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NEW PROPERTIES UNDER GENERALIZED CONTRACTIVE CONDITIONS

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Abstract. The aim of this contribution is to establish some common fixed point theorems for single and set-valued maps under contractive conditions of integral type on a symmetric space. These maps are assumed to satisfy new properties which extend the results of Aliouche [3], Aamri and El Moutawakil [2] and references therein, also they generalize the notion of non-compatible and non-δ-compatible maps in the setting of symmetric spaces.

1. Introduction and preliminaries

In 1982, Sessa [8] generalized the concept of commuting maps by giving the notion of weakly commuting maps. Two self-maps f and g of a metric space $(\mathcal{X}, d)$ are said to be weakly commuting if, for all $x \in \mathcal{X}$ we have

$$d(fgx, gfx) \leq d(gx, fx).$$

Further, in 1986, Jungck [5] gave a generalization of commuting and weakly commuting maps by introducing the concept of compatible maps. Self-maps f and g of a metric space $(\mathcal{X}, d)$ are compatible if and only if whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $fx_n,\ gx_n \to t \in \mathcal{X}$, then $d(fx_n, gx_n) \to 0$.

Later, the same author with Rhoades [6] extended the concept of compatible maps to maps $f : \mathcal{X} \to \mathcal{X}$ and $F : \mathcal{X} \to B(\mathcal{X})$ by requiring that $fFx \in B(\mathcal{X})$ for $x \in \mathcal{X}$ and $\delta(fFx_n, Ffx_n) \to 0$ whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $Fx_n \to \{t\}$ $(\delta(Fx_n, t) \to 0)$ and $fx_n \to t$ for some $t \in \mathcal{X}$.

This last definition motivated the definition of weakly compatible maps [7] mentioned below.

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On the other hand, Aamri and El Moutawakil [1] have established the notion of property $(E.A)$ for single valued maps.

To generalize this property, Djoudi and Khemis [4] introduced the definition of the so-called D-maps as follows: maps $f : \mathcal{X} \to \mathcal{X}$ and $F : \mathcal{X} \to B(\mathcal{X})$ are said to be D-maps if and only if there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that, $\lim_{n\to\infty} fx_n = t$ and $\lim_{n\to\infty} Fx_n = \{t\}$ for some $t \in \mathcal{X}$.

Let $\mathcal{X}$ be a set. Recall that a symmetric on $\mathcal{X}$ is a nonnegative real function d on $\mathcal{X} \times \mathcal{X}$ into $[0, \infty)$ such that

(1) $d(x, y) = 0$ if and only if $x = y$, and
(2) $d(x, y) = d(y, x)$ for all x, y in $\mathcal{X}$.

Let d be a symmetric on a set $\mathcal{X}$ and for $r > 0$ and any $x \in \mathcal{X}$, let $B(x, r) = \{y \in \mathcal{X} : d(x, y) < r\}$. A topology $t(d)$ on $\mathcal{X}$ is given by $U \in t(d)$ if and only if, for each $x \in U$, $B(x, r) \subset U$ for some $r > 0$. A symmetric d is a semi-metric if for each $x \in \mathcal{X}$ and each $r > 0$, $B(x, r)$ is a neighborhood of x in the topology $t(d)$. Note that $\lim_{n\to\infty} d(x_n, x) = 0$ if and only if $x_n \to x$ in the topology $t(d)$ [2].

Definition 1.1. [7] let $(\mathcal{X}, d)$ be a metric space, and let $f : \mathcal{X} \to \mathcal{X}$ and $F : \mathcal{X} \to B(\mathcal{X})$. The pair $\{F, f\}$ is a weakly compatible pair if and only if $Fx = \{fx\}$ implies that $fFx = Ffx$.

Definition 1.2. [1] Let f and g be two self-maps of a metric space $(\mathcal{X}, d)$. We say that f and g satisfy the property $(E.A)$ if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in \mathcal{X}$.

