NEW PROPERTIES UNDER GENERALIZED CONTRACTIVE CONDITIONS

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Abstract. The aim of this contribution is to establish some common fixed point theorems for single and set-valued maps under contractive conditions of integral type on a symmetric space. These maps are assumed to satisfy new properties which extend the results of Aliouche [3], Aamri and El Moutawakil [2] and references therein, also they generalize the notion of non-compatible and non-δ-compatible maps in the setting of symmetric spaces.

1. Introduction and preliminaries

In 1982, Sessa [8] generalized the concept of commuting maps by giving the notion of weakly commuting maps. Two self-maps \( f \) and \( g \) of a metric space \((\mathcal{X}, d)\) are said to be weakly commuting if, for all \( x \in \mathcal{X} \) we have

\[
d(fgx, gfx) \leq d(gx, fx).
\]

Further, in 1986, Jungck [5] gave a generalization of commuting and weakly commuting maps by introducing the concept of compatible maps. Self-maps \( f \) and \( g \) of a metric space \((\mathcal{X}, d)\) are compatible if and only if whenever \( \{x_n\} \) is a sequence in \( \mathcal{X} \) such that \( fx_n, gx_n \to t \in \mathcal{X} \), then \( d(fx_n, gx_n) \to 0 \).

Later, the same author with Rhoades [6] extended the concept of compatible maps to maps \( f : \mathcal{X} \to \mathcal{X} \) and \( F : \mathcal{X} \to B(\mathcal{X}) \) by requiring that \( FFx \in B(\mathcal{X}) \) for \( x \in \mathcal{X} \) and \( \delta(FFx_n, Ffx_n) \to 0 \) whenever \( \{x_n\} \) is a sequence in \( \mathcal{X} \) such that \( Fx_n \to \{t\} \) (\( \delta(Fx_n, t) \to 0 \)) and \( fx_n \to t \) for some \( t \in \mathcal{X} \).

This last definition motivated the definition of weakly compatible maps [7] mentioned below.

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On the other hand, Aamri and El Moutawakil [1] have established the notion of property \((E.A)\) for single valued maps.

To generalize this property, Djoudi and Khemis [4] introduced the definition of the so-called \(D\)-maps as follows: maps \(f : \mathcal{X} \rightarrow \mathcal{X}\) and \(F : \mathcal{X} \rightarrow B(\mathcal{X})\) are said to be \(D\)-maps if and only if there exists a sequence \(\{x_n\}\) in \(\mathcal{X}\) such that, 
\[
\lim_{n \to \infty} fx_n = t \quad \text{and} \quad \lim_{n \to \infty} Fx_n = \{t\} \quad \text{for some} \; t \in \mathcal{X}.
\]

Let \(\mathcal{X}\) be a set. Recall that a symmetric on \(\mathcal{X}\) is a nonnegative real function \(d\) on \(\mathcal{X} \times \mathcal{X}\) into \([0, \infty)\) such that
\[(1) \quad d(x, y) = 0 \quad \text{if and only if} \quad x = y, \quad \text{and} \]
\[(2) \quad d(x, y) = d(y, x) \quad \text{for all} \quad x, y \in \mathcal{X}.
\]

Let \(d\) be a symmetric on a set \(\mathcal{X}\) and for \(r > 0\) and any \(x \in \mathcal{X}\), let \(B(x, r) = \{y \in \mathcal{X} : d(x, y) < r\}\). A topology \(t(d)\) on \(\mathcal{X}\) is given by \(U \in t(d)\) if and only if, for each \(x \in U\), \(B(x, r) \subseteq U\) for some \(r > 0\). A symmetric \(d\) is a semi-metric if for each \(x \in \mathcal{X}\) and each \(r > 0\), \(B(x, r)\) is a neighborhood of \(x\) in the topology \(t(d)\). Note that \(\lim_{n \to \infty} d(x_n, x) = 0\) if and only if \(x_n \to x\) in the topology \(t(d)\) [2].

