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CONSTRUCTING AN ELEMENT OF A BANACH SPACE WITH GIVEN DEVIATION FROM ITS NESTED SUBSPACES

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Abstract. This paper contains two improvements on a theorem of S. N. Bernstein for Banach spaces. We show that if X is an arbitrary infinite-dimensional Banach space, $\{Y_n\}$ is a sequence of strictly nested subspaces of X and if $\{d_n\}$ is a non-increasing sequence of non-negative numbers tending to 0, then for any $c \in (0,1]$ we can find $x_c \in X$, such that the distance $\rho(x_c, Y_n)$ from x_c to Y_n satisfies

$$cd_n \leq \rho(x_c, Y_n) \leq 4cd_n, \text{ for all } n \in \mathbb{N}.$$

We prove the above inequality by first improving Borodin (2006)'s result for Banach spaces by weakening his condition on the sequence $\{d_n\}$. The weakened condition on d_n requires refinement of Borodin's construction to extract an element in X, whose distances from the nested subspaces are precisely the given values d_n.

1. Introduction

For a subspace A of a normed linear space $(X, \|\cdot\|)$, we define the distance from $x \in X$ to A by

$$\rho(x, A) := \inf\{\|x - a\| : a \in A\}.$$

In 1938, S. N. Bernstein [8] proved that if $\{d_n\}_{n\geq 1}$ is a non-increasing null sequence (i.e., $\lim_{n\to\infty} d_n = 0$) of positive numbers, and Π_n is the vector space of all real polynomials of degree at most equal to n, then there exists a function $f \in C[0,1]$ such that

$$\rho(f, \Pi_n) = d_n, \text{ for all } n \geq 1.$$

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This remarkable result is called Bernstein's Lethargy Theorem (BLT) and is used in the constructive theory of functions [19]. Then it has been applied to the theory of quasi analytic functions in several complex variables [17]. Note that the density of polynomials in $C[0,1]$ (the Weierstrass Approximation Theorem) implies that

$$\lim_{n\to\infty} \rho(f,\Pi_n) = 0.$$

However, the Weierstrass Approximation Theorem gives no information about the speed of convergence for $\rho(f,\Pi_n)$. Following the proof of Bernstein [8], Timan [20] extended his result to an arbitrary system of strictly nested *finite-dimensional* subspaces $\{Y_n\}$. Later Shapiro [18], replacing $C[0,1]$ with an arbitrary Banach space $(X,\|\cdot\|)$ and $\{\Pi_n\}$ the sequence of n-dimensional subspaces of polynomials of degree up to n, with a sequence $\{Y_n\}$ where $Y_1 \subset Y_2 \subset \cdots$ are strictly nested *closed subspaces* of X, showed that in this setting, for each null sequence $\{d_n\}$ of non-negative numbers, there exists a vector $x \in X$ such that

$$\rho(x,Y_n) \neq O(d_n), \text{ as } n \to \infty.$$

That is, there is no $M > 0$ such that

$$\rho(x,Y_n) \leq M d_n, \quad \text{for all } n \geq 1.$$

In other words $\rho(x,Y_n)$ decays arbitrarily slowly. This result was later strengthened by Tyuriemskih [21] who established that the sequence of errors of best approximation from x to Y_n, $\{\rho(x,Y_n)\}$, may converge to zero at an arbitrary slow rate up to some choice of $x \in X$. More precisely, for any expanding sequence $\{Y_n\}$ of subspaces of X and for any null sequence $\{d_n\}$ of positive numbers, he constructed an element $x \in X$ such that

$$\lim_{n\to\infty} \rho(x,Y_n) = 0, \text{ and } \rho(x,Y_n) \geq d_n \text{ for all } n \geq 1.$$

However, it is also possible that the errors of best approximation $\{\rho(x,Y_n)\}$ may converge to zero arbitrarily fast. For example, it is shown by Theorem 2.2 in [6] that, under some conditions imposed on $\{Y_n\}$ and $\{d_n\}$, for any null sequence $\{c_n\}$ of positive numbers, there exists an element $x \in X$ such that

$$\lim_{n\to\infty} \frac{\rho(x,Y_n)}{d_n} = 0 \text{ but } \frac{\rho(x,Y_n)}{d_n} \neq O(c_n) \text{ as } n \to \infty.$$

We refer the reader to [10] for an application of Tyuriemskih's Theorem to convergence of sequence of bounded linear operators and to [7] for a generalization of Shapiro's Theorem. We also refer to [1, 2, 4, 6, 13, 14] for other versions of Bernstein's Lethargy Theorem and to [3, 5, 15, 16, 23] for Bernstein's Lethargy Theorem for Fréchet spaces. Given an arbitrary Banach space X, a strictly increasing sequence $\{Y_n\}$ of subspaces of X and a non-increasing null sequence $\{d_n\}$ of non-negative numbers, one can ask the question whether there exists $x \in X$ such that $\rho(x,Y_n) = d_n$ for each n? For a long time no sequence $\{d_n\}$ of this type was known for which such an element x exists for *all* possible Banach spaces X. The only known spaces X in which the answer is always "yes" are the Hilbert spaces (see [22]). For a general (separable) Banach space X, a solution x is known to exist whenever all Y_n are finite-dimensional (see [20]). Moreover, it is known that if X has the above property, then it is reflexive (see [22]).

In this framework we provide two improvements on a theorem of S. N. Bernstein for Banach spaces. First we improve Borodin's Theorem 1 in [9]. We include the statement as Theorem 2.2 below. Namely, we obtain the same errors of best approximations as in Theorem 2.2 below but under a weaker condition on the sequence $\{d_n\}$. Then we use our first improvement to show that, if X is an arbitrary infinite-dimensional Banach space, and if $\{d_n\}$ is a decreasing null sequence of non-negative numbers, then under a natural condition on the subspaces $\{Y_n\}$, for any $c \in (0, 1]$, there exists $x_c \in X$ such that

$$cd_n \leq \rho(x_c, Y_n) \leq 4cd_n, \text{ for all } n \geq 1.$$

2. Preliminaries

Given a Banach space X and its subspaces $Y_1 \subset Y_2 \subset \cdots \subset Y_n \subset \ldots$, it is clear that

$$\rho(f, Y_1) \geq \rho(f, Y_2) \geq \cdots, \text{ for any } f \in X$$

and thus $\{\rho(f, Y_n)\}_{n \geq 1}$ form a non-increasing sequence of errors of best approximation from f to Y_n, $n \geq 1$. Furthermore we have:

Property 1: $\rho(\lambda x, Y_n) = |\lambda| \rho(x, Y_n)$ for any $x \in X$ and $\lambda \in \mathbb{R}$;
Property 2: $\rho(x + v, Y_n) = \rho(x, \overline{Y}_n)$ for any $x \in X$ and $v \in \overline{Y}_n$;
Property 3: $\rho(x_1 + x_2, Y_n) \leq \rho(x_1, Y_n) + \rho(x_2, Y_n)$ and consequently

$$\rho(x_1 + x_2, Y_n) \geq |\rho(x_1, Y_n) - \rho(x_2, Y_n)| \quad \text{for any} \quad x_1, x_2 \in X.$$

Note that we also have:

$$|\rho(x_1, Y_n) - \rho(x_2, Y_n)| \leq \|x_1 - x_2\| \text{ for } x_1, x_2 \in X,$$

which implies that the mapping $X \longrightarrow \mathbb{R}^+$ defined by $x \longmapsto \rho(x, Y_n)$ is continuous and thus properties of continuous mappings such as the intermediate value theorem can be used.

