. Then, it follows that

$$\lim_{n\to\infty} L_{n,\rho,c}^{(\alpha,\beta)}(\xi^2(t,x);x)=\xi^2(x,x)=0. \tag{3.5}$$

Now, from (3.4) and (3.5) we obtain

$$\lim_{n\to\infty} nL_{n,\rho,c}^{(\alpha,\beta)}\left(\xi(t,x)(t-x)^2;x\right)=0. \tag{3.6}$$

From (3.2), (3.3) and (3.6), we get the required result. □

3.1. **Local approximation.** For $C_B[0,\infty)$, let us consider the following K-functional:

$$K_2(f,\delta) \;=\; \inf_{g\in W^2}\{\| f-g \| +\delta \| g'' \|\},$$

where $\delta>0$ and $W^2=\{g\in C_B[0,\infty): g', g'' \in C_B[0,\infty)\}$. By, p. 177, Theorem 2.4 in [3], there exists an absolute constant $M>0$ such that

$$K_2(f,\delta)\le M\omega_2(f,\sqrt{\delta}), \tag{3.7}$$

where

$$\omega_2(f,\sqrt{\delta}) \;=\; \sup_{0<|h|\le\sqrt{\delta}}\ \sup_{x\in[0,\infty)} \mid f(x+2h)-2f(x+h)+f(x) \mid$$

is the second order modulus of smoothness of f. By

$$\omega(f,\delta) \;=\; \sup_{0<|h|\le\delta}\ \sup_{x\in[0,\infty)} \mid f(x+h)-f(x) \mid,$$

we denote the usual modulus of continuity of $f\in C_B[0,\infty)$.

Theorem 3.2. *Let $f\in C_B[0,\infty)$. Then, for every $x\in[0,\infty)$, we have*

$$\mid L_{n,\rho,c}^{(\alpha,\beta)}(f;x)-f(x)\mid \;\le\; M\omega_2\left(f,\delta_{n,\rho,c}^{(\alpha,\beta)}(x)\right)+\omega\left(f,\mu_{n,\rho,c}^{(\alpha,\beta)}(x)\right),$$

where M is an absolute constant and

$$\delta_{n,\rho,c}^{(\alpha,\beta)}(x) \;=\; \left(\gamma_{n,\rho,c}^{(\alpha,\beta)}(x)+\left(\mu_{n,\rho,c}^{(\alpha,\beta)}(x)\right)^2\right)^{1/2}.$$

Proof. For $x\in[0,\infty)$, we consider the auxiliary operators $\overline{L}_{n,\rho,c}^{(\alpha,\beta)}$ defined by

$$\overline{L}_{n,\rho,c}^{(\alpha,\beta)}(f;x) \;=\; L_{n,\rho,c}^{(\alpha,\beta)}(f;x)-f\left(\frac{n^2\rho x+\alpha(n\rho-c)}{(n\rho-c)(n+\beta)}\right)+f(x). \tag{3.8}$$

From Lemma 2.2, we observe that the operators $\overline{L}_{n,\rho,c}^{(\alpha,\beta)}$ are linear and reproduce the linear functions.
Hence

$$\overline{L}_{n,\rho,c}^{(\alpha,\beta)}((t-x);x) \;=\; 0. \tag{3.9}$$

Let $g\in W^2$. By Taylor's theorem, we have

$$g(t)=g(x)+(t-x)g'(x)+\int_x^t (t-v)g''(v)dv,\ \ t\in[0,\infty).$$

Applying $\overline{L}_{n,\rho,c}^{(\alpha,\beta)}$ on both sides of the above equation and using (3.9), we have

$$\overline{L}_{n,\rho,c}^{(\alpha,\beta)}(g;x) \;=\; g(x)+\overline{L}_{n,\rho,c}^{(\alpha,\beta)}\left(\int_x^t (t-v)g''(v)dv;x\right).$$