**Definition 1.1.** [7] Let \((\mathcal{X}, d)\) be a metric space, and let \(f : \mathcal{X} \rightarrow \mathcal{X}\) and \(F : \mathcal{X} \rightarrow B(\mathcal{X})\). The pair \(\{F, f\}\) is a weakly compatible pair if and only if \(Fx = \{fx\}\) implies that \(FFx = Ffx\).

**Definition 1.2.** [1] Let \(f\) and \(g\) be two self-maps of a metric space \((\mathcal{X}, d)\). We say that \(f\) and \(g\) satisfy the property \((E.A)\) if there exists a sequence \(\{x_n\}\) such that \(\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t\) for some \(t \in \mathcal{X}\).

In 2003, Aamri and El Moutawakil [2] introduced the notion of compatible and weakly compatible maps in a symmetric space, also, they gave new definitions of properties \((E.A)\) and \((H_E)\) in the same space.

**Definition 1.3.** [2] Let \(f\) and \(g\) be two self-maps of a symmetric space \((\mathcal{X}, d)\). \(f\) and \(g\) are said to be compatible if \(\lim_{n \to \infty} d(fgx_n, gfx_n) = 0\) whenever \(\{x_n\}\) is a sequence in \(\mathcal{X}\) such that \(\lim_{n \to \infty} d(fx_n, t) = \lim_{n \to \infty} d(gx_n, t) = 0\) for some \(t \in \mathcal{X}\).

**Definition 1.4.** [2] Two self-maps \(f\) and \(g\) of a symmetric space \((\mathcal{X}, d)\) are said to be weakly compatible if they commute at their coincidence points.

**Definition 1.5.** [2] Let \(f\) and \(g\) be two self-maps of a symmetric space \((\mathcal{X}, d)\). We say that \(f\) and \(g\) satisfy the property \((E.A)\) if there exists a sequence \(\{x_n\}\) such that \(\lim_{n \to \infty} fx_n, t = 0\) and \(\lim_{n \to \infty} gx_n, t = 0\) for some \(t \in \mathcal{X}\).

**Definition 1.6.** [2] Let \((\mathcal{X}, d)\) be a symmetric space. We say that \((\mathcal{X}, d)\) satisfies the property \((H_E)\) if given \(\{x_n\}, \{y_n\}\) and \(x\) in \(\mathcal{X}\), \(\lim_{n \to \infty} dx_n, x = 0\) and \(\lim_{n \to \infty} dy_n, x = 0\) imply \(\lim_{n \to \infty} dy_n, x_n = 0\).

In their paper [2], Aamri and El Moutawakil gave some common fixed point theorems for self-maps of a symmetric space under a generalized contractive condition. Their self-maps were assumed to satisfy properties \((E.A)\), \((H_E)\) and axioms \((W.3), (W.4)\) of Wilson [10].

The main purpose of the present paper is to establish some common fixed point theorems for single and set-valued maps under a generalized contractive condition of integral type. These maps are assumed to satisfy new properties introduced on a symmetric space. Our results extend the results of Aamri and El Moutawakil [2], Aliouche [3] and others to the setting of single and set-valued maps.

2. COMMON FIXED POINT THEOREMS UNDER A GENERALIZED CONTRACTIVE CONDITION

Following the established symbology of the literature, \( \mathcal{X} \) stands for a symmetric space and \( B(\mathcal{X}) \) denotes the family of all nonempty, bounded subsets of \( \mathcal{X} \). Define

\[
\delta(A,B) = \sup\{d(a,b) : a \in A, b \in B\},
\]

for all \( A, B \) in \( B(\mathcal{X}) \). When \( A \) consists of a single point \( a \), we write \( \delta(A,B) = \delta(a,B) \) and \( \delta(A,B) = d(a,b) \) if \( B \) also consists of a single point \( b \). The definition of the function \( \delta \) yields the next properties:

\[
\begin{align*}
\delta(A,B) &= \delta(B,A) \geq 0, \\
\delta(A,A) &= \text{diam} A, \\
\delta(A,B) &= 0 \text{ if and only if } A = B = \{a\}, \\
\delta(A,B) &\leq \delta(A,C) + \delta(C,B),
\end{align*}
\]

for all \( A, B \) and \( C \) in \( B(\mathcal{X}) \).