Next, we state a basic BLT result concerning *finite number* of subspaces, for the proof of the following lemma we refer the reader to Timan's book [20].

Lemma 2.1. *Let $(X, \|\cdot\|)$ be a normed linear space, $Y_1 \subset Y_2 \subset \ldots \subset Y_n \subset X$ be a finite system of strictly nested subspaces, $d_1 > d_2 > \ldots > d_n \geq 0$ and $z \in X \backslash Y_n$. Then, there is an element $x \in X$ for which $\rho(x, Y_k) = d_k$ $(k = 1, \ldots, n)$, $\|x\| \leq d_1 + 1$, and $x - \lambda z \in Y_n$ for some $\lambda > 0$.*

It is worth noting that, Borodin [9] proved Lemma 2.1 where $(X, \|\cdot\|)$ is assumed to be a Banach space, however with the same proof the result still holds for a normed linear space.

An element $x \in X$ satisfying $\rho(x, Y_n) = d_n$, $n \geq 1$ may exist if the sequence $\{d_n\}$ decreases strictly to zero. Borodin in [9] uses the above lemma for Banach space to establish the existence of such an element in case of rapidly decreasing sequences; more precisely, in 2006 he proves the following theorem:

Theorem 2.2 (Borodin [9], Theorem 1)**.** *Let X be an arbitrary Banach space (with finite or infinite dimension), $Y_1 \subset Y_2 \subset \cdots$ be an arbitrary countable system of strictly nested subspaces in X, and fix a numerical sequence $\{d_n\}_{n\geq 1}$ satisfying: there exists a natural number $n_0 \geq 1$ such that*

$$d_n > \sum_{k=n+1}^{\infty} d_k \text{ for all } n \geq n_0 \text{ at which } d_n > 0. \tag{2.1}$$

Then there is an element $x \in X$ such that

$$\rho(x, Y_n) = d_n, \text{ for all } n \geq 1. \tag{2.2}$$

The condition (2.1) on the sequence $\{d_n\}$ is the key to the derivation of (2.2) in Theorem 2.2. Based on this result, Konyagin [12] in 2013 takes a further step to show that, for a general non-increasing null sequence $\{d_n\}$, the deviation of $x \in X$ from each subspace Y_n can range in some interval depending on d_n.

Theorem 2.3 (Konyagin [12], Theorem 1)**.** *Let X be a real Banach space, $Y_1 \subset Y_2 \subset \cdots$ be a sequence of strictly nested closed linear subspaces of X, and $d_1 \geq d_2 \geq \cdots$ be a non-increasing sequence converging to zero, then there exists an element $x \in X$ such that the distance $\rho(x, Y_n)$ satisfies the inequalities*

$$d_n \leq \rho(x, Y_n) \leq 8d_n, \text{ for } n \geq 1. \tag{2.3}$$

Note that the condition (2.1) is satisfied when $d_n = (2+\epsilon)^{-n}$ for $\epsilon > 0$ arbitrarily small, however it is not satisfied when $d_n = 2^{-n}$. Of course there are two natural questions to ask:

Question 1: Is the condition (2.1) necessary for the results in Theorem 2.2 to hold, or does Theorem 2.2 still hold for the sequence $d_n = 2^{-n}$, $n \geq 1$?
Question 2: Under the same conditions given in Theorem 2.3, can the lower and upper bounds of $\rho(x, Y_n)$ in (2.3) be improved?

The aim of this paper is to show that the above two questions have affirmative answers. We have weakened the condition (2.1) in Theorem 2.2 and obtained the same result given by Theorem 2.2. We were also able to improve the bounds given in (2.3), provided some additional weak subspace condition.

Before we proceed with our results, observe that in Konyagin's paper [12] it is assumed that $\{Y_n\}$ are closed and strictly increasing (we will show this assumption can be weakened to $\overline{Y}_n \subset Y_{n+1}$). In Borodin's paper [9], this is not specified, but from the proof of Theorem 2.2 it is clear that his proof works only under assumption that $\overline{Y}_n$ is strictly included in Y_{n+1}. The necessity of this assumption on subspaces is illustrated by the following example.

Example 2.4. Let $X = L^\infty[0,1]$ and consider $C[0,1] \subset L^\infty[0,1]$ and define the subspaces of X as follows:

(1) $Y_1 =$ space of all polynomials;
(2) $Y_{n+1} =$span$[Y_n \cup \{f_n\}]$ where $f_n \in C[0,1] \setminus Y_n$, for $n \geq 1$.

Observe that by the Weierstrass Theorem we have $\overline{Y}_n = C[0,1]$ for all $n \geq 1$. Take any $f \in L^\infty[0,1]$ and consider the following cases:

a) If $f \in C[0,1]$, then

$$\rho(f, Y_n) = \rho(f, C[0,1]) = 0 \text{ for all } n \geq 1.$$

b) If $f \in L^\infty[0,1] \setminus C[0,1]$, then

$$\rho(f, Y_n) = \rho(f, C[0,1]) = d > 0 \quad \text{(independent of } n\text{)}.$$

Note that in above, we have used the fact that $\rho(f, Y_n) = \rho(f, \overline{Y}_n)$. Hence in this case BLT does not hold.

We will assume that the subspaces $\{Y_n\}$ satisfy $\overline{Y}_n \subset Y_{n+1}$ for $n \geq 1$ for the rest of the paper.

3. Improvement of Borodin's Result

Our first main result gives a positive answer to Question 1, by showing that Theorem 2.2 can be extended by weakening the strict inequality in (2.1) to a non-strict one:

$$d_n \geq \sum_{k=n+1}^{\infty} d_k, \text{ for every } n \geq n_0. \tag{3.1}$$

Clearly the condition (3.1) is weaker than (2.1), but unlike the condition (2.1), (3.1) is satisfied by the sequences $\{d_n\}_{n\geq 1}$ verifying $d_n = \sum\limits_{k=n+1}^{\infty} d_k$ for all $n \geq n_0$. For a typical example of such sequence one can take $\{d_n\} = \{2^{-n}\}$. As a consequence, the proof of Theorem 3.4 requires a finer construction of the element x than that of Theorem 2.2. Before proving Theorem 3.4, we provide the following technical lemmas.