Thus, by (3.8) we get

$$|\overline{L}_{n,\rho,c}^{(\alpha,\beta)}(g;x)-g(x)|$$

$$\leq L_{n,\rho,c}^{(\alpha,\beta)}\left(\left|\int_x^t (t-v)g''(v)dv\right|;x\right)+\left|\int_x^{\frac{n^2\rho x+\alpha(n\rho-c)}{(n\rho-c)(n+\beta)}}\left(\frac{n^2\rho x+\alpha(n\rho-c)}{(n\rho-c)(n+\beta)}-v\right)g''(v)dv\right|$$

$$\leq L_{n,\rho,c}^{(\alpha,\beta)}\left(\int_x^t |t-v||g''(v)|dv;x\right)+\int_x^{\frac{n^2\rho x+\alpha(n\rho-c)}{(n\rho-c)(n+\beta)}}\left|\frac{n^2\rho x+\alpha(n\rho-c)}{(n\rho-c)(n+\beta)}-v\right||g''(v)|dv$$

$$\leq \left[\gamma_{n,\rho,c}^{(\alpha,\beta)}(x)+\left(\mu_{n,\rho,c}^{(\alpha,\beta)}(x)\right)^2\right]\parallel g''\parallel$$

$$\leq \left(\delta_{n,\rho,c}^{(\alpha,\beta)}(x)\right)^2\parallel g''\parallel. \tag{3.10}$$

On the other hand, by (3.8) and Lemma 2.3, we have

$$|\overline{L}_{n,\rho,c}^{(\alpha,\beta)}(f;x)| \leq \parallel f\parallel. \tag{3.11}$$

Using (3.10) and (3.11) in (3.8), we obtain

$$\begin{aligned}|L_{n,\rho,c}^{(\alpha,\beta)}(f;x)-f(x)| &\leq |\overline{L}_{n,\rho,c}^{(\alpha,\beta)}(f-g;x)|+|(f-g)(x)|+|\overline{L}_{n,\rho,c}^{(\alpha,\beta)}(g;x)-g(x)|\\ &\quad+\left|f\left(\frac{n^2\rho x+\alpha(n\rho-c)}{(n\rho-c)(n+\beta)}\right)-f(x)\right|\\ &\leq 2\parallel f-g\parallel+\left(\delta_{n,\rho,c}^{(\alpha,\beta)}(x)\right)^2\parallel g''\parallel\\ &\quad+\left|f\left(\frac{n^2\rho x+\alpha(n\rho-c)}{(n\rho-c)(n+\beta)}\right)-f(x)\right|.\end{aligned}$$

Hence, taking infimum on the right hand side over all $g\in W^2$, we get

$$|\,L_{n,\rho,c}^{(\alpha,\beta)}(f;x)-f(x)\,| \leq K_2\left(f,(\delta_{n,\rho,c}^{(\alpha,\beta)}(x))^2\right)+\omega\left(f,\mu_{n,\rho,c}^{(\alpha,\beta)}(x)\right).$$

In view of (3.7), we get

$$|\,L_{n,\rho,c}^{(\alpha,\beta)}(f;x)-f(x)\,| \leq M\omega_2\left(f,\delta_{n,\rho,c}^{(\alpha,\beta)}(x)\right)+\omega\left(f,\mu_{n,\rho,c}^{(\alpha,\beta)}(x)\right).$$

Hence, the proof is completed. □

3.2. **Rate of convergence.** Let $\omega_a(f,\delta)$ denote the modulus of continuity of f on the closed interval $[0,a], a>0$, and defined as

$$\omega_a(f,\delta)=\sup_{|t-x|\leq\delta}\ \sup_{x,t\in[0,a]}|f(t)-f(x)|.$$

We observe that for a function $f\in C_B[0,\infty)$, the modulus of continuity $\omega_a(f,\delta)$ tends to zero.
Now, we give a rate of convergence theorem for the operators $L_{n,\rho,c}^{(\alpha,\beta)}$.