In 2003, Aamri and El Moutawakil [2] introduced the notion of compatible and weakly compatible maps in a symmetric space, also, they gave new definitions of properties $(E.A)$ and (H_E) in the same space.

Definition 1.3. [2] Let f and g be two self-maps of a symmetric space $(\mathcal{X}, d)$. f and g are said to be compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} d(fx_n, t) = \lim_{n\to\infty} d(gx_n, t) = 0$ for some $t \in \mathcal{X}$.

Definition 1.4. [2] Two self-maps f and g of a symmetric space $(\mathcal{X}, d)$ are said to be weakly compatible if they commute at their coincidence points.

Definition 1.5. [2] Let f and g be two self-maps of a symmetric space $(\mathcal{X}, d)$. We say that f and g satisfy the property $(E.A)$ if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} d(fx_n, t) = 0$ and $\lim_{n\to\infty} d(gx_n, t) = 0$ for some $t \in \mathcal{X}$.

Definition 1.6. [2] Let $(\mathcal{X}, d)$ be a symmetric space. We say that $(\mathcal{X}, d)$ satisfies the property (H_E) if given $\{x_n\}$, $\{y_n\}$ and x in $\mathcal{X}$, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(y_n, x) = 0$ imply $\lim_{n\to\infty} d(y_n, x_n) = 0$.

In their paper [2], Aamri and El Moutawakil gave some common fixed point theorems for self-maps of a symmetric space under a generalized contractive condition. Their self-maps were assumed to satisfy properties $(E.A)$, (H_E) and axioms $(W.3)$, $(W.4)$ of Wilson [10].

In 2006, Aliouche [3] generalized the results of [2] by using a contractive condition of integral type.

The main purpose of the present paper is to establish some common fixed point theorems for single and set-valued maps under a generalized contractive condition of integral type. These maps are assumed to satisfy new properties introduced on a symmetric space. Our results extend the results of Aamri and El Moutawakil [2], Aliouche [3] and others to the setting of single and set-valued maps.

2. Common fixed point theorems under a generalized contractive condition

Following the established symbology of the literature, $\mathcal{X}$ stands for a symmetric space and $B(\mathcal{X})$ denotes the family of all nonempty, bounded subsets of $\mathcal{X}$. Define

$$\delta(A,B)=\sup\{d(a,b):a\in A,b\in B\},$$

for all A, B in $B(\mathcal{X})$. When A consists of a single point a, we write $\delta(A,B)=\delta(a,B)$ and $\delta(A,B)=d(a,b)$ if B also consists of a single point b. The definition of the function δ yields the next properties:

$$\begin{aligned}
&\delta(A,B)=\delta(B,A)\geq 0,\\
&\delta(A,A)=diamA,\\
&\delta(A,B)=0 \text{ if and only if } A=B=\{a\},\\
&\delta(A,B)\leq\delta(A,C)+\delta(C,B),
\end{aligned}$$

for all A, B and C in $B(\mathcal{X})$.

A subset A of $\mathcal{X}$ is the limit of a sequence $\{A_n\}$ of non-empty subsets of $\mathcal{X}$ if each point a in A is the limit of a convergent sequence $\{a_n\}$, where a_n is in A_n for $n=1,2,\ldots$, and if for arbitrary $\epsilon>0$, there exists an integer N such that $A_n\subseteq A_\epsilon$ for $n>N$, where A_ϵ is the union of all open spheres with centers in A and radius ϵ [9].

Lemma 2.1. [9] *If $\{A_n\}$ and $\{B_n\}$ are sequences of bounded subsets of $(\mathcal{X},d)$ which converge to the bounded sets A and B respectively, then the sequence $\{\delta(A_n,B_n)\}$ converges to $\delta(A,B)$.*

Let F be a map of $\mathcal{X}$ into $B(\mathcal{X})$. F is continuous at the point x in $\mathcal{X}$ if whenever $\{x_n\}$ is a sequence of points in $\mathcal{X}$ converging to x, the sequence $\{Fx_n\}$ in $B(\mathcal{X})$ converges to Fx in $B(\mathcal{X})$ [9].