Lemma 3.1. *Let $(X, \|\cdot\|)$ be a normed linear space and let Q be a subspace of X with $\overline{Q} \subset X$. Pick two elements $x_1, x_2 \in X \backslash \overline{Q}$ such that $x_2 \notin span[\{x_1\} \cup Q]$, and $\delta \geq 0$ such that*

$$\rho(x_2 - \delta x_1, Q) \leq \rho(x_2 - a x_1, Q), \text{ for all } a \geq \delta. \tag{3.2}$$

Then there exists a nonzero linear functional $f: X \longmapsto \mathbb{R}$ such that

$$f(q) = 0, \text{ for } q \in Q, \ \|f\| = \frac{1}{\rho(x_1, Q)}, \ f(x_1) = 1 \tag{3.3}$$

and

$$f(x_2) = \delta - \frac{\rho(x_2 - \delta x_1, Q)}{\rho(x_1, Q)}. \tag{3.4}$$

Proof. First we note that the existence of δ satisfying the condition (3.2) follows from a standard convexity argument. Now, let U be the linear subspace of X spanned by $\{x_1\} \cup Q$, then any element in U has unique decomposition of the form $q + \alpha x_1$, for some $q \in Q$, $\alpha \in \mathbb{R}$. From Hahn-Banach Theorem, we have the linear functional $g: U \longmapsto \mathbb{R}$ defined by

$$g(q + \alpha x_1) = \alpha, \text{ for any } q \in Q, \ \alpha \in \mathbb{R} \tag{3.5}$$

satisfies

$$g(q) = 0, \text{ for all } q \in Q, \ \|g\| = \frac{1}{\rho(x_1, Q)}, \text{ and } g(x_1) = 1. \tag{3.6}$$

Since $x_2 \notin U$, we then let S be the linear subspace of X, spanned by $\{x_2\} \cup U$. Define the *sublinear functional* p to be

$$p(x) = \|g\|\|x\|, \text{ for } x \in S. \tag{3.7}$$

We then can extend g to a linear functional $\tilde{g}: S \longmapsto \mathbb{R}$, by taking

$$\tilde{g}(q + \alpha x_1 + \beta x_2) = \alpha + \beta\nu, \text{ for all } q \in Q \text{ and } \alpha, \beta \in \mathbb{R}, \tag{3.8}$$

where the real number ν is chosen so that

$$\tilde{g}(x) \le p(x), \text{ for } x \in S. \tag{3.9}$$

From Hahn–Banach Theorem (see [11], Page 63, inequality (i)), any ν satisfying the inequalities (3.10) below yields (3.9). Note that

$$\begin{cases} \nu \ge \max_{q\in Q,\ \alpha\in\mathbb{R}} \{-p(-q-\alpha x_1 - x_2) - g(q+\alpha x_1)\}\,; \\ \nu \le \min_{q'\in Q,\ \alpha'\in\mathbb{R}} \{p(q'+\alpha' x_1 + x_2) - g(q'+\alpha' x_1)\}\,. \end{cases} \tag{3.10}$$

Next, we show that ν can take the value $\delta - \frac{\rho(x_2-\delta x_1,Q)}{\rho(x_1,Q)}$. Using (3.5), we know that it is enough to prove

$$-p(-q-\alpha x_1 - x_2) - \alpha \le \delta - \frac{\rho(x_2-\delta x_1,Q)}{\rho(x_1,Q)} \le p(q'+\alpha' x_1 + x_2) - \alpha', \tag{3.11}$$

for all $q, q' \in Q$ and $\alpha, \alpha' \in \mathbb{R}$. Equivalently, we will show

$$\delta - \frac{\rho(x_2-\delta x_1,Q)}{\rho(x_1,Q)} \ge -p(-q-\alpha x_1 - x_2) - \alpha, \text{ for } q \in Q \text{ and } \alpha \in \mathbb{R} \tag{3.12}$$

and

$$\delta - \frac{\rho(x_2-\delta x_1,Q)}{\rho(x_1,Q)} \le p(q'+\alpha' x_1 + x_2) - \alpha', \text{ for } q' \in Q \text{ and } \alpha' \in \mathbb{R}. \tag{3.13}$$

To show (3.12), we combine (3.7), (3.6), Property 3 of $\rho(\cdot, Q)$ and the fact that $q \in Q$ to obtain:

$$\begin{aligned} -p(-q-\alpha x_1 - x_2) - \alpha &= -\frac{\|q+\alpha x_1 + x_2\|}{\rho(x_1,Q)} - \alpha \\ \le -\frac{\rho(q+\alpha x_1 + x_2, Q)}{\rho(x_1,Q)} - \alpha &= -\frac{\rho(\alpha x_1 + x_2, Q)}{\rho(x_1,Q)} - \alpha \\ \le -\frac{||\alpha+\delta|\rho(x_1,Q) - \rho(x_2-\delta x_1,Q)|}{\rho(x_1,Q)} - \alpha & \end{aligned} \tag{3.14}$$

is true for all $q \in Q$ and $\alpha \in \mathbb{R}$. Now we have two cases to consider for the value of α:

(1) If $|\alpha+\delta|\rho(x_1,Q) - \rho(x_2-\delta x_1,Q) \le 0$, then (3.14) yields

$$\begin{aligned} -p(-q-\alpha x_1 - x_2) - \alpha &\le \frac{|\alpha+\delta|\rho(x_1,Q) - \rho(x_2-\delta x_1,Q)}{\rho(x_1,Q)} - \alpha \\ &= |\alpha+\delta| - \alpha - \frac{\rho(x_2-\delta x_1,Q)}{\rho(x_1,Q)}. \end{aligned} \tag{3.15}$$

(a) If $\alpha+\delta>0$, then (3.15) becomes

$$-p(-q-\alpha x_1-x_2)-\alpha\leq\delta-\frac{\rho(x_2-\delta x_1,Q)}{\rho(x_1,Q)}. \tag{3.16}$$

(b) If $\alpha+\delta\leq 0$, then by the assumption (3.2) we have

$$\rho(\alpha x_1+x_2,Q)\geq\rho(x_2-\delta x_1,Q).$$

This together with (3.14) and the fact that $\alpha\leq-\delta$ implies

$$\begin{aligned}-p(-q-\alpha x_1-x_2)-\alpha &\leq -\frac{\rho(\alpha x_1+x_2,Q)}{\rho(x_1,Q)}-\alpha\\ \leq -\frac{\rho(x_2-\delta x_1,Q)}{\rho(x_1,Q)}-\alpha &\leq \delta-\frac{\rho(x_2-\delta x_1,Q)}{\rho(x_1,Q)}.\end{aligned} \tag{3.17}$$

(2) If $|\alpha+\delta|\rho(x_1,Q)-\rho(x_2-\delta x_1,Q)>0$, then

(a) If $\alpha+\delta>0$, we solve for α to obtain

$$-\alpha<\delta-\frac{\rho(x_2-\delta x_1,Q)}{\rho(x_1,Q)}.$$

This fact together with $p(x)\geq 0$ (see (3.7)), straightforwardly implies

$$-p(-q-\alpha x_1-x_2)-\alpha\leq-\alpha<\delta-\frac{\rho(x_2-\delta x_1,Q)}{\rho(x_1,Q)}. \tag{3.18}$$

(b) If $\alpha+\delta\leq 0$, then again (3.17) holds.

Therefore (3.12) follows from (3.16), (3.17) and (3.18).