Theorem 3.3. *Let $f \in C_B[0,\infty)$ and $\omega_{a+1}(f,\delta)$ be its modulus of continuity on the finite interval $[0,a+1] \subset [0,\infty)$, where $a>0$. Then, we have*

$$|L_{n,\rho,c}^{(\alpha,\beta)}(f;x)-f(x)| \leq 6M_f(1+a^2)\gamma_{n,\rho,c}^{(\alpha,\beta)}(a)+2\omega_{a+1}\left(f,\sqrt{\gamma_{n,\rho,c}^{(\alpha,\beta)}(a)}\right),$$

where $\gamma_{n,\rho,c}^{(\alpha,\beta)}(a)$ is defined in Remark 2.4 and M_f is a constant depending only on f.

Proof. For $x\in[0,a]$ and $t>a+1$. Since $t-x>1$, we have

$$\begin{aligned} |f(t)-f(x)| &\leq M_f(2+x^2+t^2) \\ &\leq M_f(t-x)^2(2+3x^2+2(t-x)^2) \\ &\leq 6M_f(1+a^2)(t-x)^2. \end{aligned}$$

For $x\in[0,a]$ and $t\leq a+1$, we have

$$|f(t)-f(x)| \leq \omega_{a+1}(f,|t-x|) \leq \left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f,\delta), \delta>0.$$

From the above, we have

$$|f(t)-f(x)| \leq 6M_f(1+a^2)(t-x)^2+\left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f,\delta), \delta>0.$$

Thus, by applying Cauchy-Schwarz inequality, we have
$|L_{n,\rho,c}^{(\alpha,\beta)}(f;x)-f(x)|$

$$\begin{aligned} &\leq 6M_f(1+a^2)(L_{n,\rho,c}^{(\alpha,\beta)}(t-x)^2;x)+\omega_{a+1}(f,\delta)\left(1+\frac{1}{\delta}(L_{n,\rho,c}^{(\alpha,\beta)}(t-x)^2;x)^{\frac{1}{2}}\right) \\ &\leq 6M_f(1+a^2)\gamma_{n,\rho,c}^{(\alpha,\beta)}(a)+2\omega_{a+1}\left(f,\sqrt{\gamma_{n,\rho,c}^{(\alpha,\beta)}(a)}\right), \end{aligned}$$

on choosing $\delta=\sqrt{\gamma_{n,\rho,c}^{(\alpha,\beta)}(a)}$. This completes the proof of the theorem. □

3.3. **Weighted approximation.** In this section, we obtain the Korovkin type weighted approximation by the operators defined in 1.3 . The weighted Korovkin-type theorems were proved by Gadzhiev [4]. A real function $\nu(x)=1+x^2$ is called a weight function if it is continuous on R and $\lim\limits_{|x|\to\infty}\nu(x)=\infty$, $\nu(x)\geq 1$ for all $x\in R$.

Let $B_\nu(R)$ denote the weighted space of real-valued functions f defined on R with the property $|f(x)|\leq M_f\nu(x)$ for all $x\in R$, where M_f is a constant depending on the function f. We also consider the weighted subspace $C_\nu(R)$ of $B_\nu(R)$ given by $C_\nu(R)=\{f\in B_\nu(R): f$ is continuous on $R\}$ and $C_\nu^*[0,\infty)$ denotes the subspace of all functions $f\in C_\nu[0,\infty)$ for which $\lim\limits_{|x|\to\infty}\frac{f(x)}{\nu(x)}$ exists finitely.

The space $B_\nu(R)$ is a normed linear space with the following norm:

$$\| f \|_\nu=\sup_{x\in[0,\infty)}\frac{|f(x)|}{\nu(x)}.$$

Theorem 3.4. *For each $f \in C_\nu^*[0,\infty)$, we have*

$$\lim_{n\to\infty} \parallel L_{n,\rho,c}^{(\alpha,\beta)}(f) - f \parallel_\nu = 0.$$

Proof. From [4], we know that it is sufficient to verify the following three conditions

$$\lim_{n\to\infty} \parallel L_{n,\rho,c}^{(\alpha,\beta)}(t^k) - x^k \parallel_\nu = 0,\ k = 0,1,2. \tag{3.12}$$