Definition 2.2. Let $(\mathcal{X},d)$ be a symmetric space and let $B(\mathcal{X})$ be the family of all nonempty bounded subsets of $\mathcal{X}$. Maps $f:\mathcal{X}\to\mathcal{X}$ and $F:\mathcal{X}\to B(\mathcal{X})$ are δ-compatible if and only if

$$\lim_{n\to\infty}\delta(Ffx_n,fFx_n)=0,$$

whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $fFx\in B(\mathcal{X})$ and $\lim_{n\to\infty}d(fx_n,t)=\lim_{n\to\infty}\delta(Fx_n,t)=0$ for some $t\in\mathcal{X}$.

Definition 2.3. Let $(\mathcal{X}, d)$ be a symmetric space and let $B(\mathcal{X})$ be the family of all nonempty bounded subsets of $\mathcal{X}$. Maps $f : \mathcal{X} \to \mathcal{X}$ and $F : \mathcal{X} \to B(\mathcal{X})$ are weakly compatible if and only if they commute at coincidence points; that is,

$$\{t \in \mathcal{X}/Ft = \{ft\}\} \subseteq \{t \in \mathcal{X}/Fft = fFt\}.$$

Definition 2.4. Let $(\mathcal{X}, d)$ be a symmetric space and let $B(\mathcal{X})$ be the family of all nonempty bounded subsets of $\mathcal{X}$. Maps $f : \mathcal{X} \to \mathcal{X}$ and $F : \mathcal{X} \to B(\mathcal{X})$ satisfy property $(E.A)$ if and only if there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that

$$\lim_{n\to\infty} d(fx_n, t) = 0 \text{ and } \lim_{n\to\infty} \delta(Fx_n, t) = 0,$$

for some $t \in \mathcal{X}$.

Example 2.5. Let $\mathcal{X} = [0, 1]$. Let d be a symmetric on $\mathcal{X}$ defined by

$$d(x, y) = \log[|x - y| + 1], \text{ for all } x, y \text{ in } \mathcal{X}.$$

First, note that the function d is not a metric.

Define $f : \mathcal{X} \to \mathcal{X}$ and $F : \mathcal{X} \to B(\mathcal{X})$ as follows:

$$fx = \frac{x}{2} \text{ and } Fx = [0, x] \text{ for all } x \in \mathcal{X}.$$

Let us consider the sequence $x_n = \frac{1}{n}$ for $n = 1, 2, 3, \dots$. Obviously

$$\lim_{n\to\infty} d(fx_n, 0) = \lim_{n\to\infty} \delta(Fx_n, 0) = 0 \in \mathcal{X}.$$

Then f and F satisfy property $(E.A)$.

Definition 2.6. Let $(\mathcal{X}, d)$ be a symmetric space and let $B(\mathcal{X})$ be the family of all nonempty bounded subsets of $\mathcal{X}$. $(\mathcal{X}, d)$ satisfies property (H_E) if and only if given $\{A_n\}$ in $B(\mathcal{X})$ and $\{x_n\}$, x in $\mathcal{X}$, $\lim_{n\to\infty} \delta(A_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, x) = 0$ imply $\lim_{n\to\infty} \delta(A_n, x_n) = 0$.

Example 2.7. (1) Every metric space $(\mathcal{X}, d)$ satisfies property (H_E).

(2) Let $\mathcal{X} = [0, 1]$ with the symmetric function d defined by

$$d(x, y) = \log[|x - y| + 1], \text{ for all } x, y \text{ in } \mathcal{X}.$$

It is easy to check that the symmetric space $(\mathcal{X}, d)$ satisfies property (H_E).

Encouraged by the Wilson's definition [10] we introduce the following notion:

Definition 2.8. Let $(\mathcal{X}, d)$ be a symmetric space.