To show (3.13), again we apply (3.7) and Property 3 of $\rho(\cdot,Q)$, to obtain for all $q'\in Q$ and $\alpha'\in\mathbb{R}$,

$$\begin{aligned}p(q'+\alpha' x_1+x_2)-\alpha' &= \frac{\|q'+\alpha' x_1+x_2\|}{\rho(x_1,Q)}-\alpha'\geq\frac{\rho(\alpha' x_1+x_2,Q)}{\rho(x_1,Q)}-\alpha'\\ &\geq \frac{||\alpha'+\delta|\rho(x_1,Q)-\rho(x_2-\delta x_1,Q)|}{\rho(x_1,Q)}-\alpha'.\end{aligned} \tag{3.19}$$

Then two different cases follow, according to the values of α':

(1) If $|\alpha'+\delta|\rho(x_1,Q)-\rho(x_2-\delta x_1,Q)>0$, (3.19) implies

$$p(q'+\alpha' x_1+x_2)-\alpha'\geq\frac{|\alpha'+\delta|\rho(x_1,Q)-\rho(x_2-\delta x_1,Q)}{\rho(x_1,Q)}-\alpha'. \tag{3.20}$$

Now we consider the following subcases

(a) If $\alpha'+\delta>0$, then (3.20) yields

$$p(q'+\alpha' x_1+x_2)-\alpha'\geq\delta-\frac{\rho(x_2-\delta x_1,Q)}{\rho(x_1,Q)}. \tag{3.21}$$

(b) If $\alpha'+\delta\leq 0$, then by using (3.2) we have

$$\begin{aligned}p(q'+\alpha' x_1+x_2)-\alpha' &= \frac{\|q'+\alpha' x_1+x_2\|}{\rho(x_1,Q)}-\alpha'\\ \geq \frac{\rho(\alpha' x_1+x_2,Q)}{\rho(x_1,Q)}+\delta &\geq \delta-\frac{\rho(x_2-\delta x_1,Q)}{\rho(x_1,Q)}.\end{aligned} \tag{3.22}$$

(2) If $|\alpha'+\delta|\rho(x_1,Q)-\rho(x_2-\delta x_1,Q)<0$, and $\alpha'+\delta>0$, then its equivalent inequality

$$-\alpha' > \delta - \frac{\rho(x_2-\delta x_1,Q)}{\rho(x_1,Q)}$$

yields

$$p(q'+\alpha' x_1+x_2)-\alpha' \geq -\alpha' > \delta - \frac{\rho(x_2-\delta x_1,Q)}{\rho(x_1,Q)}. \tag{3.23}$$

If $\alpha'+\delta \leq 0$, then the result is the same as in (3.22).

Therefore combining (3.21), (3.22) and (3.23), we obtain (3.13).

Having proved (3.12) and (3.13), we see that (3.11) holds. Thus we can take $\nu = \delta - \dfrac{\rho(x_2-\delta x_1,Q)}{\rho(x_1,Q)}$ in (3.8), to obtain (3.4).

Now, we show $\|\tilde{g}\| = \|g\| = \dfrac{1}{\rho(x_1,Q)}$. This is true, because the fact that $\tilde{g}(q)=g(q)$ for $q\in Q$ implies $\|\tilde{g}\|\geq\|g\|$; and the fact that $\tilde{g}(x)\leq p(x)=\|g\|\|x\|$ leads to $\|\tilde{g}\|\leq\|g\|$. It then follows from (3.6) that $\|\tilde{g}\|=\|g\|=\dfrac{1}{\rho(x_1,Q)}$.

Finally, from the Hahn–Banach Extension Theorem $\tilde{g}$ can be further extended to a linear functional $f: X \longmapsto \mathbb{R}$, that satisfies $f(x)=\tilde{g}(x)$ for $x\in S$ and $\|f\|=\|\tilde{g}\|$. □

Remark 3.2. It is worth noting that, with a similar idea one can prove the following : Let $(X,\|\cdot\|)$ be a normed linear space and let Q be a subspace of X with $\overline{Q}\subset X$. Pick two elements $x_1,x_2\in X\backslash\overline{Q}$ such that $x_2\notin \operatorname{span}[\{x_1\}\cup Q]$, and assume there exists $\delta\geq 0$ such that

$$\rho(x_2+\delta x_1,Q)\leq\rho(x_2+ax_1,Q),\ \text{for all } a\geq\delta.$$

Then there exists a nonzero linear functional $f: X\longmapsto\mathbb{R}$ such that

$$f(q)=0,\ \text{for } q\in Q,\ \|f\|=\frac{1}{\rho(x_1,Q)},\ f(x_1)=1$$

and

$$f(x_2)=-\delta+\frac{\rho(x_2+\delta x_1,Q)}{\rho(x_1,Q)}.$$

Lemma 3.3. *Let Q_1,Q_2 be two subspaces of an arbitrary normed linear space $(Q_3,\|\cdot\|)$, such that $\overline{Q}_k\subset Q_{k+1}$ for $k=1,2$. Let $\{u_m\}_{m\geq1}$ and $\{v_m\}_{m\geq1}$ be two sequences of non-negative numbers, with $u_m>v_m$ for all $m\geq1$. Then there exist a sequence of elements $\{q_m\}_{m\geq1}\subset Q_3$ and a constant $c\geq1$ such that*

$$\rho(q_m,Q_1)=u_m,\ \rho(q_m,Q_2)=v_m,\ \textit{for all } m\geq1 \tag{3.24}$$

and

$$\|q_m-q_n\|\leq c\left(\max\{u_m,u_n\}-\min\{v_m,v_n\}\right),\ \textit{for all } m,n\geq1. \tag{3.25}$$

Proof. Since $\overline{Q}_k\subset Q_{k+1}$ for $k=1,2$, then by Lemma 2.1 there exists an element $z\in Q_3\backslash\overline{Q}_2$ such that

$$\rho(z,Q_1)=2 \text{ and } \rho(z,Q_2)=1. \tag{3.26}$$

The fact that $\rho(z, Q_2) = 1$ implies that, for any $\varepsilon > 0$ arbitrarily small, one can find a corresponding element $w \in \overline{Q}_2$ such that

$$\|z - w\| = 1 + \varepsilon. \tag{3.27}$$

Since by using Property 3 of $\rho(\cdot, Q_1)$ and (3.26),

$$\rho(w, Q_1) \geq \rho(z, Q_1) - \rho(z - w, Q_1) \geq \rho(z, Q_1) - \|z - w\| = 1 - \varepsilon \neq 0,$$

we then let

$$\delta_{min} = 1 + \frac{1 + \varepsilon - \rho(z - w, Q_1)}{\rho(w, Q_1)} \text{ and } \delta_{max} = \frac{3 + \varepsilon}{\rho(w, Q_1)}. \tag{3.28}$$

It is clear that $1 \leq \delta_{min} \leq \delta_{max}$, thanks to the fact that $\rho(z - w, Q_1) \leq \|z - w\| = 1 + \varepsilon$ and Property 3 of $\rho(\cdot, Q_1)$. Next observe from (3.28) and (3.26) that,

$$\rho\left(z - \delta_{min} w, Q_1\right) \leq \rho(z - w, Q_1) + (\delta_{min} - 1)\rho(w, Q_1) = 1 + \varepsilon \tag{3.29}$$

and

$$\rho\left(z - \delta_{max} w, Q_1\right) \geq \delta_{max}\rho(w, Q_1) - \rho(z, Q_1) = 3 + \varepsilon - 2 = 1 + \varepsilon. \tag{3.30}$$

The mapping $\lambda \longmapsto \rho(z - \lambda w, Q_1)$ is continuous, then by (3.29), (3.30) and the intermediate value theorem, there exists a set of $\{\delta_i\} \subset [\delta_{min}, \delta_{max}]$ such that $\delta_i < \delta_{i+1}$ for all i and

$$\rho(z - \delta_i w, Q_1) = 1 + \varepsilon. \tag{3.31}$$

If the set $\{\delta_i\}$ is finite, say $\{\delta_i\} = \{\delta_1, \delta_2, \ldots, \delta_K\}$, then we denote

$$\delta = \delta_K. \tag{3.32}$$

If the set $\{\delta_i\}$ is an infinite sequence, then since it is strictly increasing and bounded, and the mapping $\lambda \to \rho(z - \lambda w, Q_1)$ is continuous, there exists a limit $\delta^* = \lim_{i\to\infty} \delta_i$ such that $\delta^* \in [\delta_{min}, \delta_{max}]$ and

$$\rho(z - \delta^* w, Q_1) = 1 + \varepsilon.$$

In this case we define

$$\delta = \delta^*. \tag{3.33}$$

It follows from (3.32) and (3.33) that

$$1 + \varepsilon = \rho(z - \delta w, Q_1) \leq \rho(z - aw, Q_1), \text{ for all } a \in [\delta, \delta_{max}]. \tag{3.34}$$