Since $L_{n,\rho,c}^{(\alpha,\beta)}(1;x) = 1$, the condition in (3.12) holds for $k = 0$.
By Lemma 2.2, we have

$$\begin{aligned} \parallel L_{n,\rho,c}^{(\alpha,\beta)}(t) - x \parallel_\nu &= \sup_{x\in[0,\infty)} \frac{|L_{n,\rho,c}^{(\alpha,\beta)}(t;x) - x|}{1+x^2} \\ &\leq \left|\frac{c(n+\beta) - n\rho\beta}{(n\rho - c)(n+\beta)}\right| \sup_{x\in[0,\infty)} \frac{x}{1+x^2} + \frac{\alpha}{n+\beta} \sup_{x\in[0,\infty)} \frac{1}{1+x^2} \\ &\leq \left|\frac{c(n+\beta) - n\rho\beta + \alpha(n\rho - c)}{(n\rho - c)(n+\beta)}\right| \end{aligned}$$

which implies that the condition in (3.12) holds for $k = 1$.
Similarly, we can write for $n\rho > 2c$

$$\begin{aligned} \parallel L_{n,\rho,c}^{(\alpha,\beta)}(t^2) - x^2 \parallel_\nu &= \sup_{x\in[0,\infty)} \frac{|L_{n,\rho,c}^{(\alpha,\beta)}(t^2;x) - x^2|}{1+x^2} \\ &\leq \left|\frac{n^3\rho^2(n+c)}{(n\rho - c)(n\rho - 2c)(n+\beta)^2} - 1\right| \\ &\quad + \left|\frac{n^3\rho(1+\rho) + 2n^2\alpha\rho(n\rho - 2c)}{(n\rho - c)(n\rho - 2c)(n+\beta)^2}\right| + \frac{\alpha^2}{(n+\beta)^2}, \end{aligned}$$

which implies that $\lim_{n\to\infty} \parallel L_{n,\rho,c}^{(\alpha,\beta)}(t^2) - x^2 \parallel_\nu = 0$, the equation (3.12) holds for $k = 2$.
This completes the proof of theorem. □

3.4. **Pointwise Estimates.** In this section, we establish some pointwise estimates of the rate of convergence of the operators $L_{n,\rho,c}^{(\alpha,\beta)}$. First, we give the relationship between the local smoothness of f and local approximation.
We know that a function $f \in C[0,\infty)$ is in $\mathrm{Lip}_M\ (r)$ on E, $r \in (0,1]$, E$\subset [0,\infty)$ if it satisfies the condition

$$|f(t) - f(x)| \leq M|t-x|^r,\ t \in [0,\infty) \ and\ x \in E,$$

where M is a constant depending only on r and f.

Theorem 3.5. *Let $f \in C[0,\infty) \cap Lip_M\ (r)$, $E \subset [0,\infty)$ and $r \in (0,1]$. Then, we have*

$$|L_{n,\rho,c}^{(\alpha,\beta)}(f;x) - f(x)| \ \leq\ M\bigg(\left(\gamma_{n,\rho,c}^{(\alpha,\beta)}(x)\right)^{r/2} + 2d^r(x,E)\bigg),\ x \in [0,\infty),$$

where M is a constant depending on r and f and $d(x,E)$ is the distance between x and E defined as

$$d(x,E)=\inf\{|t-x|:t\in E\}.$$

Proof. Let $\overline{E}$ be the closure of E in $[0,\infty)$. Then, there exists at least one point $t_0\in\overline{E}$ such that

$$d(x,E)=|x-t_0|.$$

By our hypothesis and the monotonicity of $L_{n,\rho,c}^{(\alpha,\beta)}$, we get

$$\begin{aligned}|L_{n,\rho,c}^{(\alpha,\beta)}(f;x)-f(x)| &\leq L_{n,\rho,c}^{(\alpha,\beta)}(|f(t)-f(t_0)|;x)+L_{n,\rho,c}^{(\alpha,\beta)}(|f(x)-f(t_0)|;x)\\ &\leq M\left(L_{n,\rho,c}^{(\alpha,\beta)}(|t-t_0|^r;x)+|x-t_0|^r\right)\\ &\leq M\left(L_{n,\rho,c}^{(\alpha,\beta)}(|t-x|^r;x)+2|x-t_0|^r\right).\end{aligned}$$