A subset \( A \) of \( \mathcal{X} \) is the limit of a sequence \( \{A_n\} \) of non-empty subsets of \( \mathcal{X} \) if each point \( a \) in \( A \) is the limit of a convergent sequence \( \{a_n\} \), where \( a_n \) is in \( A_n \) for \( n = 1, 2, \ldots \), and if for arbitrary \( \epsilon > 0 \), there exists an integer \( N \) such that \( A_n \subseteq A_\epsilon \) for \( n > N \), where \( A_\epsilon \) is the union of all open spheres with centers in \( A \) and radius \( \epsilon \) [9].

**Lemma 2.1.** [9] If \( \{A_n\} \) and \( \{B_n\} \) are sequences of bounded subsets of \( (\mathcal{X},d) \) which converge to the bounded sets \( A \) and \( B \) respectively, then the sequence \( \{\delta(A_n,B_n)\} \) converges to \( \delta(A,B) \).

Let \( F \) be a map of \( \mathcal{X} \) into \( B(\mathcal{X}) \). \( F \) is continuous at the point \( x \) in \( \mathcal{X} \) if whenever \( \{x_n\} \) is a sequence of points in \( \mathcal{X} \) converging to \( x \), the sequence \( \{Fx_n\} \) in \( B(\mathcal{X}) \) converges to \( Fx \) in \( B(\mathcal{X}) \) [9].

**Definition 2.2.** Let \( (\mathcal{X},d) \) be a symmetric space and let \( B(\mathcal{X}) \) be the family of all nonempty bounded subsets of \( \mathcal{X} \). Maps \( f : \mathcal{X} \to \mathcal{X} \) and \( F : \mathcal{X} \to B(\mathcal{X}) \) are \( \delta \)-compatible if and only if

\[
\lim_{n \to \infty} \delta(Ffx_n,ffx_n) = 0,
\]

whenever \( \{x_n\} \) is a sequence in \( \mathcal{X} \) such that \( ffx \in B(\mathcal{X}) \) and \( \lim_{n \to \infty} d(fx_n,t) = \lim_{n \to \infty} \delta(Fx_n,t) = 0 \) for some \( t \in \mathcal{X} \).
Definition 2.3. Let $(\mathcal{X}, d)$ be a symmetric space and let $B(\mathcal{X})$ be the family of all nonempty bounded subsets of $\mathcal{X}$. Maps $f : \mathcal{X} \to \mathcal{X}$ and $F : \mathcal{X} \to B(\mathcal{X})$ are weakly compatible if and only if they commute at coincidence points; that is,
\[
\{ t \in \mathcal{X} / Ft = \{ ft \} \} \subseteq \{ t \in \mathcal{X} / Fft = fFt \}.
\]

Definition 2.4. Let $(\mathcal{X}, d)$ be a symmetric space and let $B(\mathcal{X})$ be the family of all nonempty bounded subsets of $\mathcal{X}$. Maps $f : \mathcal{X} \to \mathcal{X}$ and $F : \mathcal{X} \to B(\mathcal{X})$ satisfy property (\textit{E.A}) if and only if there exists a sequence $\{ x_n \}$ in $\mathcal{X}$ such that
\[
\lim_{n \to \infty} d(fx_n, t) = 0 \quad \text{and} \quad \lim_{n \to \infty} \delta(Fx_n, t) = 0,
\]
for some $t \in \mathcal{X}$.

Example 2.5. Let $\mathcal{X} = [0, 1]$. Let $d$ be a symmetric on $\mathcal{X}$ defined by
\[
d(x, y) = \log(|x - y| + 1), \quad \text{for all } x, y \in \mathcal{X}.
\]
First, note that the function $d$ is not a metric.

Define $f : \mathcal{X} \to \mathcal{X}$ and $F : \mathcal{X} \to B(\mathcal{X})$ as follows:
\[
f(x) = \frac{x}{2} \quad \text{and} \quad F(x) = [0, x] \quad \text{for all } x \in \mathcal{X}.
\]
Let us consider the sequence $x_n = \frac{1}{n}$ for $n = 1, 2, 3, \ldots$. Obviously
\[
\lim_{n \to \infty} d(fx_n, 0) = \lim_{n \to \infty} \delta(Fx_n, 0) = 0 \quad \text{in } \mathcal{X}.
\]
Then $f$ and $F$ satisfy property (\textit{E.A}).