Construction of $\{q_m\}_{m\geq 1}$: The fact that $z \in Q_3 \setminus \overline{Q}_2$ results in

$$z \notin \operatorname{span}[\{w\} \cup Q_1] \subseteq Q_2.$$

Observe the triangle inequality

$$\rho\left(z - aw, Q_1\right) \geq a\rho(w, Q_1) - \rho(z, Q_1) \geq 1 + \varepsilon, \text{ for all } a \geq \delta_{max}.$$

Then from the above inequality and (3.34), we know that

$$1 + \varepsilon = \rho(z - \delta w, Q_1) \leq \rho(z - aw, Q_1), \text{ for all } a \geq \delta.$$

Then we can apply Lemma 3.1 to confirm the existence of a nonzero real-valued linear functional $f:\ Q_3 \to \mathbb{R}$ such that

$$\begin{aligned} &Q_1 \subset kerf,\ \|f\| = \frac{1}{\rho(w,Q_1)},\ f(w)=1,\\ &f(z) = \delta - \frac{\rho(z-\delta w, Q_1)}{\rho(w,Q_1)} = \delta - \frac{1}{\rho(w,Q_1)}. \end{aligned} \tag{3.35}$$

We define

$$x_1 = (f(z)-\delta)w,\ x_2 = z - f(z)w. \tag{3.36}$$

We will show that the sequence $\{q_m\}_{m\geq 1}$ satisfying (3.24) and (3.25) can be found in span$[\{x_1,x_2\}]$ (see (3.41) below). Using (3.36) and (3.26) we obtain

$$\rho(v_m x_2, Q_2) = v_m. \tag{3.37}$$

First, by using (3.36), (3.31) and the fact that $v_m < u_m$, we have

$$\rho(v_m x_2 + v_m x_1, Q_1) = |v_m|\rho(z-\delta w, Q_1) = v_m \leq u_m. \tag{3.38}$$

And, since the kernel $kerf$ satisfies

$$\rho(x, kerf) = \frac{|f(x)|}{\|f\|},\ \text{for all } x \in Q_3, \tag{3.39}$$

then by using (3.39), (3.36) and (3.35), we obtain

$$\begin{aligned} &\rho(v_m x_2 + u_m x_1, Q_1) \geq \rho(v_m x_2 + u_m x_1, kerf)\\ &= \frac{|f(v_m x_2 + u_m x_1)|}{\|f\|} = \rho(w,Q_1)\,|v_m f(z - f(z)w) + u_m f((f(z)-\delta w))|\\ &= \rho(w,Q_1)u_m\,|f(z)-\delta| = \rho(w,Q_1)u_m \left|\delta - \frac{1}{\rho(w,Q_1)} - \delta\right|\\ &= \rho(w,Q_1)u_m\left(\frac{1}{\rho(w,Q_1)}\right) = u_m. \end{aligned} \tag{3.40}$$

Since the mapping $\lambda \longmapsto \rho(v_m x_2 + \lambda x_1, Q_1)$ is continuous, it follows from (3.38), (3.40) and the intermediate value theorem that there is a real number $\mu_m \in [v_m, u_m]$ such that

$$\rho(v_m x_2 + \mu_m x_1, Q_1) = u_m \text{ and } \rho(v_m x_2 + \mu_m x_1, Q_2) = v_m.$$

We then denote by

$$q_m = v_m x_2 + \mu_m x_1 = v_m(z - f(z)w) + \mu_m(f(z)-\delta)w,\ \text{for all } m \geq 1. \tag{3.41}$$

As a consequence (3.24) holds:

$$\rho(q_m, Q_1) = u_m,\ \rho(q_m, Q_2) = v_m,\ \text{for all } m \geq 1.$$

Now we show (3.25) holds. To this end we first state the following 2 evident facts:

(1) If $\mu_m \in [v_m, u_m]$ for any $m \geq 1$, then we have

$$|\mu_m - \mu_n| \leq \max\{u_m, u_n\} - \min\{v_m, v_n\},\ \text{for any } m,n \geq 1. \tag{3.42}$$

(2) For any 4 real numbers u_m, v_m, u_n, v_n such that $u_m > v_m,\ u_n > v_n$, the following inequality holds:

$$|v_m - v_n| \leq \max\{u_m, u_n\} - \min\{v_m, v_n\}. \tag{3.43}$$

It results from (3.41), the triangle inequality, (3.42) and (3.43) that for any $m, n \geq 1$,

$$\begin{aligned} \|q_m - q_n\| &\leq |v_m - v_n| \|z - f(z)w\| + |\mu_m - \mu_n| |f(z) - \delta| \|w\| \\ &\leq c\left(\max\{u_m, u_n\} - \min\{v_m, v_n\}\right), \end{aligned}$$

where

$$c = \max\left\{\|z - f(z)w\|, |f(z) - \delta| \|w\|\right\} \geq |f(z) - \delta| \|w\| \geq \frac{\|w\|}{\rho(w, Q_1)} \geq 1.$$

The last inequality follows from (3.35) and hence (3.25) holds. Lemma 3.3 is proved by combining (3.24) and (3.25). □

Now, we are ready to prove the following theorem which improves the theorem of Borodin [9].

Theorem 3.4. *Let X be an arbitrary infinite-dimensional Banach space, $\{Y_n\}_{n\geq 1}$ be an arbitrary system of strictly nested subspaces with the property $\overline{Y}_n \subset Y_{n+1}$ for all $n \geq 1$, and let the non-negative numbers $\{d_n\}_{n\geq 1}$ satisfy the following property: there is an integer $n_0 \geq 1$ such that*

$$d_n \geq \sum_{k=n+1}^{\infty} d_k, \text{ for every } n \geq n_0.$$

Then there exists an element $x \in X$ such that $\rho(x, Y_n) = d_n$ for all $n \geq 1$.

Proof. We start by observing the following four cases:

(1) If $n_0 \geq 2$, the problem is easily converted to the case $n_0 = 1$: having constructed an element z with $\rho(z, Y_n) = d_n$ for all $n \geq n_0$, we can use Lemma 2.1 to construct an element x with $\rho(x, Y_k) = d_k$ at $k = 1, \ldots, n_0$ and such that $x - \lambda z \in Y_{n_0}$ for some $\lambda > 0$. But then observe that

$$d_{n_0} = \rho(x, Y_{n_0}) = \rho(\lambda z, Y_{n_0}) = \lambda d_{n_0};$$

therefore, $\lambda = 1$ and

$$\rho(x, Y_n) = \rho(z, Y_n) = d_n \text{ for all } n \geq n_0.$$

Finally $\rho(x, Y_n) = d_n$ for all $n \geq 1$.

(2) If $Y_1 = \{0\}$, we first convert the problem to the case $n_0 = 2$ (notice that $Y_2 \neq \{0\}$), then by using the above argument, convert the problem to the case $n_0 = 1$.

(3) If $d_n = 0$ starting from some n, then the desired element exists by applying Lemma 2.1 to $X = Y_n$ and the subspaces $Y_1 \subset Y_2 \subset \ldots \subset Y_{n-1}$ within it.