Now, applying Hölder's inequality with $p=\dfrac{2}{r}$ and $\dfrac{1}{q}=1-\dfrac{1}{p}$, we obtain

$$|L_{n,\rho,c}^{(\alpha,\beta)}((f;x)-f(x)|\leq M\left((L_{n,\rho,c}^{(\alpha,\beta)}(|t-x|^2;x))^{r/2}+2d^r(x,E)\right),$$

from which the desired result immediate. □

Next, we obtain the local direct estimate of the operators defined in (1.3), using the Lipschitz-type maximal function of order r introduced by B. Lenze [12] as

$$\widetilde{\omega}_r(f,x)=\sup_{t\neq x,\ t\in[0,\infty)}\frac{|f(t)-f(x)|}{|t-x|^r},\quad x\in[0,\infty)\ \text{ and }\ r\in(0,1]. \tag{3.13}$$

Theorem 3.6. *Let $f\in C_B[0,\infty)$ and $0<r\leq 1$. Then, for all $x\in[0,\infty)$ we have*

$$|L_{n,\rho,c}^{(\alpha,\beta)}(f;x)-f(x)|\leq\widetilde{\omega}_r(f,x)\left(\gamma_{n,\rho,c}^{(\alpha,\beta)}(x)\right)^{r/2}.$$

Proof. From the equation (3.13), we have

$$|L_{n,\rho,c}^{(\alpha,\beta)}(f;x)-f(x)|\leq\widetilde{\omega}_r(f,x)L_{n,\rho,c}^{(\alpha,\beta)}(|t-x|^r;x).$$

Applying the Hölder's inequality with $p=\dfrac{2}{r}$ and $\dfrac{1}{q}=1-\dfrac{1}{p}$, we get

$$|L_{n,\rho,c}^{(\alpha,\beta)}(f;x)-f(x)|\leq\widetilde{\omega}_r(f,x)L_{n,\rho,c}^{(\alpha,\beta)}((t-x)^2;x)^{\frac{r}{2}}\leq\widetilde{\omega}_r(f,x)\left(\gamma_{n,\rho,c}^{(\alpha,\beta)}(x)\right)^{r/2}.$$

Thus, the proof is completed. □

For $a,b>0$, Özarslan and Aktuğlu [17] consider the Lipschitz-type space with two parameters:

$$Lip_M^{(a,b)}(r)=\left(f\in C[0,\infty):|f(t)-f(x)|\leq M\frac{|t-x|^r}{(t+ax^2+bx)^{r/2}};\ x,t\in[0,\infty)\right),$$

where M is any positive constant and $0<r\leq 1$.

Theorem 3.7. *For $f \in Lip_M^{(a,b)}(r)$. Then, for all $x > 0$, we have*

$$|L_{n,\rho,c}^{(\alpha,\beta)}(f;x) - f(x)| \leq M \left(\frac{\gamma_{n,\rho,c}^{(\alpha,\beta)}(x)}{ax^2 + bx} \right)^{r/2}.$$

Proof. First we prove the theorem for $r = 1$. Then, for $f \in Lip_M^{(a,b)}(1)$, and $x \in [0, \infty)$, we have

$$\begin{aligned} |L_{n,\rho,c}^{(\alpha,\beta)}(f;x) - f(x)| &\leq L_{n,\rho,c}^{(\alpha,\beta)}(|f(t) - f(x)|;x) \\ &\leq M L_{n,\rho,c}^{(\alpha,\beta)} \left(\frac{|t-x|}{(t + ax^2 + bx)^{1/2}}; x \right) \\ &\leq \frac{M}{(ax^2 + bx)^{1/2}} L_{n,\rho,c}^{(\alpha,\beta)}(|t-x|;x). \end{aligned}$$