Definition 2.6. Let $(\mathcal{X}, d)$ be a symmetric space and let $B(\mathcal{X})$ be the family of all nonempty bounded subsets of $\mathcal{X}$. $(\mathcal{X}, d)$ satisfies property (\textit{H.E}) if and only if given $\{ A_n \}$ in $B(\mathcal{X})$ and $\{ x_n \}$ in $\mathcal{X}$, $\lim_{n \to \infty} \delta(A_n, x) = 0$ and $\lim_{n \to \infty} d(x_n, x) = 0$ imply $\lim_{n \to \infty} \delta(A_n, x_n) = 0$.

Example 2.7. (1) Every metric space $(\mathcal{X}, d)$ satisfies property (\textit{H.E}).
(2) Let $\mathcal{X} = [0, 1]$ with the symmetric function $d$ defined by
\[
d(x, y) = \log(|x - y| + 1), \quad \text{for all } x, y \in \mathcal{X}.
\]
It is easy to check that the symmetric space $(\mathcal{X}, d)$ satisfies property (\textit{H.E}).

Encouraged by the Wilson’s definition \cite{10} we introduce the following notion:

Definition 2.8. Let $(\mathcal{X}, d)$ be a symmetric space.
(\textit{HB.1}) Given $A$, $\{ A_n \}$ in $B(\mathcal{X})$ and $x$ in $\mathcal{X}$, $\lim_{n \to \infty} \delta(A_n, x) = 0$ and $\lim_{n \to \infty} \delta(A_n, A) = 0$ imply $A = \{ x \}$.
(\textit{HB.2}) Given $\{ A_n \}$, $\{ B_n \}$ in $B(\mathcal{X})$ and $x$ in $\mathcal{X}$, $\lim_{n \to \infty} \delta(A_n, x) = 0$ and $\lim_{n \to \infty} \delta(A_n, B_n) = 0$ imply that $\lim_{n \to \infty} \delta(B_n, x) = 0$.

Before giving our main results, we introduce the definition of non-$\delta$-compatible maps as follows:
Definition 2.9. Let $(X, d)$ be a symmetric space and let $B(X)$ be the family of all nonempty bounded subsets of $X$. Maps $f : X \to X$ and $F : X \to B(X)$ are said to be non-δ-compatible if and only if there exists at least one sequence $\{x_n\}$ in $X$ such that $FfX \in B(X)$ and $\lim_{n \to \infty} d(fx_n, t) = \lim_{n \to \infty} \delta(Fx_n, t) = 0$ for some $t \in X$ but $\lim_{n \to \infty} \delta(Ffx_n, Ffx_n)$ is either non-zero or does not exist.

Therefore, two non-δ-compatible maps satisfy property (E.A).

Let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the condition $0 < \Phi(t) < t$ for each $t > 0$.

2.1. A common fixed point theorem for two maps.

Theorem 2.10. Let $d$ be a symmetric for $X$ that satisfies $(HB.1)$ and $(H_E)$. Let $f : X \to X$ and $F : X \to B(X)$ be a single and a set-valued map, respectively such that

$$
\int_0^\delta(Fx,Fy) \varphi(t)dt \leq \Phi \left( \int_0^{\max\{d(fx,fy),\delta(fx,Fy),\delta(Fy,fy)\}} \varphi(t)dt \right),
$$

(2.1)

for all $(x, y) \in X^2$, where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a Lebesgue-integrable map which is summable and such that $\int_0^\epsilon \varphi(t)dt > 0$ for all $\epsilon > 0$,

1. $f$ and $F$ satisfy property (E.A),
2. $F \subseteq f X$,
3. $F$ and $f$ are weakly compatible.

If the range of $F$ or $f$ is a complete subspace of $X$, then $F$ and $f$ have a unique common fixed point in $X$.