(HB.1) Given A, $\{A_n\}$ in $B(\mathcal{X})$ and x in $\mathcal{X}$, $\lim_{n\to\infty} \delta(A_n, x) = 0$ and $\lim_{n\to\infty} \delta(A_n, A) = 0$ imply $A = \{x\}$.

(HB.2) Given $\{A_n\}$, $\{B_n\}$ in $B(\mathcal{X})$ and x in $\mathcal{X}$, $\lim_{n\to\infty} \delta(A_n, x) = 0$ and $\lim_{n\to\infty} \delta(A_n, B_n) = 0$ imply that $\lim_{n\to\infty} \delta(B_n, x) = 0$.

Before giving our main results, we introduce the definition of non-δ-compatible maps as follows:

Definition 2.9. Let $(\mathcal{X}, d)$ be a symmetric space and let $B(\mathcal{X})$ be the family of all nonempty bounded subsets of $\mathcal{X}$. Maps $f : \mathcal{X} \to \mathcal{X}$ and $F : \mathcal{X} \to B(\mathcal{X})$ are said to be **non-δ-compatible** if and only if there exists at least one sequence $\{x_n\}$ in $\mathcal{X}$ such that $fFx \in B(\mathcal{X})$ and $\lim_{n\to\infty} d(fx_n, t) = \lim_{n\to\infty} \delta(Fx_n, t) = 0$ for some $t \in \mathcal{X}$ but $\lim_{n\to\infty} \delta(Ffx_n, fFx_n)$ is either non zero or does not exist.

Therefore, two non-δ-compatible maps satisfy property $(E.A)$.
Let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the condition $0 < \Phi(t) < t$ for each $t > 0$.

2.1. **A common fixed point theorem for two maps.**

Theorem 2.10. *Let d be a symmetric for $\mathcal{X}$ that satisfies $(HB.1)$ and (H_E). Let $f : \mathcal{X} \to \mathcal{X}$ and $F : \mathcal{X} \to B(\mathcal{X})$ be a single and a set-valued map, respectively such that*

$$\int_0^{\delta(Fx,Fy)} \varphi(t)dt \leq \Phi\left(\int_0^{\max\{d(fx,fy),\delta(fx,Fy),\delta(Fy,fy)\}} \varphi(t)dt\right), \tag{2.1}$$

for all $(x, y) \in \mathcal{X}^2$, where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a Lebesgue-integrable map which is summable and such that $\int_0^{\epsilon} \varphi(t)dt > 0$ for all $\epsilon > 0$,

1. *f and F satisfy property $(E.A)$,*
2. *$F\mathcal{X} \subset f\mathcal{X}$,*
3. *F and f are weakly compatible.*

If the range of F or f is a complete subspace of $\mathcal{X}$, then F and f have a unique common fixed point in $\mathcal{X}$.

Proof. Since F and f satisfy property $(E.A)$, there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that $\lim_{n\to\infty} \delta(Fx_n, t) = \lim_{n\to\infty} d(fx_n, t) = 0$ for some $t \in \mathcal{X}$. Therefore, by (H_E), we have $\lim_{n\to\infty} \delta(Fx_n, fx_n) = 0$.

Suppose that $f\mathcal{X}$ is a complete subspace of $\mathcal{X}$. Then $t = fu$ for some $u \in \mathcal{X}$. We claim that $Fu = \{fu\}$. Indeed, by (2.1), we have

$$\begin{aligned} \int_0^{\delta(Fu,Fx_n)} \varphi(t)dt &\leq \Phi\left(\int_0^{\max\{d(fu,fx_n),\delta(fu,Fx_n),\delta(Fx_n,fx_n)\}} \varphi(t)dt\right) \\ &< \int_0^{\max\{d(fu,fx_n),\delta(fu,Fx_n),\delta(Fx_n,fx_n)\}} \varphi(t)dt. \end{aligned}$$

Letting $n \to \infty$, we have $\lim_{n\to\infty} \delta(Fu, Fx_n) = 0$. Hence, by $(HB.1)$, we have $Fu = \{t\} = \{fu\}$. The weak compatibility of F and f implies that $Ffu = fFu$ and then $FFu = Ffu = fFu = \{ffu\}$.