Applying Cauchy-Schwarz inequality, we get

$$\begin{aligned} |L_{n,\rho,c}^{(\alpha,\beta)}(f;x) - f(x)| &\leq \frac{M}{(ax^2 + bx)^{1/2}} \left(L_{n,\rho,c}^{(\alpha,\beta)}((t-x)^2;x) \right)^{1/2} \\ &\leq M \left(\frac{\gamma_{n,\rho,c}^{(\alpha,\beta)}(x)}{ax^2 + bx} \right)^{1/2}. \end{aligned}$$

Thus the result holds for $r = 1$.
Now, we prove that the result is true for $0 < r < 1$. Then, for $f \in Lip_M^{(a,b)}(r)$, and $x \in [0, \infty)$, we get

$$|L_{n,\rho,c}^{(\alpha,\beta)}(f;x) - f(x)| \leq \frac{M}{(ax^2 + bx)^{r/2}} L_{n,\rho,c}^{(\alpha,\beta)}(|t-x|^r;x).$$

Taking $p = \frac{1}{r}$ and $q = \frac{p}{p-1}$, applying the Hölders inequality, we have

$$|L_{n,\rho,c}^{(\alpha,\beta)}(f;x) - f(x)| \leq \frac{M}{(ax^2 + bx)^{r/2}} \left(L_{n,\rho,c}^{(\alpha,\beta)}(|t-x|;x) \right)^r.$$

Finally by Cauchy-Schwarz inequality, we get

$$|L_{n,\rho,c}^{(\alpha,\beta)}(f;x) - f(x)| \leq M \left(\frac{\gamma_{n,\rho,c}^{(\alpha,\beta)}(x)}{ax^2 + bx} \right)^{r/2}.$$

Thus, the proof is completed. ☐

4. Better Estimates

It is well known that the classical Bernstein polynomial preserve constant as well as linear functions. To make the convergence faster, King [11] proposed an approach to modify the Bernstein polynomial, so that the sequence preserve test functions e_0 and e_2, where $e_i(t) = t^i, i = 0, 1, 2$. As the operator $L_{n,\rho,c}^{(\alpha,\beta)}(f;x)$ defined in (1.3) preserve only the constant functions so further modification of these operators is proposed to be made so that the modified operators preserve

the constant as well as linear functions.
For this purpose the modification of (1.3) is defined as

$$\hat{L}_{n,\rho,c}^{(\alpha,\beta)}(f;x) = \sum_{k=1}^{\infty} p_{n,k}(r_n(x),c)\int_0^{\infty}\Theta_{n,k}^{\rho}(t,c)f\left(\frac{nt+\alpha}{n+\beta}\right)dt + p_{n,0}(r_n(x),c)f\left(\frac{\alpha}{n+\beta}\right), \tag{4.1}$$

where $r_n(x)=\frac{(n\rho-c)[(n+\beta)x-\alpha]}{n^2\rho}$ for $x\in I_n=[\frac{\alpha}{n+\beta},\infty)$ and $n\rho>2c$.

Lemma 4.1. *For each $x\in I_n$, by simple computations, we have*

(1) $\hat{L}_{n,\rho,c}^{(\alpha,\beta)}(1;x)=1$;
(2) $\hat{L}_{n,\rho,c}^{(\alpha,\beta)}(t;x)=x$;
(3) $\hat{L}_{n,\rho,c}^{(\alpha,\beta)}(t^2;x)=\left\{\frac{(n\rho-c)(n+c)}{n(n\rho-2c)}\right\}x^2+\left\{\frac{n^2(1+\rho)+2\alpha(c^2-n\rho c-nc)}{n(n\rho-2c)(n+\beta)}\right\}x + \frac{\alpha^2(n\rho c+nc-c^2)-n^2\alpha(1+\rho)}{n(n\rho-2c)(n+\beta)^2}.$