Proof. Since $F$ and $f$ satisfy property (E.A), there exists a sequence $\{x_n\}$ in $X$ such that $\lim_{n \to \infty} \delta(Fx_n, t) = \lim_{n \to \infty} d(fx_n, t) = 0$ for some $t \in X$. Therefore, by $(H_E)$, we have $\lim_{n \to \infty} \delta(Fx_n, fx_n) = 0$.

Suppose that $fX$ is a complete subspace of $X$. Then $t = fu$ for some $u \in X$. We claim that $Fu = \{fu\}$. Indeed, by (2.1), we have

$$
\int_0^{\delta(Fu,Fx_n)} \varphi(t)dt \leq \Phi \left( \int_0^{\max\{d(fu,fx_n),\delta(fu,Fx_n),\delta(Fx_n,fx_n)\}} \varphi(t)dt \right)
$$

$$
< \int_0^{\max\{d(fu,fx_n),\delta(fu,Fx_n),\delta(Fx_n,fx_n)\}} \varphi(t)dt.
$$

Letting $n \to \infty$, we have $\lim_{n \to \infty} \delta(Fu, Fx_n) = 0$. Hence, by $(HB.1)$, we have $Fu = \{t\} = \{fu\}$. The weak compatibility of $F$ and $f$ implies that $Ffu = fFf$ and then $FFu = Ffu = fFf = \{ffu\}$. 
Let us show that \( fu \) is a common fixed point of \( F \) and \( f \). Suppose that \( ffu \neq fu \). In view of (2.1), it follows
\[
\int_{0}^{d(fu,ffu)} \varphi(t) \, dt = \int_{0}^{\delta(Fu,Ffu)} \varphi(t) \, dt \leq \Phi \left( \int_{0}^{\max\{d(fu,ffu),\delta(fu,Ffu),\delta(Ffu,ffu)\}} \varphi(t) \, dt \right)
\leq \Phi \left( \int_{0}^{d(fu,ffu)} \varphi(t) \, dt \right) < \int_{0}^{d(fu,ffu)} \varphi(t) \, dt,
\]
which is a contradiction. Therefore \( Ffu = \{ffu\} = \{fu\} \) and \( fu \) is a common fixed point of \( F \) and \( f \). The proof is similar when \( FX \) is assumed to be a complete subspace of \( X \) since \( FX \subset fX \). If \( Fu = \{fu\} = \{u\} \), \( Fv = \{fv\} = \{v\} \) and \( u \neq v \), then (2.1) gives
\[
\int_{0}^{d(u,v)} \varphi(t) \, dt = \int_{0}^{\delta(Fu,Fv)} \varphi(t) \, dt \leq \Phi \left( \int_{0}^{\max\{d(fu,fv),\delta(fu,Fv),\delta(Fv,fv)\}} \varphi(t) \, dt \right) \leq \Phi \left( \int_{0}^{d(fu,fv)} \varphi(t) \, dt \right) < \int_{0}^{d(u,v)} \varphi(t) \, dt,
\]
which is a contradiction. Therefore \( u = v \) and the common fixed point is unique. \( \square \)

Since two non-\( \delta \)-compatible maps of a symmetric space \( (X,d) \) satisfy property \( (E.A) \), we get the following result.

**Corollary 2.11.** Let \( d \) be a symmetric for \( X \) that satisfies \( (HB.1) \) and \( (HE) \). Let \( f : X \to X; F : X \to B(X) \) be two non-\( \delta \)-compatible maps such that
\[
\int_{0}^{\delta(Fx,Fy)} \varphi(t) \, dt \leq \Phi \left( \int_{0}^{\max\{d(fx,fy),\delta(fx,Fy),\delta(Fy,fy)\}} \varphi(t) \, dt \right), \tag{2.2}
\]
for all \( (x,y) \in X^2 \), where \( \varphi \) is as in Theorem 2.10, and \( FX \subset fX \). If the range of \( F \) or \( f \) is a complete subspace of \( X \), then \( F \) and \( f \) have a unique common fixed point.
2.2. A common fixed point theorem for four maps.