Let us show that fu is a common fixed point of F and f. Suppose that $ffu \neq fu$. In view of (2.1), it follows

$$\begin{aligned}\int_0^{d(fu,ffu)} \varphi(t)dt &= \int_0^{\delta(Fu,Ffu)} \varphi(t)dt \\ &\leq \Phi\left(\int_0^{\max\{d(fu,ffu),\delta(fu,Ffu),\delta(Ffu,ffu)\}} \varphi(t)dt\right) \\ &\leq \Phi\left(\int_0^{d(fu,ffu)} \varphi(t)dt\right) \\ &< \int_0^{d(fu,ffu)} \varphi(t)dt,\end{aligned}$$

which is a contradiction. Therefore $Ffu = \{ffu\} = \{fu\}$ and fu is a common fixed point of F and f. The proof is similar when $F\mathcal{X}$ is assumed to be a complete subspace of $\mathcal{X}$ since $F\mathcal{X} \subset f\mathcal{X}$. If $Fu = \{fu\} = \{u\}$, $Fv = \{fv\} = \{v\}$ and $u \neq v$, then (2.1) gives

$$\begin{aligned}\int_0^{d(u,v)} \varphi(t)dt &= \int_0^{\delta(Fu,Fv)} \varphi(t)dt \\ &\leq \Phi\left(\int_0^{\max\{d(fu,fv),\delta(fu,Fv),\delta(Fv,fv)\}} \varphi(t)dt\right) \\ &\leq \Phi\left(\int_0^{d(fu,fv)} \varphi(t)dt\right) \\ &< \int_0^{d(u,v)} \varphi(t)dt,\end{aligned}$$

which is a contradiction. Therefore $u = v$ and the common fixed point is unique. □

Since two non-δ-compatible maps of a symmetric space $(\mathcal{X}, d)$ satisfy property $(E.A)$, we get the following result.

Corollary 2.11. *Let d be a symmetric for $\mathcal{X}$ that satisfies $(HB.1)$ and (H_E). Let $f : \mathcal{X} \to \mathcal{X}$; $F : \mathcal{X} \to B(\mathcal{X})$ be two non-δ-compatible maps such that*

$$\int_0^{\delta(Fx,Fy)} \varphi(t)dt \leq \Phi\left(\int_0^{\max\{d(fx,fy),\delta(fx,Fy),\delta(Fy,fy)\}} \varphi(t)dt\right), \tag{2.2}$$

for all $(x, y) \in \mathcal{X}^2$, where φ is as in Theorem 2.10, and $F\mathcal{X} \subset f\mathcal{X}$. If the range of F or f is a complete subspace of $\mathcal{X}$, then F and f have a unique common fixed point.

2.2. A common fixed point theorem for four maps.

Theorem 2.12. *Let d be a symmetric for $\mathcal{X}$ that satisfies $(HB.1)$, $(HB.2)$ and (H_E). Let f, $g : \mathcal{X} \to \mathcal{X}$; F, $G : \mathcal{X} \to B(\mathcal{X})$ be maps such that*

$$\int_0^{\delta(Fx,Gy)} \varphi(t)dt \leq \Phi\left(\int_0^{\max\{d(fx,gy),\delta(fx,Gy),\delta(gy,Gy)\}} \varphi(t)dt\right), \tag{2.3}$$

for all $(x, y) \in \mathcal{X}^2$, where φ is as in Theorem 2.10,

1. *(F, f) and (G, g) are weakly compatible,*
2. *(F, f) or (G, g) satisfies property $(E.A)$, and*
3. *$F\mathcal{X} \subset g\mathcal{X}$ and $G\mathcal{X} \subset f\mathcal{X}$.*

If the range of one of maps F, G, f or g is a complete subspace of $\mathcal{X}$, then F, G, f and g have a unique common fixed point.