Consequently, for each $x\in I_n$, we have the following equalities

$$\hat{L}_{n,\rho,c}^{(\alpha,\beta)}(t-x;x)=0$$

$$\hat{L}_{n,\rho,c}^{(\alpha,\beta)}((t-x)^2;x) = \left\{\frac{n\rho c+nc-c^2}{n(n\rho-2c)}\right\}x^2+\left\{\frac{n^2(1+\rho)+2\alpha(c^2-n\rho c-nc)}{n(n\rho-2c)(n+\beta)}\right\}x + \frac{\alpha^2(n\rho c+nc-c^2)-n^2\alpha(1+\rho)}{n(n\rho-2c)(n+\beta)^2} = \zeta_{n,\rho,c}^{(\alpha,\beta)}(x). \tag{4.2}$$

Theorem 4.2. *Let $f\in C_B(I_n)$ and $x\in I_n$. Then for $n\rho>2c$, there exists a positive constant M' such that*

$$|\hat{L}_{n,\rho,c}^{(\alpha,\beta)}(f;x)-f(x)|\le M'\omega_2\left(f,\sqrt{\zeta_{n,\rho,c}^{(\alpha,\beta)}(x)}\right),$$

where $\zeta_{n,\rho,c}^{(\alpha,\beta)}(x)$ is given by (4.2).

Proof. Let $g\in W^2$ and $x,t\in I_n$. Using the Taylor's expansion we have

$$g(t)=g(x)+(t-x)g'(x)+\int_x^t(t-v)g''(v)dv.$$

Applying $\hat{L}_{n,\rho,c}^{(\alpha,\beta)}$ on both sides and using Lemma 4.1, we get

$$\hat{L}_{n,\rho,c}^{(\alpha,\beta)}(g;x)-g(x) = \hat{L}_{n,\rho,c}^{(\alpha,\beta)}\left(\int_x^t(t-v)g''(v)dv;x\right).$$

Obviously, we have

$$\left|\int_x^t(t-v)g''(v)dv\right|\le(t-x)^2\|g''\|.$$

Therefore

$$| \hat{L}_{n,\rho,c}^{(\alpha,\beta)}(g;x) - g(x) | \leq \hat{L}_{n,\rho,c}^{(\alpha,\beta)}((t-x)^2;x) \parallel g'' \parallel = \zeta_{n,\rho,c}^{(\alpha,\beta)}(x) \parallel g'' \parallel .$$

Since $| \hat{L}_{n,\rho,c}^{(\alpha,\beta)}(f;x) | \leq \|f\|$, we get

$$\begin{aligned} &| \hat{L}_{n,\rho,c}^{(\alpha,\beta)}(f;x) - f(x) | \\ &\quad\leq \ | \hat{L}_{n,\rho,c}^{(\alpha,\beta)}(f-g;x) | + | (f-g)(x) | + | \hat{L}_{n,\rho,c}^{(\alpha,\beta)}(g;x) - g(x) | \\ &\quad\leq \ 2\|f-g\| + \zeta_{n,\rho,c}^{(\alpha,\beta)}(x)\|g''\|. \end{aligned}$$

Finally, taking the infimum over all $g \in W^2$ and using (3.7) we obtain

$$| \hat{L}_{n,\rho,c}^{(\alpha,\beta)}(f;x) - f(x) | \leq M'\omega_2 \left(f, \sqrt{\zeta_{n,\rho,c}^{(\alpha,\beta)}(x)} \right),$$

which proves the theorem. □

Theorem 4.3. *Let $f \in C_B(I_n)$. If f', f'' exists at a fixed point $x \in I_n$, then we have*

$$\lim_{n\to\infty} n \left(\hat{L}_{n,\rho,c}^{(\alpha,\beta)}(f;x) - f(x) \right) = \frac{x(1+cx)}{2} \left(1 + \frac{1}{\rho} \right) f''(x).$$

The proof follows along the lines of Theorem 3.1.

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