(4) Thus we assume $n_0 = 1$, $Y_1 \neq \{0\}$ and $d_n > 0$, $n \geq 1$ for the rest of the proof.

For each $j \geq 1$, we define

$$\tau_j = \sum_{k=j+1}^{\infty} d_k.$$

In view of the above assumptions, we know $\tau_j > 0$ and the sequence $\{\tau_j\}_{j\geq 1}$ is monotonically decreasing to 0. Since $Y_j \neq \{0\}$, then for any integers j, n with $1 \leq j \leq n$, we can set

$$Q_1 = \{0\},\ Q_2 = Y_j,\ Q_3 = Y_{j+1};\ u_n = 1 + \frac{\tau_n}{2^j d_j},\ v_n = 1$$

and apply Lemma 3.3 to obtain, the existence of a sequence $\{q_{j,n}\}_{n\geq j} \subset Y_{j+1} \backslash Y_j$ and a constant $c > 0$ such that

$$\rho(q_{j,n}, \{0\}) = 1 + \frac{\tau_n}{2^j d_j},\ \rho(q_{j,n}, Y_j) = 1 \tag{3.44}$$

and

$$\|q_{j,m} - q_{j,n}\| \leq c\frac{\tau_m}{2^j d_j},\ \text{for all } n \geq m \geq j. \tag{3.45}$$

Now we fix $n \geq 1$. Take $\lambda_{n,n} := d_n$. We see clearly from (3.44) that

$$\rho(\lambda_{n,n} q_{n,n}, Y_n) = \lambda_{n,n} \rho(q_{n,n}, Y_n) = d_n. \tag{3.46}$$

Since $\rho(q_{j,n}, Y_j) = 1 > 0$ for $j \leq n$ (see (3.44)), then there is a nonzero real-valued linear functional (see [11], Page 64) $f_{j,n} :\ Y_{j+1} \longmapsto \mathbb{R}$ such that

$$Y_j \subset ker f_{j,n} \text{ and } \|f_{j,n}\| = f_{j,n}(q_{j,n}) = 1. \tag{3.47}$$

We first assume $f_{n-1,n}(q_{n,n}) \geq 0$. Then first by using (3.44) and (3.47), we obtain

$$\begin{aligned} \rho(\lambda_{n,n} q_{n,n}, Y_{n-1}) &\leq \rho(\lambda_{n,n} q_{n,n}, \{0\}) = \lambda_{n,n} \rho(q_{n,n}, \{0\}) \\ &= d_n \left(1 + \frac{\tau_n}{2^n d_n}\right) \leq d_n + \tau_n = \sum_{k=n}^{\infty} d_k \leq d_{n-1}; \end{aligned} \tag{3.48}$$

and then, by the properties of $f_{n-1,n}$ in (3.47), we see

$$\rho(\lambda_{n,n} q_{n,n} + d_{n-1} q_{n-1,n}, Y_{n-1}) \geq d_{n-1} f_{n-1,n}(q_{n-1,n}) = d_{n-1}. \tag{3.49}$$

It follows from Property 3, (3.48) and (3.49) that the mapping

$$\lambda \longmapsto \rho(\lambda_{n,n} q_{n,n} + \lambda q_{n-1,n}, Y_{n-1})$$

is continuous, through which the image of $[0, d_{n-1}]$ contains d_{n-1}. Then by the intermediate value theorem there exists $\lambda_{n-1,n} \in [0, d_{n-1}]$ such that

$$\rho(\lambda_{n,n} q_{n,n} + \lambda_{n-1,n} q_{n-1,n}, Y_{n-1}) = d_{n-1}. \tag{3.50}$$

If in contrast $f_{n-1,n}(q_{n,n}) < 0$, we still have (3.48) and moreover,

$$\begin{aligned} \rho(\lambda_{n,n} q_{n,n} - d_{n-1} q_{n-1,n}, Y_{n-1}) &\geq -\lambda_{n,n} f_{n-1,n}(q_{n,n}) + d_{n-1} f_{n-1,n}(q_{n-1,n}) \\ &= d_{n-1}. \end{aligned}$$

This implies that we can find $\lambda_{n-1,n}$ which lies in the interval $[-d_{n-1}, 0]$ and satisfies the above equation (3.50). Furthermore, in both cases of $f_{n-1,n}(q_{n,n})$, by the fact that $q_{n-1,n} \in Y_n$ and (3.46), we have

$$\rho(\lambda_{n,n} q_{n,n} + \lambda_{n-1,n} q_{n-1,n}, Y_n) = \rho(\lambda_{n,n} q_{n,n}, Y_n) = d_n.$$

For any $2 \le k \le n$, assume that we can find the real numbers

$$\lambda_{n,n} \in [-d_n, d_n],\ \lambda_{n-1,n} \in [-d_{n-1}, d_{n-1}], \dots, \lambda_{k,n} \in [-d_k, d_k] \tag{3.51}$$

such that

$$\rho(\lambda_{n,n} q_{n,n} + \lambda_{n-1,n} q_{n-1,n} + \dots + \lambda_{k,n} q_{k,n}, Y_m) = d_m,\ \text{ for } m = k, k+1, \dots, n.$$

Let

$$z_{k,n} = \lambda_{n,n} q_{n,n} + \dots + \lambda_{k,n} q_{k,n}.$$

Without loss of generality, suppose $f_{k-1,n}(z_{k,n}) \ge 0$. Then first by using the triangle inequality, (3.51) and (3.44), we obtain

$$\begin{aligned} \rho(z_{k,n}, Y_{k-1}) &\le \|z_{k,n}\| \le \sum_{j=k}^{n} |\lambda_{j,n}| \|q_{j,n}\| \le \sum_{j=k}^{n} d_j \left(1 + \frac{\tau_n}{2^j d_j}\right) \\ &\le \sum_{j=k}^{n} d_j + \tau_n \sum_{j=1}^{\infty} 2^{-j} = \sum_{j=k}^{n} d_j + \tau_n \le d_{k-1} \end{aligned} \tag{3.52}$$

and

$$\rho(z_{k,n} + d_{k-1} q_{k-1,n}, Y_{k-1}) \ge d_{k-1} f_{k-1,n}(q_{k-1,n}) = d_{k-1}.$$

Therefore, there is $\lambda_{k-1,n} \in [0, d_{k-1}]$ such that

$$\rho(z_{k,n} + \lambda_{k-1,n} q_{k-1,n}, Y_{k-1}) = d_{k-1}.$$

(If $f_{k-1,n}(z_{k,n}) < 0$, then the number $\lambda_{k-1,n}$ must be found in $[-d_{k-1}, 0]$.) Furthermore,

$$\rho(z_{k,n} + \lambda_{k-1,n} q_{k-1,n}, Y_m) = \rho(z_{k,n}, Y_m) = d_m \text{ for } m = k, \dots, n.$$