Proof. Suppose that (G, g) satisfies property $(E.A)$. Then there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that $\lim_{n\to\infty} \delta(Gx_n, t) = \lim_{n\to\infty} d(gx_n, t) = 0$, for some $t \in \mathcal{X}$. Since $G\mathcal{X} \subset f\mathcal{X}$, there exists in $\mathcal{X}$ a sequence $\{y_n\}$ such that $Gx_n = \{fy_n\}$. Hence $\lim_{n\to\infty} d(fx_n, t) = 0$. Let us show that $\lim_{n\to\infty} \delta(Fy_n, t) = 0$. Indeed, in view of (2.3), we have

$$\begin{aligned}
\int_0^{\delta(Fy_n,Gx_n)} \varphi(t)dt &\leq \Phi\left(\int_0^{\max\{d(fy_n,gx_n),\delta(fy_n,Gx_n),\delta(gx_n,Gx_n)\}} \varphi(t)dt\right)\\
&\leq \Phi\left(\int_0^{\max\{\delta(Gx_n,gx_n),0,\delta(gx_n,Gx_n)\}} \varphi(t)dt\right)\\
&\leq \Phi\left(\int_0^{\delta(Gx_n,gx_n)} \varphi(t)dt\right).
\end{aligned}$$

Therefore, by (H_E), one has $\lim_{n\to\infty} \delta(Fy_n, Gx_n) = 0$. By $(HB.2)$, we deduce that $\lim_{n\to\infty} \delta(Fy_n, t) = 0$. Suppose that $f\mathcal{X}$ is a complete subspace of $\mathcal{X}$. Then $t = fu$ for some $u \in \mathcal{X}$. Subsequently, we have

$$\lim_{n\to\infty} \delta(Fy_n, fu) = \lim_{n\to\infty} \delta(Gx_n, fu) = \lim_{n\to\infty} d(gx_n, fu) = \lim_{n\to\infty} d(fy_n, fu) = 0.$$

Using (2.3), it follows

$$\int_0^{\delta(Fu,Gx_n)} \varphi(t)dt \leq \Phi\left(\int_0^{\max\{d(fu,gx_n),\delta(fu,Gx_n),\delta(gx_n,Gx_n)\}} \varphi(t)dt\right).$$

Letting $n \to \infty$, we have $\lim_{n\to\infty} \delta(Fu, Gx_n) = 0$. By $(HB.1)$, we have $Fu = \{fu\}$. The weak compatibility of F and f implies that $Ffu = fFu$ and then $FFu = Ffu = fFu = \{ffu\}$.

On the other hand, since $F\mathcal{X} \subset g\mathcal{X}$, there exists $v \in \mathcal{X}$ such that $Fu = \{gv\}$. We claim that $\{gv\} = Gv$. If not, condition (2.3) gives

$$\begin{aligned}\int_0^{\delta(Fu,Gv)} \varphi(t)dt &\leq \Phi\left(\int_0^{\max\{d(fu,gv),\delta(fu,Gv),\delta(gv,Gv)\}} \varphi(t)dt\right)\\ &\leq \Phi\left(\int_0^{\delta(Fu,Gv)} \varphi(t)dt\right)\\ &< \int_0^{\delta(Fu,Gv)} \varphi(t)dt,\end{aligned}$$

which is a contradiction. Hence $Fu = Gv = \{gv\} = \{fu\}$. The weak compatibility of G and g implies that $Ggv = gGv$ and $GGv = Ggv = gGv = \{ggv\}$.