**Theorem 2.12.** Let \( d \) be a symmetric for \( X \) that satisfies (HB.1), (HB.2) and (HE). Let \( f, g : X \to X; F, G : X \to B(X) \) be maps such that

\[
\int_0^\delta(Fx,Gy) \varphi(t) dt \leq \Phi \left( \int_0^{\max\{d(fx,gy),\delta(fx,Gy),\delta(gy,Gy)\}} \varphi(t) dt \right),
\]

for all \((x, y) \in X^2\), where \( \varphi \) is as in Theorem 2.10,

(1) \((F, f)\) and \((G, g)\) are weakly compatible,
(2) \((F, f)\) or \((G, g)\) satisfies property (E.A), and
(3) \(FX \subseteq gX\) and \(GX \subseteq fX\).

If the range of one of maps \(F, G, f\) or \(g\) is a complete subspace of \(X\), then \(F, G, f\) and \(g\) have a unique common fixed point.

**Proof.** Suppose that \((G, g)\) satisfies property (E.A). Then there exists a sequence \(\{x_n\}\) in \(X\) such that \(\lim_{n \to \infty} \delta(Gx_n, t) = \lim_{n \to \infty} d(gx_n, t) = 0\), for some \(t \in X\). Since \(GX \subseteq fX\), there exists in \(X\) a sequence \(\{y_n\}\) such that \(Gx_n = \{fy_n\}\). Hence \(\lim_{n \to \infty} d(fx_n, t) = 0\). Let us show that \(\lim_{n \to \infty} \delta(Fy_n, t) = 0\). Indeed, in view of (2.3), we have

\[
\int_0^\delta(Fy_n,Gx_n) \varphi(t) dt \leq \Phi \left( \int_0^{\max\{d(fy_n,gx_n),\delta(fy_n,Gx_n),\delta(gx_n,Gx_n)\}} \varphi(t) dt \right) 
\]

\[
\leq \Phi \left( \int_0^{\delta(Gx_n,gx_n)} \varphi(t) dt \right) 
\]

\[
\leq \Phi \left( \int_0^{\delta(Gx_n,x_n)} \varphi(t) dt \right). 
\]

Therefore, by (HE), one has \(\lim_{n \to \infty} \delta(Fy_n, Gx_n) = 0\). By (HB.2), we deduce that \(\lim_{n \to \infty} \delta(Fy_n, t) = 0\). Suppose that \(fX\) is a complete subspace of \(X\). Then \(t = fu\) for some \(u \in X\). Subsequently, we have

\[
\lim_{n \to \infty} \delta(Fy_n, fu) = \lim_{n \to \infty} \delta(Gx_n, fu) = \lim_{n \to \infty} d(gx_n, fu) = \lim_{n \to \infty} d(fy_n, fu) = 0. 
\]

Using (2.3), it follows

\[
\int_0^\delta(Fu,Gx_n) \varphi(t) dt \leq \Phi \left( \int_0^{\max\{d(fu,gx_n),\delta(fu,Gx_n),\delta(gx_n,Gx_n)\}} \varphi(t) dt \right) . 
\]

Letting \(n \to \infty\), we have \(\lim_{n \to \infty} \delta(Fu, Gx_n) = 0\). By (HB.1), we have \(Fu = \{fu\}\). The weak compatibility of \(F\) and \(f\) implies that \(Ffu = fFu\) and then \(FFu = Ffu = fFu = \{ffu\}\).
On the other hand, since $F\mathcal{X} \subset g\mathcal{X}$, there exists $v \in \mathcal{X}$ such that $Fu = \{gv\}$. We claim that $\{gv\} = Gv$. If not, condition (2.3) gives

$$
\int_0^{\delta(Fu,Gv)} \varphi(t) dt \leq \Phi \left( \int_0^{\max\{d(fu,gv),\delta(fu,Gv),\delta(gv,Gv)\}} \varphi(t) dt \right)
$$

$$
\leq \Phi \left( \int_0^{\delta(Fu,Gv)} \varphi(t) dt \right)
$$

$$
< \int_0^{\delta(Fu,Gv)} \varphi(t) dt,
$$

which is a contradiction. Hence $Fu = Gv = \{gv\} = \{fu\}$. The weak compatibility of $G$ and $g$ implies that $Ggv = gGv$ and $GGv = Ggv = gGv = \{ggv\}$.