Continuing this procedure until $k = 1$ is included, we obtain the element

$$x_{n,n} = \lambda_{n,n} q_{n,n} + \dots + \lambda_{1,n} q_{1,n},$$

for which $\rho(x_{n,n}, Y_k) = d_k$ and $|\lambda_{k,n}| \le d_k$ for $1 \le k \le n$. Using the usual diagonalization process, we choose a sequence Λ of indices n such that, for all $k \ge 1$, $\lambda_{k,n}$ converges to the limit λ_k as $n \to \infty$, $n \in \Lambda$. We then claim that $|\lambda_k| \le d_k$ and the limit $\lim_{n\to\infty} \sum_{k=1}^{n} \lambda_k q_{k,n}$ exists in X. As a matter of fact, by using the triangle inequality, we have for $m \le n$,

$$\begin{aligned} \Big\| \sum_{k=1}^{n} \lambda_k q_{k,n} - \sum_{k=1}^{m} \lambda_k q_{k,m} \Big\| &= \Big\| \sum_{k=1}^{m} \lambda_k (q_{k,n} - q_{k,m}) + \sum_{k=m+1}^{n} \lambda_k q_{k,n} \Big\| \\ &\le \sum_{k=1}^{m} |\lambda_k| \|q_{k,n} - q_{k,m}\| + \sum_{k=m+1}^{n} |\lambda_k| \|q_{k,n}\|. \end{aligned} \tag{3.53}$$

First by using $|\lambda_k| \le d_k$ and (3.45), we obtain

$$\sum_{k=1}^{m} |\lambda_k| \|q_{k,n} - q_{k,m}\| \le c\tau_m \sum_{k=1}^{m} 2^{-k} \xrightarrow[n,m\to\infty]{} 0, \tag{3.54}$$

next from (3.52) we see

$$\sum_{k=m}^{n} |\lambda_k| \|q_{k,n}\| \leq d_{m-1} \xrightarrow[n,m\to\infty]{} 0. \tag{3.55}$$

Combining (3.53), (3.54) and (3.55) yields

$$\Big\| \sum_{k=1}^{n} \lambda_k q_{k,n} - \sum_{k=1}^{m} \lambda_k q_{k,m} \Big\| \xrightarrow[n,m\to\infty]{} 0,$$

i.e., $\left\{ \sum_{k=1}^{n} \lambda_k q_{k,n} \right\}_{n\geq 1}$ is a Cauchy sequence in the Banach space X, therefore it has a limit in X.

Further, we claim that the element

$$x := \lim_{n\to\infty} \sum_{k=1}^{n} \lambda_k q_{k,n}$$

is the limit of the sequence $\{x_{n,n}\}_{n\in\Lambda}$ as $n \to \infty$. By using the facts that $\|q_{k,n}\| \leq 2$, $|\lambda_{k,n}| \leq d_k$ for all $n \geq 1$, $k \leq n$, and $|\lambda_k| \leq d_k$ for all $k \geq 1$, we obtain

$$\begin{aligned}
&\|x - x_{n,n}\| \leq \|x - x_n\| + \|x_n - x_{n,n}\| \\
&\leq \|x - x_n\| + \sum_{k=1}^{n} |\lambda_{k,n} - \lambda_k| \|q_{k,n}\| \\
&\leq \|x - x_n\| + \sum_{k=1}^{N} |\lambda_{k,n} - \lambda_k| \|q_{k,n}\| + \sum_{k=N+1}^{n} (|\lambda_{k,n}| + |\lambda_k|) \|q_{k,n}\| \\
&\leq \|x - x_n\| + 2 \sum_{k=1}^{N} |\lambda_{k,n} - \lambda_k| + 4 \sum_{k=N+1}^{n} d_k \\
&\leq \|x - x_n\| + 2N \max_{1\leq k\leq N, N<n} |\lambda_{k,n} - \lambda_k| + 4 d_N \\
&\xrightarrow[n\to\infty, n\in\Lambda, N\to\infty]{} 0.
\end{aligned}$$

Finally, by the continuity of $x \longmapsto \rho(x, Y_k)$, we obtain

$$\rho(x, Y_k) = \lim_{n\to\infty, n\in\Lambda} \rho(x_{n,n}, Y_k) = d_k, \text{ for all } k \geq 1.$$

□

Remark 3.5. In the above theorem, we improve Borodin's hypothesis comparing d_n with the tail of the sequence from a strict inequality to $\geq$. This means that our result applies when $d_n = \frac{1}{2^n}$, which was not the case in Borodin's formulation. However, our proof is not a mere existence result. We also give an explicit construction of an element $x \in X$, where the sequence $\rho(x, Y_n)$ is exactly the sequence d_n. The condition $\overline{Y_n} \subset Y_{n+1}$ does not come at the expense of our assumption to weaken the condition on the sequence d_n. This is a natural condition. To clarify the reason why almost all Lethargy Theorems have this condition on the subspaces, we gave a simple example before Theorem 3.4, (see Example 2.4).

4. Improvement of Konyagin's result

As an application of Theorem 3.4, we derive our second main result as an improvement of Konyagin's result in [12], which provides a positive answer to Question 2 that we posed in Section 2. In Konyagin [12] the interval for the error of best approximation was $[d_n, 8d_n]$, according to our Theorem 4.1 below this range can be reduced to $[d_n, 4d_n]$, under a natural subspace condition on $\{Y_n\}$. Notice that in the theorem below, we exclude the cases (1) $d_1 \geq d_2 \geq \ldots \geq d_{n_0} > d_{n_0+1} = 0$ for some $n_0 \geq 1$; (2) All $\{Y_n\}$ are finite-dimensional or Hilbert spaces; (3) $\{d_n\}$ satisfies the condition (3.1). It has been proved that in Cases (2) and (3), there is an element $x \in X$ such that $\rho(x, Y_n) = d_n$ for $n \geq 1$. In Case (1), analogous to the remark in Konyagin [12], Page 206, for every $\varepsilon > 0$ arbitrarily small, we can define

$$d'_n = \left(1 + \frac{(n_0 - n)\varepsilon}{n_0}\right) d_n, \text{ for } n = 1, 2, \ldots, n_0.$$

We then observe that

$$d'_1 > d'_2 > \ldots > d'_{n_0} = d_{n_0}, \quad \text{and} \quad Y_1 \subset Y_2 \subset \ldots \subset Y_{n_0} \subset Y_{n_0+1}.$$

Applying Lemma 2.1 to $\{Y_n\}_{1 \leq n \leq n_0}$ and $\{d'_n\}_{1 \leq n \leq n_0}$, we obtain an element $x \in Y_{n_0+1}$ such that

$$\rho(x, Y_n) = d'_n \in [d_n, (1+\varepsilon)d_n], \text{ for } n = 1, 2, \ldots, n_0,$$

and

$$\rho(x, Y_n) = 0 = d_n, \text{ for all } n > n_0.$$

Therefore $d_n \leq \rho(x, Y_n) \leq (1+\varepsilon)d_n$ for all $n \geq 1$.

Theorem 4.1. *Let X be an infinite-dimensional Banach space, $\{Y_n\}$ be a system of strictly nested subspaces of X satisfying the condition $\overline{Y}_n \subset Y_{n+1}$ for all $n \geq 1$. Let $\{d_n\}_{n \geq 1}$ be a non-increasing null sequence of strictly positive numbers. Assume that there exists an extension $\{(\tilde{d}_n, \tilde{Y}_n)\}_{n \geq 1} \supseteq \{(d_n, Y_n)\}_{n \geq 1}$ satisfying: $\{\tilde{d}_n\}_{n \geq 1}$ is a non-increasing null sequence of strictly positive values, $\overline{\tilde{Y}}_n \subset \tilde{Y}_{n+1}$ for $n \geq 1$; and there are some integer $i_0 \geq 1$ and a constant $K > 0$ such that*

$$\{K2^{-n}\}_{n \geq i_0} \subseteq \{\tilde{d}_n\}_{n \geq 1}.$$

Then for any $c \in (0, 1]$, there exists an element $x_c \in X$ (depending on c) such that

$$cd_n \leq \rho(x_c, Y_n) \leq 4cd_n, \text{ for } n \geq 1. \tag{4.1}$$

Proof. We first show (4.1) holds for $c = 1$.