Let us show that fu is a common fixed point of F, G, f and g. Suppose that $ffu \neq fu$. We have

$$\begin{aligned}\int_0^{d(ffu,fu)} \varphi(t)dt &= \int_0^{\delta(Ffu,Gv)} \varphi(t)dt\\ &\leq \Phi\left(\int_0^{\max\{d(ffu,gv),\delta(ffu,Gv),\delta(gv,Gv)\}} \varphi(t)dt\right)\\ &\leq \Phi\left(\int_0^{d(ffu,fu)} \varphi(t)dt\right)\\ &< \int_0^{d(ffu,fu)} \varphi(t)dt,\end{aligned}$$

which is a contradiction. Therefore $Ffu = \{ffu\} = \{fu\}$ and fu is a common fixed point of F and f. Similarly, we prove that gv is a common fixed point of G and g. Since $fu = gv$, we conclude that fu is a common fixed point of F, G, f and g. The proof is similar when $g\mathcal{X}$ is assumed to be a complete subspace of $\mathcal{X}$. The cases in which $F\mathcal{X}$ or $G\mathcal{X}$ is a complete subspace of X are similar to the cases in which $g\mathcal{X}$ or $f\mathcal{X}$, respectively, is complete since $F\mathcal{X} \subset g\mathcal{X}$ and $G\mathcal{X} \subset f\mathcal{X}$. If $Fu = Gu = \{gu\} = \{fu\} = \{u\}$ and $Fv = Gv = \{gv\} = \{fv\} = \{v\}$ and $u \neq v$, then (2.3) gives

$$\begin{aligned}\int_0^{d(u,v)} \varphi(t)dt &= \int_0^{\delta(Fu,Gv)} \varphi(t)dt\\ &\leq \Phi\left(\int_0^{\max\{d(fu,gv),\delta(fu,Gv),\delta(gv,Gv)\}} \varphi(t)dt\right)\\ &\leq \Phi\left(\int_0^{d(u,v)} \varphi(t)dt\right)\\ &< \int_0^{d(u,v)} \varphi(t)dt,\end{aligned}$$

which is a contradiction. Therefore $u = v$ and the common fixed point is unique. □

Corollary 2.13. *Let $(\mathcal{X}, d)$ be a metric space, $B(\mathcal{X})$ be the family of all nonempty bounded subsets of $\mathcal{X}$ and let f, $g : \mathcal{X} \to \mathcal{X}$; F, $G : \mathcal{X} \to B(\mathcal{X})$ be single and set-valued maps such that for all $(x, y) \in \mathcal{X}^2$,*

$$\int_0^{\delta(Fx,Gy)} \varphi(t)dt \leq \Phi\left(\int_0^{\max\{d(fx,gy),\delta(fx,Gy),\delta(gy,Gy)\}} \varphi(t)dt\right), \tag{2.4}$$

(1) *(F, f) and (G, g) are weakly compatible,*
(2) *F and f or G and g are D-maps, and*
(3) *$F\mathcal{X} \subset g\mathcal{X}$ and $G\mathcal{X} \subset f\mathcal{X}$.*

If the range of the one of maps F, G, f and g is a complete subspace of $\mathcal{X}$, then F, G, f and g have a unique common fixed point.

Remark 2.14. (1) Theorem 2.12 is an extension of Theorem 1 of Aliouche [3].
(2) If $\varphi(t) = 1$ in Theorem 2.12, we obtain an extension of Theorem 2.2 of Aamri and El Moutawakil [2].
(3) If $\varphi(t) = 1$ in Theorem 2.10, we get an extension of Theorem 2.1 of [2].
(4) Corollary 2.11 is an extension of Corollary 2 of [3].
(5) If we put $\varphi(t) = 1$ in Corollary 2.11, we obtain an extension of Corollary 2.1 of [2].
(6) Corollary 2.13 is an extension of Corollary 3 of Aliouche [3].
(7) If we let $\varphi(t) = 1$ in Corollary 2.13, we get an extension of Corollary 2.2 of Aamri and El Moutawakil [2].

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