Let us show that $fu$ is a common fixed point of $F$, $G$, $f$ and $g$. Suppose that $ffu \neq fu$. We have

$$
\int_0^{d(ffu,fu)} \varphi(t) dt = \int_0^{\delta(Ffu,Gv)} \varphi(t) dt
$$

$$
\leq \Phi \left( \int_0^{\max\{d(ffu,gv),\delta(ffu,Gv),\delta(gv,Gv)\}} \varphi(t) dt \right)
$$

$$
\leq \Phi \left( \int_0^{d(ffu,fu)} \varphi(t) dt \right)
$$

$$
< \int_0^{d(ffu,fu)} \varphi(t) dt,
$$

which is a contradiction. Therefore $Ffu = \{ffu\} = \{fu\}$ and $fu$ is a common fixed point of $F$ and $f$. Similarly, we prove that $gv$ is a common fixed point of $G$ and $g$. Since $fu = gv$, we conclude that $fu$ is a common fixed point of $F$, $G$, $f$ and $g$. The proof is similar when $g\mathcal{X}$ is assumed to be a complete subspace of $\mathcal{X}$. The cases in which $F\mathcal{X}$ or $G\mathcal{X}$ is a complete subspace of $X$ are similar to the cases in which $g\mathcal{X}$ or $f\mathcal{X}$, respectively, is complete since $F\mathcal{X} \subset g\mathcal{X}$ and $G\mathcal{X} \subset f\mathcal{X}$. If $Fu = Gu = \{gu\} = \{fu\} = \{u\}$ and $Fv = Gv = \{gv\} = \{fv\} = \{v\}$ and $u \neq v$, then (2.3) gives

$$
\int_0^{d(u,v)} \varphi(t) dt = \int_0^{\delta(Fu,Gv)} \varphi(t) dt
$$

$$
\leq \Phi \left( \int_0^{\max\{d(fu,gv),\delta(fu,Gv),\delta(gv,Gv)\}} \varphi(t) dt \right)
$$

$$
\leq \Phi \left( \int_0^{d(u,v)} \varphi(t) dt \right)
$$

$$
< \int_0^{d(u,v)} \varphi(t) dt,
$$
which is a contradiction. Therefore \( u = v \) and the common fixed point is unique. \( \square \)

**Corollary 2.13.** Let \((X, d)\) be a metric space, \(B(X)\) be the family of all nonempty bounded subsets of \(X\) and let \(f, g : X \rightarrow X; F, G : X \rightarrow B(X)\) be single and set-valued maps such that for all \((x, y) \in X^2\),

\[
\int_0^{\delta(Fx, Gy)} \varphi(t)dt \leq \Phi \left( \int_0^{\max\{d(fx, gy), \delta(fx, Gy), \delta(gy, Gy)\}} \varphi(t)dt \right), \tag{2.4}
\]

(1) \((F, f)\) and \((G, g)\) are weakly compatible,
(2) \(F\) and \(f\) or \(G\) and \(g\) are \(D\)-maps, and
(3) \(F X \subset g X\) and \(G X \subset f X\).

If the range of the one of maps \(F\), \(G\), \(f\) and \(g\) is a complete subspace of \(X\), then \(F\), \(G\), \(f\) and \(g\) have a unique common fixed point.

**Remark 2.14.**
(1) Theorem 2.12 is an extension of Theorem 1 of Aliouche [3].
(2) If \(\varphi(t) = 1\) in Theorem 2.12, we obtain an extension of Theorem 2.2 of Aamri and El Moutawakil [2].
(3) If \(\varphi(t) = 1\) in Theorem 2.10, we get an extension of Theorem 2.1 of [2].
(4) Corollary 2.11 is an extension of Corollary 2 of [3].
(5) If we put \(\varphi(t) = 1\) in Corollary 2.11, we obtain an extension of Corollary 2.1 of [2].
(6) Corollary 2.13 is an extension of Corollary 3 of Aliouche [3].
(7) If we let \(\varphi(t) = 1\) in Corollary 2.13, we get an extension of Corollary 2.2 of Aamri and El Moutawakil [2].

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**References**

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