By assumption, there is a subsequence $\{n_i\}_{i \geq i_0}$ of $\mathbb{N}$ such that

$$\tilde{d}_{n_i} = K2^{-i}, \text{ for } i \geq i_0.$$

Since the sequence $\{\tilde{d}_n\}_{n=1,2,\ldots,n_{i_0}-1} \cup \{\tilde{d}_{n_i}\}_{i \geq i_0}$ satisfies the condition (3.1) and $\overline{\tilde{Y}}_n \subset \tilde{Y}_{n+1}$ for all $n \geq 1$, then we can apply Theorem 3.4 to get $x \in X$ so that

$$\rho(x, \tilde{Y}_n) = \tilde{d}_n, \text{ for } n = 1, \ldots, n_{i_0} - 1, \text{ and } \rho(x, \tilde{Y}_{n_i}) = \tilde{d}_{n_i}, \text{ for all } i \geq i_0. \tag{4.2}$$

Therefore for any integer $n \geq 1$,

Case 1: if $n \leq n_{i_0} - 1$ or $n = n_i$ for some $i \geq i_0$, it then follows from (4.2) that
$$\rho(x, \tilde{Y}_n) = \tilde{d}_n;$$
Case 2: if $n_i < n < n_{i+1}$ for some $i \geq i_0$, then the facts that $\{\tilde{d}_n\}$ is non-increasing and $\tilde{Y}_{n_i} \subset \tilde{Y}_n \subset \tilde{Y}_{n_{i+1}}$ lead to
$$\rho(x, \tilde{Y}_n) \in \left(\rho(x, \tilde{Y}_{n_{i+1}}), \rho(x, \tilde{Y}_{n_i})\right) = \left(K2^{-(i+1)}, K2^{-i}\right)$$
and
$$\tilde{d}_n \in \left[K2^{-i}, K2^{-i+1}\right].$$
It follows that
$$\frac{\rho(x, \tilde{Y}_n)}{\tilde{d}_n} \in \left(\frac{K2^{-i-1}}{K2^{-i+1}}, \frac{K2^{-i}}{K2^{-i}}\right) = \left(\frac{1}{4}, 1\right).$$
Putting together the 2 above cases yields
$$\frac{1}{4}\tilde{d}_n \leq \rho(x, \tilde{Y}_n) \leq \tilde{d}_n \text{ for all } n \geq 1.$$
For $c \in (0, 1]$, taking $x_c = 4cx$ in the above inequalities, we obtain
$$c\tilde{d}_n \leq \rho(x_c, \tilde{Y}_n) \leq 4c\tilde{d}_n, \text{ for all } n \geq 1.$$
Remembering that $\{(d_n, Y_n)\}_{n\geq 1} \subseteq \{(\tilde{d}_n, \tilde{Y}_n)\}_{n\geq 1}$, we then necessarily have
$$cd_n \leq \rho(x_c, Y_n) \leq 4cd_n, \text{ for all } n \geq 1.$$
Hence Theorem 4.1 is proved. □

Remark 4.2. The subspace condition given in Theorem 4.1 states that the nested sequence $\{Y_n\}$ has "enough gaps" so that the sequence
$$\{(d'_n, Y'_n)\}_{n\geq 1} = \{(d_n, Y_n)\}_{n\geq 1} \cup \{(K2^{-i}, \tilde{Y}_{n_i})\}_{i\geq i_0}$$
satisfies $d'_n \geq d'_{n+1} \to 0$ and $\overline{Y'_n} \subset Y'_{n+1}$ for all $n \geq 1$. A counterexample to this situation can be obtained by considering say Y_1 be an infinite-dimensional Banach but not Hilbert space and $Y_n = \text{span}[Y_1 \cup \{y_1, \ldots, y_{n-1}\}]$, with $y_n \notin Y_n$ for $n \geq 2$. In this case, there is no gap between any pair of Y_n and Y_{n+1}, so it is impossible to find any $\{\tilde{Y}_{n_i}\}_{i\geq i_0}$ such that $\{Y'_n\} = \{Y_n\}_{n\geq 1} \cup \{\tilde{Y}_{n_i}\}_{i\geq i_0}$ satisfies $\overline{Y'_n} \subset Y'_{n+1}$ for all $n \geq 1$.

Remark 4.3. There are several straightforward consequences arising from Theorem 4.1 which we list below.

(1) Taking $c = 1$ in Theorem 4.1, we obtain there exists $x \in X$ such that for $n \geq 1$,
$$d_n \leq \rho(x, Y_n) \leq 4d_n \text{ for all } n \geq 1. \tag{4.3}$$
Note that the inequalities (4.3) improve the upper bound of $\dfrac{\rho(x, Y_n)}{d_n}$ in Theorem 1 of Konyagin [12] from 8 to 4. However we should point out that, Konyagin's approach in [12] involves extracting a subsequence $\{d_{n_i}\}$ of $\{d_n\}$ satisfying (2.1), as a result the element x is selected such that $\rho(x, Y_n) = d_n$ for an infinite number of n. Our construction of x in (4.3)

does not necessarily satisfy this equality for infinitely many n, since our method involves extending $\{d_n\}$ to $\{d_n\} \cup \{K2^{-n}\}$, and Theorem 4.1 is applied to the "inserted sequence" $\{K2^{-n}\}$. Hence in view of our approach the obtained element x satisfies $\rho(x, Y_n) = d_n$ only when the intersection $\{d_n\}_{n\geq 1} \cap \{K2^{-n}\}_{n\geq i_0}$ contains an infinite number of elements.

(2) Taking $c = \frac{1}{4}$ in Theorem 4.1, we obtain existence of $x \in X$ for which

$$\frac{1}{4} \leq \frac{\rho(x, Y_n)}{d_n} \leq 1, \text{ for all } n \geq 1.$$

The interval length $\frac{3}{4}$ makes $[\frac{1}{4}, 1]$ the "narrowest" estimating interval of $\frac{\rho(x, Y_n)}{d_n}$ that Theorem 4.1 could provide.

(3) Bernstein Lethargy for Fréchet spaces is given in [3], thus an improvement on Konyagin's result can be deduced. However, our proof is constructive in explaining the relationship between the sequence of subspaces and the "lethargic" sequence we take.

(4) By using Borodin's result Theorem 2.2 and under the same subspace condition on $\{Y_n\}$, we can apply the same approach as in the proof of Theorem 4.1 to show that for any $\varepsilon > 0$ arbitrarily small, there exists $x \in X$ (depending on ε) such that

$$d_n \leq \rho(x, Y_n) \leq (2+\varepsilon)^2 d_n \text{ for all } n \geq 1. \tag{4.4}$$

Note that (4.4) above, in fact can be obtained by replacing $\{K2^{-n}\}$ with $\{K(2+\varepsilon)^{-n}\}$ in the proof of Theorem 4.1. Our approximation interval of $\rho(x, Y_n)$ in (4.3) presents less deviation than that in (4.4), due to the fact that unlike the condition (2.1), (3.1) allows to take $d_n = K2^{-n}$, $n \geq 1$.

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